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Coupled KdV Equations and Their Explicit Solutions Through Two-Dimensional Hamiltonian System with a Quartic Potential*

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Abstract *Based on the second integrable case of known two-dimensional Hamiltonian system with a quartic potential, we propose a 4×4 matrix spectral problem and derive a hierarchy of coupled KdV equations and their Hamiltonian structures. It is shown that solutions of the coupled KdV equations in the hierarchy are reduced to solving two compatible systems of ordinary differential equations. As an application, quite a few explicit solutions of the coupled KdV equations are obtained via using separability for the second integrable case of the two-dimensional Hamiltonian system.*

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1 Introduction

It is well known that finite-dimensional integrable systems arise as the reduction of integrable soliton equations. In recent years, a considerable number of new finite-dimensional completely integrable systems have been discovered through various reduction techniques from soliton hierarchies.[1−8] A natural problem is whether the corresponding soliton hierarchy can be found from a known finite-dimensional integrable system. This paper is devoted to a two-dimensional integrable Hamiltonian system, from which we successfully find a spectral problem and its associated hierarchy of coupled KdV equations.

Let us consider the two-dimensional integrable Hamiltonian system,[9−16]

$$q_{jx} = \partial H/\partial p_j , \quad p_{jx} = -\partial H/\partial q_j , \quad j = 1, 2 , \tag{1}$$

with the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \xi q_1^2 + \zeta q_2^2 + \alpha q_1^4 + \gamma q_1^2 q_2^2 + \beta q_2^4 ,$$

where ξ, ζ, α, γ, and β are constant parameters. There are four nontrivial integrable cases which are separable:

(i) $\beta = \alpha$, $\gamma = 2\alpha$, with α, ξ, and ζ arbitrary;
(ii) $\beta = \alpha$, $\gamma = 6\alpha$, $\zeta = \xi$, with α and ξ arbitrary;
(iii) $\beta = 16\alpha$, $\gamma = 12\alpha$, $\zeta = 4\xi$, with α and ξ arbitrary;
(iv) $\beta = 8\alpha$, $\gamma = 6\alpha$, $\zeta = 4\xi$, with α and ξ arbitrary.

Case (i) is related to the KdV equation, which can be obtained by nonlinearization of Lax pair of the KdV equation.[5] A connection between Cases (ii) and (iii) and some stationary flows associated with the fourth-order Lax operators was discussed in Ref. [17].

In this paper, based on the investigation for the integrable Case (ii), we propose a 4×4 matrix spectral problem and derive a new hierarchy of coupled KdV equations and their bi-Hamiltonian structures. The typical systems of coupled equations in the hierarchy are reduced to

$$u_t = -u_{xxx} + 6uu_x + 6vv_x , \quad v_t = -v_{xxx} + 6(uv)_x \tag{2}$$

and

$$u_t = -v_{xxx} + 6(uv)_x , \quad v_t = -u_{xxx} + 6uu_x + 6vv_x , \tag{3}$$

which are two generalizations of the KdV equation. It is shown that solutions of the typical system of coupled KdV equations are reduced to solving two compatible systems of ordinary differential equations. Resorting to separability for integrable Case (ii), quite a few explicit solutions of the coupled KdV equations, including periodic solutions, are obtained.

2 Hierarchy and Hamiltonian Structures

In order to search for evolution equations associated with the integrable Case (ii), we first consider the canonical equations of integrable Case (ii)

$$\begin{aligned} & q_{1x} = p_1 , \quad q_{2x} = p_2 , \\ & p_{1x} = -\lambda q_1 - 4\alpha(q_1^2 + q_2^2)q_1 - 8\alpha q_1 q_2^2 , \\ & p_{2x} = -\lambda q_2 - 4\alpha(q_1^2 + q_2^2)q_2 - 8\alpha q_1^2 q_2 , \end{aligned} \tag{4}$$

with $\lambda = 2\xi$. A possible assumption is

$$y = (q_1, q_2, p_1, p_2)^{\mathrm{T}} , \quad u = -4\alpha(q_1^2 + q_2^2) , \quad v = -8\alpha q_1 q_2 . \tag{5}$$

Then we get a 4×4 matrix spectral problem

$$y_x = Uy , \quad U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ u-\lambda & v & 0 & 0 \\ v & u-\lambda & 0 & 0 \end{pmatrix} , \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} . \tag{6}$$

Let us first solve the stationary zero-curvature equation,

$$V_x - [U, V] = 0 , \quad V = (V_{ij})_{4\times 4} , \tag{7}$$

where each entry $V_{ij} = V_{ij}(A, B)$ is a Laurent expansion in λ:

$$\begin{aligned} & V_{11} = -A_x , \quad V_{12} = -B_x , \quad V_{13} = 2A , \\ & V_{14} = 2B , \quad V_{21} = -B_x , \quad V_{22} = -A_x , \\ & V_{23} = 2B , \quad V_{24} = 2A , \\ & V_{31} = -A_{xx} + 2(u-\lambda)A + 2vB , \\ & V_{32} = -B_{xx} + 2(u-\lambda)B + 2vA , \\ & V_{33} = A_x , \quad V_{34} = B_x , \\ & V_{41} = -B_{xx} + 2(u-\lambda)B + 2vA , \\ & V_{42} = -A_{xx} + 2(u-\lambda)A + 2vB , \\ & V_{43} = B_x , \quad V_{44} = A_x , \\ & A = \sum_{j\geq 0} A_{j-1}\lambda^{-j} , \quad B = \sum_{j\geq 0} B_{j-1}\lambda^{-j} . \end{aligned}$$

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The stationary zero-curvature equation is equivalent to the Lenard's equation

$$KG = \lambda JG, \quad G = (A, B)^{\mathrm{T}}, \tag{8}$$

that is,

$$KG_{j-1} = JG_j, \quad JG_{-1} = 0, \quad G_j = (A_j, B_j)^{\mathrm{T}}, \tag{9}$$

where K and J are two skew-symmetric operators,

$$K = \begin{pmatrix} -\partial^3 + 2\partial u + 2u\partial & 2\partial v + 2v\partial \\ 2\partial v + 2v\partial & -\partial^3 + 2\partial u + 2u\partial \end{pmatrix},$$

$$J = 4\begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}.$$

Since the equation $JG_{-1} = 0$ has two basic solutions

$$g_{-1} = (2, 0)^{\mathrm{T}}, \quad \hat{g}_{-1} = (0, 2)^{\mathrm{T}}.$$

We define $\{g_j\}$ and $\{\hat{g}_j\}$ by the following two systems of recurrence equations:

$$Kg_{j-1} = Jg_j, \quad g_j|_{(u,v)=0} = 0, \quad j \geq 0, \tag{10}$$

$$K\hat{g}_{j-1} = J\hat{g}_j, \quad \hat{g}_j|_{(u,v)=0} = 0, \quad j \geq 0, \tag{11}$$

which are uniquely determined. For example,

$$g_0 = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \hat{g}_0 = \begin{pmatrix} v \\ u \end{pmatrix}. \tag{12}$$

Therefore, if we choose the general solution, $G_{-1} = c_0 g_{-1} + \hat{c}_0 \hat{g}_{-1}$, of equation $JG_{-1} = 0$ as a starting point, G_j determined by Eq. (9) can be expressed as

$$\begin{aligned} G_j = {} & c_0 g_j + \hat{c}_0 \hat{g}_j + c_1 g_{j-1} + \hat{c}_1 \hat{g}_{j-1} + \cdots \\ & + c_{j+1} g_{-1} + \hat{c}_{j+1} \hat{g}_{-1}, \end{aligned} \tag{13}$$

where c_j and $\hat{c}_j$ $(j \geq 0)$ are arbitrary constants.

Let y satisfy the spectral problem (6) and its auxiliary problem

$$y_t = V^{(m)} y, \quad V^{(m)} = (V_{ij}^{(m)})_{4\times 4}, \tag{14}$$

where

$$V_{ij}^{(m)} = V_{ij}(A^{(m)}(\lambda), B^{(m)}(\lambda)),$$

$$A^{(m)}(\lambda) = \sum_{j=0}^{m} A_{j-1}\lambda^{m-j},$$

$$B^{(m)}(\lambda) = \sum_{j=0}^{m} B_{j-1}\lambda^{m-j}.$$

Then the compatibility condition between Eqs. (6) and (14) yields the zero-curvature equation $U_t - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$(u_t, v_t)^{\mathrm{T}} = X_m, \quad m \geq 0 \tag{15}$$

with $X_j = KG_{j-1} = JG_j$, $j \geq 0$. The first nontrivial system of evolution equations in the hierarchy (15) is

$$\begin{aligned} u_t = {} & c_0(-u_{xxx} + 6uu_x + 6vv_x) + \hat{c}_0(-v_{xxx} + 6(uv)_x) \\ & + 4c_1 u_x + 4\hat{c}_1 v_x, \\ v_t = {} & c_0(-v_{xxx} + 6(uv)_x) + \hat{c}_0(-u_{xxx} + 6uu_x + 6vv_x) \\ & + 4c_1 v_x + 4\hat{c}_1 u_x. \end{aligned} \tag{16}$$

which can be reduced to the coupled KdV equations (2) and (3).

To establish the Hamiltonian structures of evolution equations (15), we first calculate the following quantities:

$$\mathrm{tr}\Big(V\frac{\partial U}{\partial \lambda}\Big) = -4A, \quad \mathrm{tr}\Big(V\frac{\partial U}{\partial u}\Big) = 4A, \quad \mathrm{tr}\Big(V\frac{\partial U}{\partial v}\Big) = 4B.$$

Noticing the trace identity[18]

$$\Big(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\Big)\mathrm{tr}\Big(V\frac{\partial U}{\partial \lambda}\Big) = \Big[\lambda^{-\varepsilon}\Big(\frac{\partial}{\partial \lambda}\Big)\lambda^{\varepsilon}\Big]\Big(\mathrm{tr}\Big(V\frac{\partial U}{\partial u}\Big), \mathrm{tr}\Big(V\frac{\partial U}{\partial v}\Big)\Big),$$

where ε is a constant to be fixed. Equating the coefficients of λ^{-j-1} on both sides we obtain

$$\Big(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\Big)(-A_j) = (\varepsilon - j)(A_{j-1}, B_{j-1}), \quad j \geq 0. \tag{17}$$

To fix the constant ε, we simply set $j = 0$ in the above equation and arrive at $(-c_0, -\hat{c}_0) = 2\varepsilon(c_0, \hat{c}_0)$, from which we see that $\varepsilon = -1/2$. Hence we deduce that

$$\Big(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\Big)H_j = (A_{j-1}, B_{j-1}), \quad H_j = \frac{2}{2j+1}A_j, \tag{18}$$

from which we obtain the desired bi-Hamiltonian form of (15)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = K\begin{pmatrix} \delta H_m/\delta u \\ \delta H_m/\delta v \end{pmatrix} = J\begin{pmatrix} \delta H_{m+1}/\delta u \\ \delta H_{m+1}/\delta v \end{pmatrix}. \tag{19}$$

3 Explicit Solutions

In this section, we shall give explicit solutions of coupled KdV equation (16). To this end, we consider the two-dimensional Hamiltonian system,

$$q_{jt} = \frac{\partial F_0}{\partial p_j}, \quad p_{jt} = -\frac{\partial F_0}{\partial q_j}, \quad j = 1, 2, \tag{20}$$

with the Hamiltonian function

$$\begin{aligned} F_0 = {} & 2(\lambda c_0 + c_1)(p_1^2 + p_2^2 + \lambda q_1^2 + \lambda q_2^2 + 2\alpha q_1^4 + 12\alpha q_1^2 q_2^2 + 2\alpha q_2^4) \\ & + 4(\lambda\hat{c}_0 + \hat{c}_1)(p_1 p_2 + \lambda q_1 q_2 + 4\alpha q_1 q_2^3 + 4\alpha q_1^3 q_2). \end{aligned}$$

Equation (20) can be obtained by substituting constraint (5) into the time part, Eq. (14) with $m = 1$, of the Lax pair of Eq. (16).

Theorem 1 Let $(p_j(x,t), q_j(x,t))$, $j = 1, 2$, be involutive solutions of the Hamiltonian systems (4) and (20). Then

$$u = -4\alpha(q_1^2 + q_2^2), \quad v = -8\alpha q_1 q_2 \tag{21}$$

satisfy the coupled KdV equation (17).

Proof A direct calculation shows that the Hamiltonian functions H and F_0 are in involution with respect to the standard Poisson bracket, $\{H, F_0\} = 0$. Therefore the systems (4) and (20) are consistent and their involutive solution is a smooth function of (x, t).[19,20] By substituting Eq. (21) into Eq. (16) and using the systems (4) and (20), we see that the coupled KdV equation turns into an identity. This means that u and v determined by Eq. (21) satisfy Eq. (16).

The two-dimensional Hamiltonian systems (4) and (20) can be rewritten as

$$\begin{aligned} q_{1xx} &= -\lambda q_1 - 4\alpha q_1^3 - 12\alpha q_1 q_2^2 \,, \\ q_{2xx} &= -\lambda q_2 - 4\alpha q_2^3 - 12\alpha q_1^2 q_2 \,, \end{aligned} \tag{22}$$

$$\begin{aligned} q_{1tt} &= \beta_0(\lambda q_1 + 4\alpha q_1^3 + 12\alpha q_1 q_2^2) \\ &\quad + \beta_1(\lambda q_2 + 4\alpha q_2^3 + 12\alpha q_1^2 q_2) \,, \\ q_{2tt} &= \beta_1(\lambda q_1 + 4\alpha q_1^3 + 12\alpha q_1 q_2^2) \\ &\quad + \beta_0(\lambda q_2 + 4\alpha q_2^3 + 12\alpha q_1^2 q_2) \,. \end{aligned} \tag{23}$$

with

$$\beta_0 = -16(\lambda c_0 + c_1)^2 - 16(\lambda \hat{c}_0 + \hat{c}_1)^2 \,,$$
$$\beta_1 = -32(\lambda c_0 + c_1)(\lambda \hat{c}_0 + \hat{c}_1) \,.$$

Now we introduce the separating variables by Cartesian coordinates

$$f = q_1 + q_2 \,, \quad g = q_1 - q_2 \,.$$

The resulting equations from Eqs. (22) and (23) are of the following form,

$$f_{xx} = -\lambda f - 4\alpha f^3 \,, \quad f_{tt} = -b_1^2(\lambda f + 4\alpha f^3) \,, \tag{24}$$

$$g_{xx} = -\lambda g - 4\alpha g^3 \,, \quad g_{tt} = -b_2^2(\lambda g + 4\alpha g^3) \,, \tag{25}$$

with

$$b_1^2 = -\beta_0 - \beta_1 \,, \quad b_2^2 = \beta_1 - \beta_0 \,.$$

Corollary 1 Let $f(x, t)$ and $g(x, t)$ be the solutions of Eqs. (24) and (25), respectively. Then $(u(x, t), v(x, t))$ given by

$$u(x, t) = -2\alpha(f^2 + g^2) \,, \quad v(x, t) = -2\alpha(f^2 - g^2) \tag{26}$$

is a solution of the coupled KdV equation (16).

In what follows, we shall apply Corollary 1 to give explicit solution of Eq. (16). For simplicity, we assume that $\alpha = -\alpha_1^2/2 < 0$. Then equations (24) and (25) are written as

$$\begin{aligned} f_x^2 &= \alpha_1^2 f^4 - \lambda f^2 + \eta_1 \,, \\ f_t^2 &= b_1^2(\alpha_1^2 f^4 - \lambda f^2 + \eta_1) \,, \end{aligned} \tag{27}$$

$$\begin{aligned} g_x^2 &= \alpha_1^2 g^4 - \lambda g^2 + \eta_2 \,, \\ g_t^2 &= b_2^2(\alpha_1^2 g^4 - \lambda g^2 + \eta_2) \,, \end{aligned} \tag{28}$$

where η_1 and η_2 are arbitrary constants.

(i) Let $\lambda > 0$, $\eta_i < 0$, $i = 1, 2$. Equation (27) can be written in the form

$$\begin{aligned} f_x &= \alpha_1 \sqrt{(f^2 - a_1)(f^2 + a_2)} \,, \\ f_t &= \alpha_1 b_1 \sqrt{(f^2 - a_1)(f^2 + a_2)} \,, \end{aligned} \tag{29}$$

with

$$a_1 = \frac{1}{2\alpha_1^2}(\sqrt{\lambda^2 - 4\alpha_1^2\eta_1} + \lambda) \,,$$
$$a_2 = \frac{1}{2\alpha_1^2}(\sqrt{\lambda^2 - 4\alpha_1^2\eta_1} - \lambda) \,.$$

From Eq. (29), we have

$$\begin{aligned} \alpha_1 x + \gamma_1 &= \int_{\sqrt{\alpha_1}}^{f} \frac{\mathrm{d}w}{\sqrt{(w^2 - a_1)(w^2 + a_2)}} \\ &= \frac{1}{\sqrt{a_1 + a_2}} \int_0^{\sqrt{1 - a_1/f^2}} \frac{\mathrm{d}s}{\sqrt{(1 - s^2)(1 - k_1^2 s^2)}} \,, \end{aligned} \tag{30}$$

where γ_1 is a function of t, $k_1^2 = a_2/(a_1 + a_2)$, $w^2 = \alpha_1/(1 - s^2)$. Equation (30) determines the Jacobi elliptic function

$$\sqrt{1 - \frac{a_1}{f^2}} = \mathrm{sn}(\sqrt{a_1 + a_2}(\alpha_1 x + \gamma_1), k_1) \,,$$

that is,

$$f = \sqrt{a_1}\,\mathrm{nc}(\sqrt{a_1 + a_2}(\alpha_1 x + \gamma_1), k_1) \,. \tag{31}$$

Equation (29) implies $b_1 f_x = f_t$, which together with Eq. (31) deduces that

$$\gamma_{1t} = \alpha_1 b_1 \,, \quad \gamma_1 = \alpha_1(b_1 t + \delta_1) \,. \tag{32}$$

Here and in the following δ_j and $\hat{\delta}_j$ $(1 \le j \le 6)$ are integral constants. Substituting the second expression of Eq. (32) into Eq. (31) yields a solution of Eq. (24)

$$f_1 = \sqrt{a_1}\,\mathrm{nc}(\alpha_1\sqrt{a_1 + a_2}(x + b_1 t + \delta_1), k_1) \,. \tag{33}$$

In a similar way, we arrive at a solution of Eq. (25)

$$g_1 = \sqrt{\hat{a}_1}\,\mathrm{nc}(\alpha_1\sqrt{\hat{a}_1 + \hat{a}_2}(x + b_2 t + \delta_2), \hat{k}_1) \tag{34}$$

with

$$\hat{k}_1^2 = \frac{\hat{a}_2}{\hat{a}_1 + \hat{a}_2} \,,$$
$$\hat{a}_1 = \frac{1}{2\alpha_1^2}(\sqrt{\lambda^2 - 4\alpha_1^2\eta_2} + \lambda) \,,$$
$$\hat{a}_2 = \frac{1}{2\alpha_1^2}(\sqrt{\lambda^2 - 4\alpha_1^2\eta_2} - \lambda) \,.$$

(ii) Let $\lambda > 0$, $0 < \eta_i < \lambda^2/(4\alpha_1^2)$, $i = 1, 2$. By Eq. (27), we have

$$\begin{aligned} f_x &= \alpha_1 \sqrt{(f^2 - a_3)(f^2 - a_4)} \,, \\ f_t &= \alpha_1 b_1 \sqrt{(f^2 - a_3)(f^2 - a_4)} \,, \end{aligned} \tag{35}$$

where

$$a_3 = \frac{1}{2\alpha_1^2}(\lambda + \sqrt{\lambda^2 - 4\alpha_1^2\eta_1}) \,,$$
$$a_4 = \frac{1}{2\alpha_1^2}(\lambda - \sqrt{\lambda^2 - 4\alpha_1^2\eta_1}) \,.$$

Assume that $k_2^2 = a_3 a_4^{-1}$, $f = \sqrt{a_4}w^{-1}$. Then the first expression of Eq. (35) turns into

$$w_x = -\alpha_1\sqrt{a_4(1 - w^2)(1 - k_2^2 w^2)} \,,$$

which implies

$$\int_0^w \frac{\mathrm{d}s}{\sqrt{(1 - s^2)(1 - k_2^2 s^2)}} = -\alpha_1\sqrt{a_4}x + \gamma_2 \,.$$

The above expression is equivalent to $w = \mathrm{sn}(-\alpha_1\sqrt{a_4}x + \gamma_2, k_2)$, that is,

$$f = \sqrt{a_4}\,\mathrm{ns}(-\alpha_1\sqrt{a_4}x + \gamma_2, k_2) \,. \tag{36}$$

Substituting Eq. (36) into Eq. (35), we have

$$\gamma_{2t} = -\alpha_1 b_1\sqrt{a_4} \,, \quad \gamma = -\alpha_1\sqrt{a_4}(b_1 t + \delta_2) \,.$$

Then we obtain a solution of Eq. (24)

$$f_2 = \sqrt{a_4}\,\mathrm{ns}(-\alpha_1\sqrt{a_4}(x + b_2 t + \delta_2, k_2) \,. \tag{37}$$

Similarly we have a solution of Eq. (25)

$$g_2 = \sqrt{\hat{a}_4}\,\mathrm{ns}(-\alpha_1\sqrt{\hat{a}_4}(x + b_2 t + \hat{\delta}_2, \hat{k}_2)) \,, \tag{38}$$

where

$$\hat{k}_2^2 = \hat{a}_3\hat{a}_4^{-1} \,,$$
$$\hat{a}_3 = \frac{1}{2\alpha_1^2}\left(\lambda + \sqrt{\lambda^2 - 4\alpha_1^2\eta_2}\right) \,,$$
$$\hat{a}_4 = \frac{1}{2\alpha_1^2}\left(\lambda - \sqrt{\lambda^2 - 4\alpha_1^2\eta_2}\right) \,.$$

(iii) Let $\lambda > 0$, $\eta_i = \lambda^2/(4\alpha_1^2)$, $i = 1, 2$. Noticing that as $\eta_i \to \lambda^2/(4\alpha_1^2)$ (so that $k_2, \hat{k}_2 \to 1$) in case (ii),

$$\lim \mathrm{ns}(u, k_2) = \lim \mathrm{ns}(u, \hat{k}_2) = \coth u \,,$$

$$\lim a_i = \lim \hat{a}_i = \frac{\lambda}{2\alpha_1^2}, \quad i = 3, 4,$$

therefore we obtain from Eqs. (37) and (38) two solutions of Eqs. (24) and (25)

$$f_3 = -\sqrt{\frac{\lambda}{2\alpha_1^2}} \coth \sqrt{\frac{\lambda}{2}}(x + b_1 t + \delta_3), \tag{39}$$

$$g_3 = -\sqrt{\frac{\lambda}{2\alpha_1^2}} \coth \sqrt{\frac{\lambda}{2}}(x + b_2 t + \hat{\delta}_3), \tag{40}$$

(iv) Let $\lambda > 0$, $\eta_i = 0$, $i = 1, 2$. Equations (27) and (28) are reduced to

$$f_x = \alpha_1 f \sqrt{f^2 - \frac{\lambda}{\alpha_1^2}}, \quad f_t = \alpha_1 b_1 f \sqrt{f^2 - \frac{\lambda}{\alpha_1^2}},$$

$$g_x = \alpha_1 g \sqrt{g^2 - \frac{\lambda}{\alpha_1^2}}, \quad g_t = \alpha_1 b_1 g \sqrt{g^2 - \frac{\lambda}{\alpha_1^2}},$$

whose solutions are

$$f_4 = -\frac{\sqrt{\lambda}}{\alpha_1 \sin \sqrt{\lambda}(x + b_1 t + \delta_4)}, \tag{41}$$

$$g_4 = -\frac{\sqrt{\lambda}}{\alpha_1 \sin \sqrt{\lambda}(x + b_2 t + \hat{\delta}_4)}. \tag{42}$$

(v) Let $\lambda < 0, 0 < \eta_i < \lambda^2/(4\alpha_1^2), i = 1, 2$. Then equation (27) implies

$$f_x = \alpha_1 \sqrt{(f^2 + a_5)(f^2 + a_6)},$$

$$f_t = \alpha_1 b_1 \sqrt{(f^2 + a_5)(f^2 + a_6)}, \tag{43}$$

where

$$a_5 = \frac{1}{2\alpha_1^2}(|\lambda| - \sqrt{\lambda^2 - 4\alpha_1^2 \eta_1}),$$

$$a_6 = \frac{1}{2\alpha_1^2}(|\lambda| + \sqrt{\lambda^2 - 4\alpha_1^2 \eta_1}).$$

Noticing Eq. (43) and the transformation $w = \sqrt{a_5} s / \sqrt{1 - s^2}$, we have

$$\alpha_1(x + \gamma_3) = \int_0^f \frac{\mathrm{d}w}{\sqrt{(w^2 + a_5)(w^2 + a_6)}} = \frac{1}{\sqrt{\alpha_6}} \int_0^{\sqrt{f^2/(a_5 + f^2)}} \frac{\mathrm{d}s}{\sqrt{(1 - s^2)(1 - k_4^2 s^2)}}.$$

This implies

$$\sqrt{\frac{f^2}{a_5 + f^2}} = \mathrm{sn}(\alpha_1 \sqrt{\alpha_6}(x + \gamma_3), k_4),$$

$$k_4^2 = 1 - \frac{a_5}{a_6},$$

that is

$$f = \sqrt{a_5}\, \mathrm{sc}(\alpha_1 \sqrt{\alpha_6}(x + \gamma_3), k_4). \tag{44}$$

Substituting Eq. (44) into $b_1 f_x = f_t$ yields

$$\gamma_{3t} = b_1, \quad \gamma_3 = b_1 t + \delta_5.$$

Therefore we arrive at a solution of Eq. (24)

$$f_5 = \sqrt{\alpha_5}\, \mathrm{sc}(\alpha_1 \sqrt{\alpha_6}(x + b_1 t + \delta_5), k_4). \tag{45}$$

Applying the same procedure to Eq. (28), we get a solution of Eq. (25)

$$g_5 = \sqrt{\hat{\alpha}_5}\, \mathrm{sc}(\alpha_1 \sqrt{\hat{\alpha}_6}(x + b_2 t + \hat{\delta}_5), \hat{k}_4), \tag{46}$$

with

$$\hat{k}_4^2 = 1 - \frac{\hat{a}_5}{\hat{a}_6}, \quad \hat{a}_5 = \frac{1}{2\alpha_1^2}(|\lambda| - \sqrt{\lambda^2 - 4\alpha_1^2 \eta_2}),$$

$$\hat{a}_6 = \frac{1}{2\alpha_1^2}(|\lambda| + \sqrt{\lambda^2 - 4\alpha_1^2 \eta_2}).$$

(vi) Let $\lambda < 0$, $\eta_i = 0, i = 1, 2$. In a way similar to case (iv), we can obtain solutions of Eqs. (24) and (25):

$$f_6 = -\frac{\sqrt{|\lambda|}}{\alpha_1 \sinh \sqrt{|\lambda|}(x + b_1 t + \delta_6)}, \tag{47}$$

$$g_6 = -\frac{\sqrt{|\lambda|}}{\alpha_1 \sinh \sqrt{|\lambda|}(x + b_2 t + \hat{\delta}_6)}. \tag{48}$$

(vii) Let $\lambda = 0, \eta_i = 0, i = 1, 2$. Then solutions of equations (24) and (25) are

$$f_7 = -\frac{1}{\alpha_1(x + b_1 t + \delta_7)}, \quad g_7 = -\frac{1}{\alpha_1(x + b_2 t + \hat{\delta}_7)}. \tag{49}$$

Using Corollary 1 of Theorem 1, we obtain 49 explicit solutions $(u_{ij}(x, t), v_{ij}(x, t))$, given by

$$u_{ij}(x, t) = \alpha_1(f_i^2 + g_j^2),$$

$$v_{ij}(x, t) = \alpha_1(f_i^2 - g_j^2), \quad 1 \le i, j \le 7, \tag{50}$$

of the coupled KdV equation (16).

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