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Two-Mode Excited Squeezed Vacuum State and Its Properties*

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Abstract *In this paper, the two-mode excited squeezed vacuum state (TESVS) is studied by using the statistical method. By calculating the normalization of the TESVS, a new form of Jacobi polynomials and some new identities are obtained. The photon number distribution of the TESVS is given and it is a simple form of Jacobi polynomials. Using the entangled state representation of Wigner operator, the Wigner function of the TESVS is obtainded and the variations of the Wigner function with the parameters m, n, and r are discussed. Here from the phase space point of view the TESVS can be well interpreted and described.*

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Key words: two-mode excited squeezed vacuum state, photon number distribution, Wigner function, entangled state representation

1 Introduction

Squeezed state[1−6] has been a major topic in quantum optics since 1970s because the quantum fluctuations of the squeezed state can be reduced below the vacuum-state limit in various forms of nonlinear optical interactions. Experimentally, the squeezed states of radiation are produced in nonlinear process, in which a "classical" electromagnetic field drives a nonlinear medium.[7] Theoretically, a two-mode squeezed state, which involves strong entanglement, is constructed by operating the squeezing operator $S_2(r) = \exp[r(a_1^\dagger a_2^\dagger - a_1 a_2)]$ on the two-mode vacuum state, i.e.,

$$S_2(r)|00\rangle = \mathrm{sech}(r)\exp(a_1^\dagger a_2^\dagger \tanh r)|00\rangle\,, \tag{1}$$

where $a_1^\dagger$ and $a_2^\dagger$ are the creation operators which obey $[a_i, a_j^\dagger] = \delta_{ij}$ $(i,j = 1,2)$. In Ref. [8], the excitation on the two-mode squeezed vacuum state is introduced,

$$a_1^{\dagger n} a_2^{\dagger m} S_2(r)|00\rangle \equiv |r,n,m\rangle\,, \tag{2}$$

its normalization constant is given by

$$\langle r,n,m|r,n,m\rangle = n!m!(\cosh^2 r)^n\, \mathrm{P}_m^{(0,n-m)}(\cosh 2r)\,, \tag{3}$$

where $\mathrm{P}_m^{(0,n-m)}(\cosh 2r)$ is a Jacobi polynomial, its normal expansion is[9]

$$\mathrm{P}_m^{(0,n-m)}(\cosh 2r) = (\cosh^2 r)^m \sum_{l=0}^{m} \frac{m!n!}{(l!)^2(m-l)!(n-l)!}(\tanh r)^{2l}\,. \tag{4}$$

By taking the normalization constant of $|r,n,m\rangle$ into account, we define the normalized TESVS as

$$[n!m!(\cosh^2 r)^n\, \mathrm{P}_m^{(0,n-m)}(\cosh 2r)]^{-1/2} a_1^{\dagger n} a_2^{\dagger m} S(r)|00\rangle \equiv ||r,n,m\rangle\,. \tag{5}$$

The TESVS is of interest because it can be generated when an excited atom passes through a cavity field which is in a two-mode squeezed vacuum state.

In this work, by virtue of the entangled state representation we shall study the two-mode excited squeezed vacuum state (TESVS) and its properties. This paper is organized as follows. In Sec. 2 we shall show that the state $|r,n,m\rangle$ provides a new approach (i.e., quantum optics method) to obtaining a new form of Jacobi polynomials. Then its applications are briefly discussed. In Sec. 3 the photon number distribution of the TESVS is obtained. In Sec. 4 the Wigner function of the TESVS is calculated based on the entangled state representation of the Wigner operator. Further, the physical meaning of the Wigner function is identified according to its marginal distributions.

2 Normalization of TESVS

From Eq. (1), we obtain the normalization function for the TESVS

$$\langle r,n,m|r,n,m\rangle = \mathrm{sech}^2 r\langle 00|\exp[a_1a_2\tanh r]a_2^m a_1^n a_1^{\dagger n} a_2^{\dagger m} \times \exp[a_1^\dagger a_2^\dagger \tanh r]|00\rangle\,. \tag{6}$$

Using the following formulae,[10]

$$\int \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2| = 1\,, \tag{7}$$

and

$$\langle z_1z_2|00\rangle = \exp\Big[-\frac{1}{2}(|z_1|^2+|z_2|^2)\Big]\,, \tag{8}$$

from Eq. (6), we have

$$\langle r,n,m|r,n,m\rangle = \mathrm{sech}^2 r\iint \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2}|z_1|^{2n}|z_2|^{2m}\exp[-|z_1|^2-|z_2|^2+(z_1z_2+z_1^*z_2^*)\tanh r]\,. \tag{9}$$

Further, using the following two well-known integral formulae,[11]

$$\int \frac{\mathrm{d}^2 z}{\pi}\exp(\zeta|z|^2+\xi z+\eta z^*)z^n z^{*m} = \mathrm{e}^{-\xi\eta/\zeta}\sum_{l=0}^{\min(m,n)}\frac{m!n!\xi^{m-l}\eta^{n-l}}{l!(m-l)!(n-l)!(-\zeta)^{m+n-l+1}}\,,\qquad (\mathrm{Re}\,\zeta<0)\,, \tag{10}$$

$$\int \frac{\mathrm{d}^2 z}{\pi}z^{*n}z^k\,\mathrm{e}^{\lambda|z|^2} = \delta_{n,k}(-)^{k+1}\lambda^{-(k+1)}k!\,, \tag{11}$$

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it is not difficult to obtain the normalization constant for $|r,n,m\rangle$ as follows:

$$\langle r,n,m|r,n,m\rangle = \mathrm{sech}^2 r\sum_{l=0}^{m}\frac{(m!)^2(\tanh^2 r)^{m-l}}{l![(m-l)!]^2}\int\frac{\mathrm{d}^2 z_1}{\pi}|z_1|^{2(m-l+n)}\exp[-|z_1|^2\,\mathrm{sech}^2 r]$$
$$=\sum_{l=0}^{m}\frac{(m!)^2(m+n-l)!}{l![(m-l)!]^2}(\cosh^2 r)^n(\sinh^2 r)^{m-l}\,. \tag{12}$$

Comparing Eq. (12) with Eq. (3) leads to

$$\mathrm{P}_m^{(0,n-m)}(\cosh 2r)=\sum_{l=0}^{m}\frac{m!\,(m+n-l)!}{n!l!\,[(m-l)!]^2}\left(\sinh^2 r\right)^{m-l}\,, \tag{13}$$

which is in sharply contrast in form to the normal form of Jacobi polynomials in Eq. (4). We point out that this form of Jacobi polynomials cannot be obtained by some series summation rearrangement technique from its original definition in Eq. (4). Hence, we think that it is a new form of Jacobi polynomials. In fact, we also check the validity of the new form of Jacobi polynomials by listing some terms and comparing them with the corresponding terms of the standard Jacobi polynomials. For instance, from Eq. (13) we can list

$$m=0\,,\quad \mathrm{P}_0^{(0,\,n)}(\cosh 2r)=1\,, \tag{14}$$

$$m=1\,,\quad \mathrm{P}_1^{(0,\,n-1)}(\cosh 2r)=1\,, \tag{15}$$

$$m=2\,,\quad \mathrm{P}_2^{(0,n-2)}(\cosh 2r)=1+2(n+1)\sinh^2 r+\frac{(n+1)(n+2)}{2}\sinh^4 r\,, \tag{16}$$

$$m=3\,,\quad \mathrm{P}_3^{(0,n-3)}(\cosh 2r)=1+3(n+1)\sinh^2 r+\frac{3(n+1)(n+2)}{2}\sinh^4 r+\frac{(n+1)(n+2)(n+3)}{6}\sinh^6 r\,, \tag{17}$$

$$m=4\,,\quad \mathrm{P}_4^{(0,n-4)}(\cosh 2r)=1+4(n+1)\sinh^2 r+3(n+1)(n+2)\sinh^4 r+\frac{2(n+1)(n+2)(n+3)}{3}\sinh^6 r$$
$$+\frac{(n+1)(n+2)(n+3)(n+4)}{24}\sinh^8 r\,. \tag{18}$$

Using the normalization constant of the TESVS and the new form of Jacobi polynomials, we can obtain some useful results. For example, comparing Eq. (13) with Eq. (4), we have the following equation,

$$\sum_{l=0}^{m}\binom{m}{l}\left[\binom{n}{l}(\cosh r)^{2m}(\tanh r)^{2l}-\binom{m+n-l}{n}(\sinh r)^{2(m-l)}\right]=0\,, \tag{19}$$

if setting $x=\sinh^2 r$, we can obtain a new identity equation

$$\sum_{l=0}^{m}\binom{m}{l}\left[\binom{n}{l}(1+x)^{m-l}-\binom{m+n-l}{n}x^{m-2l}\right]x^l=0\,. \tag{20}$$

If setting $y\to\coth^2 r$, we can rewrite Eq. (12) as

$$\langle r,n,m|r,n,m\rangle=\sum_{l=0}^{m}\frac{(m!)^2(m+n-l)!}{l![(m-l)!]^2}\frac{y^n}{(y-1)^{m+n-l}}\,. \tag{21}$$

Comparing Eq. (12) with the following formula,[8]

$$\langle r,n,m|r,n,m\rangle=n!(y-1)y^m\Big(-\frac{\mathrm{d}}{\mathrm{d}y}\Big)^m\frac{y^n}{(y-1)^{n+1}}\Big|_{y\to\coth^2 r}\,, \tag{22}$$

it leads to a new mathematical differential formula, i.e.,

$$\frac{\mathrm{d}^m}{\mathrm{d}y^m}\frac{y^n}{(y-1)^{n+1}}=\sum_{l=0}^{m}(-)^m\frac{(m!)^2(m+n-l)!}{n!l![(m-l)!]^2}\frac{y^{n-m}}{(y-1)^{m+n-l+1}}\,, \tag{23}$$

which is difficult to be found in the ordinary mathematics handbooks.

3 Photon Number Distribution of TESVS

In order to obtain the photon number distribution of the TESVS, firstly in terms of the two-mode number state we expand $|r,n,m\rangle$ as

$$C_{l,k}=\langle l,k|r,n,m\rangle=\langle l,k|a_1^{\dagger n}a_2^{\dagger m}S_2(r)|00\rangle\,. \tag{24}$$

Then, we construct generating function $\chi(\lambda,\sigma)$ of the expansion coefficient $C_{l,k}$

$$\chi(\lambda,\sigma)=\langle l,k|\,\mathrm{e}^{\lambda a_1^\dagger}\,\mathrm{e}^{\sigma a_2^\dagger}S_2(r)|00\rangle=\frac{\mathrm{sech}\,r}{\sqrt{l!k!}}\langle 00|a_1^l a_2^k\,\mathrm{e}^{\lambda a_1^\dagger}\,\mathrm{e}^{\sigma a_2^\dagger}\exp[a_1^\dagger a_2^\dagger\tanh r]|00\rangle$$
$$=\frac{\mathrm{sech}\,r}{\sqrt{l!k!}}\frac{\partial^{l+k}}{\partial\xi^l\partial\eta^k}\langle 00|\,\mathrm{e}^{\xi a_1}\,\mathrm{e}^{\eta a_2}\,\mathrm{e}^{\lambda a_1^\dagger}\,\mathrm{e}^{\sigma a_2^\dagger}\exp[a_1^\dagger a_2^\dagger\tanh r]|00\rangle|_{\xi=\eta=0}\,. \tag{25}$$

Inserting Eq. (7) into Eq. (25), we have

$$\chi(\lambda,\sigma)=\frac{\operatorname{sech} r}{\sqrt{l!k!}}\frac{\partial^{l+k}}{\partial\xi^l\partial\eta^k}\langle 00|\,\mathrm{e}^{\xi a_1}\,\mathrm{e}^{\eta a_2}\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2|\,\mathrm{e}^{\lambda a_1^\dagger}\,\mathrm{e}^{\sigma a_2^\dagger}\exp[a_1^\dagger a_2^\dagger\tanh r]|00\rangle|_{\xi=\eta=0}$$
$$=\frac{\operatorname{sech} r}{\sqrt{l!k!}}\frac{\partial^{l+k}}{\partial\xi^l\partial\eta^k}\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}\exp(-|z_1|^2-|z_2|^2+\xi z_1+\eta z_2+\lambda z_1^*+\sigma z_2^*+z_1^*z_2^*\tanh r)|_{\xi=\eta=0}$$
$$=\frac{\operatorname{sech} r}{\sqrt{l!k!}}\frac{\partial^{l+k}}{\partial\xi^l\partial\eta^k}\exp[\eta(\sigma+\xi\tanh r)+\lambda\xi]|_{\xi=\eta=0}=\frac{\operatorname{sech} r}{\sqrt{l!k!}}\sum_{i=0}^{\min(l,k)}\frac{l!k!}{i!(l-i)!(k-i)!}\tanh^i(r)\sigma^{k-i}\lambda^{l-i}\,,\qquad(26)$$

where we have used the following integration formula[11]

$$\int\frac{\mathrm{d}^2z}{\pi}\exp(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2})=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\Big[\frac{-\zeta\xi\eta+\xi^2g+\eta^2f}{\zeta^2-4fg}\Big]\,,\qquad(27)$$

its convergence condition is

$$\mathrm{Re}\,(\zeta\pm f\pm g)<0\,,\quad \mathrm{Re}\Big(\frac{\zeta^2-4fg}{\zeta\pm f\pm g}\Big)<0\,.\qquad(28)$$

Further, using the expansion of two-mode Hermite polynomial $\mathrm{H}_{m,n}(\lambda,\lambda^*)$,[12]

$$\mathrm{H}_{m,n}(\lambda,\lambda^*)=\sum_{k=0}^{\min(m,n)}(-)^k\frac{m!n!}{k!(m-k)!(n-k)!}\lambda^{m-k}\lambda^{*n-k}\,,\qquad(29)$$

we can simplify $\chi(\lambda,\sigma)$ as

$$\chi(\lambda,\sigma)=\frac{\operatorname{sech} r}{\sqrt{l!k!}}(-\tanh r)^l\,\mathrm{H}_{l,k}\Big(-\frac{\lambda}{\tanh r},\sigma\Big)\,.\qquad(30)$$

Using the following differential relations of $\mathrm{H}_{m,n}(\lambda,\lambda^*)$

$$\frac{\partial}{\partial\lambda}\mathrm{H}_{m,n}(\lambda,\lambda^*)=m\,\mathrm{H}_{m-1,n}(\lambda,\lambda^*)\,,\qquad(31)$$
$$\frac{\partial}{\partial\lambda^*}\mathrm{H}_{m,n}(\lambda,\lambda^*)=n\,\mathrm{H}_{m,n-1}(\lambda,\lambda^*)\,,\qquad(32)$$

we have

$$C_{l,k}=\frac{\partial^{m+n}}{\partial\lambda^n\partial\sigma^m}\chi(\lambda,\sigma)|_{\lambda=\sigma=0}=\frac{\operatorname{sech} r}{\sqrt{l!k!}}(-\tanh r)^l\frac{\partial^{m+n}}{\partial\lambda^n\partial\sigma^m}\mathrm{H}_{l,k}\Big(-\frac{\lambda}{\tanh r},\sigma\Big)\Big|_{\lambda=\sigma=0}$$
$$=(-\tanh r)^{l-n}\operatorname{sech} r\frac{\sqrt{k!l!}}{(k-m)!(l-n)!}\mathrm{H}_{l-n,k-m}(0,0)\,.\qquad(33)$$

From Eq. (5) the photon number distribution of $|r,n,m\rangle$ is

$$\mathrm{P}(n)=\big[n!m!(\cosh^2 r)^n\,\mathrm{P}_m^{(0,n-m)}(\cosh 2r)\big]^{-1}|C_{l,k}|^2$$
$$=\frac{\operatorname{sech}^2 r(\tanh^2 r)^l k!l!}{n!m!(\sinh^2 r)^n\,\mathrm{P}_m^{(0,n-m)}(\cosh 2r)[(k-m)!(l-n)!]^2}\mathrm{H}^2_{l-n,k-m}(0,0)$$
$$=\frac{\operatorname{sech}^2 r(\tanh^2 r)^l}{n!m!(\sinh^2 r)^n\,\mathrm{P}_m^{(0,n-m)}(\cosh 2r)}\frac{k!l!}{[(k-m)!]^2}\,,\qquad(34)$$

which is only a simple form of Jacobi polynomials and where we have used the formula[12]

$$\mathrm{H}_{l-n,k-m}(0,0)=(-)^{k-m}(l-n)!\,.\qquad(35)$$

4 Wigner Function for $|r,n,m\rangle$

Now we want to derive the Wigner function for $|r,n,m\rangle$. The Wigner operator $\Delta_{1,2}(\rho,\gamma)$ under the entangled state $|\eta\rangle$ representation was obtained,[13]

$$\Delta_{1,2}(\rho,\gamma)=\int\frac{\mathrm{d}^2\eta}{\pi^3}|\rho-\eta\rangle\langle\rho+\eta|\exp(\eta\gamma^*-\eta^*\gamma)\,,\qquad(36)$$

where the state $|\eta\rangle$ is defined as

$$|\eta\rangle=\exp\Big[-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*a_2^\dagger+a_2^\dagger a_1^\dagger\Big]|00\rangle\,,\quad \eta=\eta_1+\mathrm{i}\eta_2\,.\qquad(37)$$

The state $|\eta\rangle$ obeys the eigenvector equations

$$(X_1-X_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle\,,\quad (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle\,.\qquad(38)$$

Therefore, the state $|\eta\rangle$ is the common eigenstate of two particles' relative position (X_1-X_2) and the total momentum (P_1+P_2) in two-mode Fock space.

From Eqs. (2) and (37), we obtain the Wigner function for $|r,n,m\rangle$

$$\langle r,n,m|\Delta_{1,2}(\rho,\gamma)|r,n,m\rangle = \langle 00|S_2^{-1}(r)a_2^m a_1^n \Delta_{1,2}(\rho,\gamma)a_1^{\dagger n}a_2^{\dagger m}S_2(r)|00\rangle\,. \tag{39}$$

It would be convenient to express

$$a_1^{\dagger n}a_2^{\dagger m}S_2(r)|00\rangle = S_2(r)b_1^{\dagger n}b_2^{\dagger m}|0\rangle\,, \tag{40}$$

where

$$b_1^{\dagger n} \equiv S_2^{-1}(r)a_1^{\dagger n}S_2(r) = (a_1^\dagger \cosh r + a_2 \sinh r)^n\,, \tag{41}$$

$$b_2^{\dagger m} \equiv S_2^{-1}(r)a_2^{\dagger m}S_2(r) = (a_2^\dagger \cosh r + a_1 \sinh r)^m\,, \tag{42}$$

such that

$$\langle r,n,m|\Delta_{1,2}(\rho,\gamma)|r,n,m\rangle = \langle 00|b_2^m b_1^n S_2^{-1}(r)\Delta_{1,2}(\rho,\gamma)S_2(r)b_1^{\dagger n}b_2^{\dagger m}|0\rangle\,, \tag{43}$$

now the Wigner operator $\Delta_{1,2}(\rho,\gamma)$ is directly sandwiched in between $S_2^{-1}(r)\cdots S_2(r)$. Since the squeezed operator $S_2(r)$ can be expressed as[13]

$$S_2(r) = \frac{1}{\mu}\int \frac{\mathrm{d}^2\eta}{\pi}\Big|\frac{\eta}{\mu}\Big\rangle\langle\eta|\,, \tag{44}$$

and satisfies the following relation

$$S_2(r)|\eta\rangle = \frac{1}{\mu}\Big|\frac{\eta}{\mu}\Big\rangle, \quad \mu = \mathrm{e}^r\,, \tag{45}$$

then

$$S_2^{-1}(r)\Delta_{1,2}(\rho,\gamma)S_2(r) = \mu^2\int\frac{\mathrm{d}^2\eta}{\pi^3}|\mu(\gamma-\eta)\rangle\langle\mu(\gamma+\eta)|\exp(\eta\rho^*-\eta^*\rho)\,, \quad \mu = \mathrm{e}^r\,. \tag{46}$$

Using Eqs. (40) $\sim$ (45), then equation (46) becomes

$$\langle r,n,m|\Delta_{1,2}(\rho,\gamma)|r,n,m\rangle = \mu^2(\cosh^2 r)^{m+n}\langle 00|a_2^m a_1^n\int\frac{\mathrm{d}^2\eta}{\pi^3}|\mu(\gamma-\eta)\rangle\langle\mu(\gamma+\eta)|\exp(\eta\rho^*-\eta^*\rho)a_1^{\dagger n}a_2^{\dagger m}|00\rangle\,. \tag{47}$$

Recalling the generating function formula of the two-variable Hermite polynomials $\mathrm{H}_{m,n}(\eta,\eta^*)$,[14]

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\mathrm{H}_{m,n}(\eta,\eta^*) = \exp(-tt'+t\eta+t'\eta^*)\,, \tag{48}$$

where

$$\mathrm{H}_{m,n}(\eta,\eta^*) = \sum_{l=0}^{\min(n,m)}\frac{(-)^l n!m!}{l!(m-l)!(n-l)!}\eta^{m-l}\eta^{*n-l} = \frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp(-tt'+t\eta+t'\eta^*)|_{t=t'=0}\,, \tag{49}$$

which yields

$$\mathrm{H}^*_{m,n}(\eta,\eta^*) = \mathrm{H}_{n,m}(\eta,\eta^*)\,. \tag{50}$$

The entangled state $|\eta\rangle$ given by Eq. (37) can be expanded as

$$|\eta\rangle = \mathrm{e}^{-|\eta|^2/2}\sum_{j,k=0}^{\infty}\frac{(-1)^k}{\sqrt{j!k!}}\mathrm{H}_{j,k}(\eta,\eta^*)|j,k\rangle\,, \tag{51}$$

where $|j,k\rangle = (a_1^{\dagger j}a_2^{\dagger k}/\sqrt{j!k!})|00\rangle$ is the two-mode number state. Therefore,

$$\langle\mu(\gamma+\eta)|a_1^{\dagger n}a_2^{\dagger m}|00\rangle = (-)^m\,\mathrm{H}_{m,n}(\mu(\gamma+\eta),\mu(\gamma+\eta)^*)\exp\Big[-\frac{1}{2}|\mu(\gamma+\eta)|^2\Big]\,, \tag{52}$$

$$\langle 00|a_2^m a_1^n|\mu(\gamma-\eta)\rangle = (-)^m\,\mathrm{H}_{n,m}(\mu(\gamma-\eta),\mu(\gamma-\eta)^*)\exp\Big[-\frac{1}{2}|\mu(\gamma-\eta)|^2\Big]\,. \tag{53}$$

Substituting Eqs. (52) and (53) into Eq. (47), then using Eqs. (27) and (49), we have

$$\begin{aligned}
&\langle r,n,m|\Delta_{1,2}(\rho,\gamma)|r,n,m\rangle \\
&= \mu^2(\cosh^2 r)^{m+n}\int\frac{\mathrm{d}^2\eta}{\pi^3}\mathrm{H}_{m,n}(\mu(\gamma+\eta),\mu(\gamma+\eta)^*)\,\mathrm{H}_{n,m}(\mu(\gamma-\eta),\mu(\gamma-\eta)^*)\exp(-\mu^2|\eta|^2-\mu^2|\gamma|^2+\eta\rho^*-\eta^*\rho) \\
&= \mu^2(\cosh^2 r)^{m+n}\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\frac{\partial^{m+n}}{\partial s^n\partial s'^m}\int\frac{\mathrm{d}^2\eta}{\pi^3}\exp(-\mu^2|\eta|^2-\mu^2|\gamma|^2+\eta\rho^*-\eta^*\rho-tt'+t\mu(\gamma+\eta) \\
&\quad + t'\mu(\gamma+\eta)^*-ss'+s\mu(\gamma-\eta)+s'\mu(\gamma-\eta)^*)|_{t=t'=s=s'=0} \\
&= \frac{\mu^2(\cosh^2 r)^{m+n}}{\pi^2}\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\frac{\partial^{m+n}}{\partial s^n\partial s'^m}\exp\Big(\Big(\mu\gamma-\frac{\rho}{\mu}-s'\Big)t+\Big(\mu\gamma^*+\frac{\rho^*}{\mu}-s\Big)t'+\Big(\mu\gamma+\frac{\rho}{\mu}\Big)s \\
&\quad + \Big(\mu\gamma^*-\frac{\rho^*}{\mu}\Big)s'-\mu^2|\gamma|^2-\frac{|\rho|^2}{\mu^2}\Big)\Big|_{t=t'=s=s'=0}
\end{aligned}$$

$$= \frac{(\cosh^2 r)^{m+n}}{\pi^2} \sum_{l=0}^{m} \sum_{k=0}^{n} \left[\binom{m}{l} \binom{n}{k} \right]^2 l!k! \left| \mu\gamma + \frac{\rho}{\mu} \right|^{2(m-l)} \left| \mu\gamma^* - \frac{\rho^*}{\mu} \right|^{2(n-k)} \exp\Big(-\mu^2|\gamma|^2 - \frac{|\rho|^2}{\mu^2} \Big) . \tag{54}$$

So the Wigner function for $||r, n, m\rangle$ is

$$\begin{aligned} W(\rho,\gamma) &= \langle r, n, m||\Delta_{1,2}(\rho,\gamma)||r, n, m\rangle \\ &= \frac{(\cosh^2 r)^m}{\pi^2 n! m! \mathrm{P}_m^{(0,n-m)}(\cosh 2r)} \sum_{l=0}^{m} \sum_{k=0}^{n} \left[\binom{m}{l} \binom{n}{k} \right]^2 l!k! \left| \mu\gamma + \frac{\rho}{\mu} \right|^{2(m-l)} \left| \mu\gamma^* - \frac{\rho^*}{\mu} \right|^{2(n-k)} \\ &\quad \times \exp\Big(-\mu^2|\gamma|^2 - \frac{|\rho|^2}{\mu^2} \Big) . \end{aligned} \tag{55}$$

Now we would like to discuss changes in the Wigner function $W(\rho,\gamma)$ of the TESVS as we vary the parameters m, n, and r. From Figs. 1 and 2 we find that the values and the number of the peaks, the moving direction of the peaks and the distance between two peaks are determined by the parameters m and n. For instance, in Fig. 1 for $m = 2$, $n = 0$, and $r = 0.1$ the single peak of the Wigner function $W(\rho,\gamma)$ moves along the direction of AB and develops two independent peaks in the end, however the values of the two peaks are small. With increasing n, the superimposed peaks A and B move respectively along the directions of AD and BC, at last they develop four independent peaks. But the values of the peaks are increased and the distance between the peaks A and D, B and C is increased. Similarly, by virtue of the numerical computation results, we also find that, with increasing m, the superimposed peaks A and B move respectively along the directions of AB and CD, and the distance between the peaks A and B, C and D is increased. So these results show that the TESVS exhibits very different quantum interference effect for different parameters m and n.

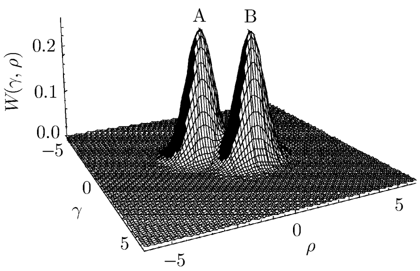

Fig. 1 Wigner functions of the TESVS for $m = 2$, $n = 0$, and $r = 0.1$.

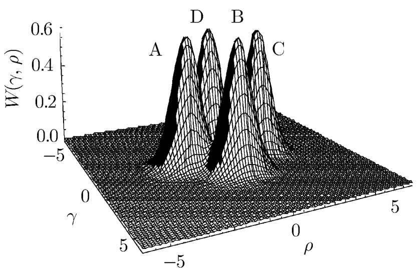

Fig. 2 Wigner functions of the TESVS for $m = n = 2$ and $r = 0.1$.

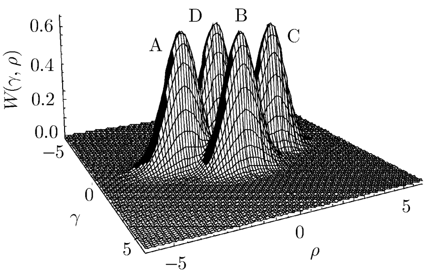

Fig. 3 Wigner functions of the TESVS for $m = n = 2$, and $r = 0.3$.

Comparing Fig. 2 with Fig. 3, in the phase space we clearly see four independent and compressed peaks of the Wigner function $W(\rho,\gamma)$ along the γ direction when the squeezing parameter r is small (for example, $r = 0.1$ in Fig. 2). With increasing r, the four peaks are further compressed along the γ direction at the expanse of an increase along the ρ-direction as can be seen in Fig. 3. Thus the phenomenon of squeezing can be well interpreted from the phase space point of view. In conclusion, the behaviour of the Wigner function $W(\rho,\gamma)$ is in a good agreement with the quantum features of the TESVS.

In the following section we shall obtain the marginal distribution of the Wigner function for $||r, n, m\rangle$ based on Eq. (55). From Eq. (36) carrying out the integral over $\mathrm{d}^2\rho$ for $\Delta_{1,2}(\rho,\gamma)$ yields[13]

$$\int \mathrm{d}^2\rho \Delta_{1,2}(\rho,\gamma) = \frac{1}{\pi} |\eta\rangle\langle\eta|_{\eta=\gamma} , \tag{56}$$

owing to

$$\begin{aligned} \langle\eta|r, n, m\rangle &= \langle 00| \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \exp\Big[-\frac{1}{2}|\eta|^2 + \eta^* a_1 - \eta a_2 + a_2 a_1 \Big] |z_1 z_2\rangle\langle z_1 z_2| a_1^{\dagger n} a_2^{\dagger m} S_2(r)|00\rangle \\ &= \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} z_1^{*n} z_2^{*m} \exp\Big[-|z_1|^2 - |z_2|^2 - \frac{|\eta|^2}{2} + \eta^* z_1 - \eta z_2 + z_1 z_2 + z_1^* z_2^* \tanh r \Big] \\ &= \sum_{l=0}^{n} \binom{n}{l} \eta^{*n-l} \int \frac{\mathrm{d}^2 z_2}{\pi} z_2^l z_2^{*m} \exp\Big[(\tanh r - 1)|z_2|^2 - \eta z_2 + \eta^* z_2^* \tanh r - \frac{|\eta|^2}{2} \Big] \\ &= \sum_{l=0}^{n} \binom{n}{l} \frac{(-)^m \eta^{*n-l}}{(1-\tanh r)^{l+1}} \mathrm{H}_{m,l}\Big(\frac{\eta}{1-\tanh r}, \eta^* \Big) \exp\Big(\frac{3-\tanh r}{2(\tanh r - 1)} |\eta|^2 \Big) , \end{aligned} \tag{57}$$

where we have employed the formula in Eq. (10). So we obtain a marginal distribution of the Wigner function for $||r, n, m\rangle$ in the γ variable

$$\int \mathrm{d}^2\rho W(\rho,\gamma) = \frac{1}{\pi} |\langle\eta||r, n, m\rangle|^2_{\eta=\gamma} = \frac{1}{\pi n! m! \, \mathrm{P}_m^{(0,n-m)}(\cosh 2r)}$$

$$\times \Big| \sum_{l=0}^{n} \binom{n}{l} \frac{(-)^m \gamma^{*n-l}}{(1-\tanh r)^{l+1}} \mathrm{H}_{m,l}\Big(\frac{\gamma}{1-\tanh r}, \gamma^*\Big) \exp\Big(\frac{3-\tanh r}{2(\tanh r - 1)}|\gamma|^2\Big)\Big|^2 . \tag{58}$$

Similarly, in the entangled state $|\xi\rangle$ representation the two-mode Wigner operator $\Delta_{1,2}(\rho,\gamma)$ is[15]

$$\Delta_{1,2}(\rho,\gamma) = \int \frac{\mathrm{d}^2\xi}{\pi^3} |\rho - \xi\rangle\langle\rho + \xi| \exp(\xi^*\gamma - \xi\gamma^*) , \tag{59}$$

where $|\xi\rangle$ is the common eigenstate of two particles' center-of-mass coordinate $(X_1 + X_2)$ and the relative momentum $(P_1 - P_2)$ in two-mode Fock space. The state $|\xi\rangle$ is given by

$$|\xi\rangle = \exp\Big[-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_2^\dagger a_1^\dagger\Big]|00\rangle , \quad \xi = \xi_1 + \mathrm{i}\xi_2 , \tag{60}$$

which is capable of making up the entangled state representation. The state $|\xi\rangle$ yields the following eigenvector equations

$$(X_1 + X_2)|\xi\rangle = \sqrt{2}\,\xi_1|\xi\rangle , \quad (P_1 - P_2)|\xi\rangle = \sqrt{2}\,\xi_2|\xi\rangle . \tag{61}$$

Performing the integration of $\Delta_{1,2}(\rho,\gamma)$ in Eq. (59) over $\mathrm{d}^2\gamma$ leads to a projection operator, i.e.,

$$\int \mathrm{d}^2\gamma \Delta_{1,2}(\rho,\gamma) = \frac{1}{\pi}|\xi\rangle\langle\xi|_{\xi=\rho} , \tag{62}$$

then using the expansion of the entangled state $|\xi\rangle$ in the Fock space

$$|\xi\rangle = \mathrm{e}^{-|\xi|^2/2} \sum_{j,k=0}^{\infty} \frac{1}{\sqrt{j!k!}} \mathrm{H}_{j,k}(\xi,\xi^*)|j,k\rangle . \tag{63}$$

Thus we obtain

$$\int \mathrm{d}^2\gamma W(\rho,\gamma) = \frac{1}{\pi}|\langle\xi||r,n,m\rangle|^2_{\xi=\rho} = \frac{1}{\pi n! m!\, \mathrm{P}_m^{(0,n-m)}(\cosh 2r)}$$

$$\times \Big| \sum_{l=0}^{n} \binom{n}{l} \frac{\rho^{*n-l}}{(1+\tanh r)^{l+1}} \mathrm{H}_{m,l}\Big(\frac{\rho}{1+\tanh r}, \rho^*\Big) \exp\Big(\frac{1-\tanh r}{2(1+\tanh r)}|\rho|^2\Big)\Big|^2 , \tag{64}$$

which is another marginal distribution of the Wigner function for the state $||r,n,m\rangle$ in the ρ variable. Equation (58) (or Eq. (64)) are proportional to the probability for finding the two particles, which have total momentum $\sqrt{2}\gamma_2$ (or relative momentum $\sqrt{2}\rho_2$) and simultaneously relative position $\sqrt{2}\gamma_1$ (or centre-of-mass position $\sqrt{2}\rho_1$), under the TESVS. Therefore, for an entangled particle system in the TESVS, the physical meaning of the Wigner function $W(\rho,\gamma)$ should lie in that its marginal distributions give the probability of finding the particles in an entangled way in the ρ-γ phase space.

In summary, in this article we have investigated the TESVS in the view of the phase space. By the normalization of the TESVS, we have obtained a new form of Jacobi polynomials, which is in sharply contrast in form to the normal form of Jacobi polynomials. With the aid of the normalization constant of the TESVS, we have derived some useful identities. The photon number distribution of the TESVS is given and it is a simple form of Jacobi polynomials. Using the entangled state representation of Wigner operator we have obtained the Wigner function of the TESVS and discussed the variations of the Wigner function with the parameters m, n and r. Here from the phase space point of view the TESVS can be well interpreted and described. Then the physical meaning of the Wigner function is given by its marginal distributions. These results may be useful for experiments as reference.

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