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Differential Representations of SO(4) Dynamical Group*

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Abstract *In this paper we present systematic differential representations for the dynamical group SO(4). These representations include the left and the right differential representations and the left and the right adjoint differential representations in both the group parameter space and its coset spaces. They are the generalization of the differential representations of the SO(3) rotation group in the Euler angles. These representations may find their applications in the study of the physical systems with SO(4) dynamical symmetry.*

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1 Introduction

Many quantum many-body systems have their low-energy collective excitations showing certain kind of dynamical symmetry.[1−5] To describe these collective phenomena effectively, some relevant collective coordinates considered as order parameters are needed. These order parameters are extracted from the operators representing collective degrees of freedom. Although some representations of the dynamical groups and their applications on many physical problems have been explored, the differential representations (continuous-variable representations) have not been well studied and the relevant collective coordinates have not been clearly identified.

The advantage of the differential representation is that the collective variables (the order parameters of the collective modes) are identified explicitly and naturally and its representation space becomes a Hilbert space, so its physics is very clear. Thus, the differential representation of the dynamic groups is a powerful tool in the study of collective behavior of quantum many-body systems (for example, nuclear many-body systems). In addition, the adjoint representations of the dynamical groups have also rich physical context and are very useful in the study of many physical problems. For example, the adjoint representation of the SU(3) dynamical group in quark model is a well-known example. As alternative interesting application of the adjoint differential representation, one can reduce the von Neumann equation of the quantum characteristic function of a linear system to a Schrödinger-like equation.[6] This fact shows that the dynamical symmetry of the von Neumann equation of the linear system is the same as that of the corresponding Schrödinger equation. Thus, it is convenient to use a technique to solve the Schrödinger equation to solve the von Neumann equation, which is much difficult to be solved in the conventional formalism. In Ref. [6], we have presented an explicit example of the adjoint differential representation of SO(3) for the Jaynes–Cummings model.

In the previous paper,[7] we have studied the representations of the dynamical group SO(3,1). These representations are the corresponding relativistic generalizations of the differential representations of the SO(3) rotation group in the Euler angles and in the polar angles. In this paper, we shall present in detail the method to study the differential representation of a dynamical group. As a further application, we discuss the left and the right differential representations of the SO(4) group in the full group space with six parameters and in its coset spaces with three parameters, and their adjoint differential representations in the corresponding parameter spaces. At the same time, the corresponding differential representation of the subgroup SO(3) are also obtained.

This paper is organized as follows. In Sec. 2, we discuss in detail the method to obtain the differential representations of a dynamical group. In Sec. 3, we present the symbolic method to calculate the differential representations. In Sec. 4 the explicit representations of the dynamical group SO(4) are presented. Finally, section 5 is devoted to a brief summary and discussion of possible applications.

2 Differential Representation of Dynamical Group

Let G be an r-dimensional complex Lie group and $U(g)$ be an element in G, where $g = (g^1, g^2, \ldots, g^r)$ are the parameters of G. Let $X_1, X_2, \ldots, X_r$ be the generators of G.

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The canonical form of the group element can be expressed as

$$U(g) = \exp(\mathrm{i}g \circ X)\,, \quad g \circ X = \sum_m g^m X_m\,.$$

Any group element $U(g)$ can also be considered as an operator acting on certain manifold M. Correspondingly, we can define a differential operation and thus derive the differential representation of this group generator, i.e., the differential representation of the corresponding Lie algebra. Hereafter, the Lie algebra of Lie group G is denoted as $LieG$.

The right and the left differential representations of a dynamical group have been introduced in Refs. [8] and [9] and the corresponding adjoint differential representation has been introduced in Refs. [6] and [7]. In this section, we present a general description for these differential representations.

Suppose that G acts on a manifold M via $U(g)(x) = F(U(g), x)$ for $(U(g), x) \in G \times M$, then there is an infinitesimal action of G on M. In other words, if $v \in LieG$, we can define $f(v)$ to be a vector field on M whose flow coincides with the action of the one-parameter subgroup $\exp(\mathrm{i} \cdot tv)$ of G on M, i.e., for $x \in M$,

$$f(v)(x) = \lim_{t \to 0} \frac{1}{t} \big[F(\exp(\mathrm{i} \cdot tv), x) - x\big]\,. \tag{1}$$

Due to the well-known theorem of Ado, we can consider G as a matrix group. If M is a sub-manifold of $\boldsymbol{R}^{n^2}$ $(n \geq 1)$, $F(\exp(\mathrm{i} \cdot tv), x)$ is always well-defined by the multiplication of matrices.

When the action of G on M is the right action (in the sense of linear action by matrix), equation (1) becomes

$$f(v)(x) = \lim_{t \to 0} \frac{1}{t}[x \exp(\mathrm{i} \cdot tv) - x] = x(\mathrm{i}v)\,. \tag{2}$$

Equation (2) defines a linear map $v \to \mathrm{i}f(v)$ and we have

$$\begin{aligned}[\mathrm{i}f(u), \mathrm{i}f(v)](x) &= (\mathrm{i}f(u))(\mathrm{i}f(v))(x) - (\mathrm{i}f(v))(\mathrm{i}f(u))(x) \\ &= x(vu) - x(uv) = -x[u, v] \\ &= \mathrm{i}f([u, v])(x)\,, \end{aligned} \tag{3}$$

which gives an isomorphic representation of differential form of $LieG$, and we call it right differential representation of G, denoted by D^r.

When the action of G on M is the left action, equation (1) becomes

$$f(v)(x) = \lim_{t \to 0} \frac{1}{t}[\exp(\mathrm{i} \cdot tv)x - x] = (\mathrm{i}v)x\,. \tag{4}$$

Similarly, equation (3) also gives an anti-isomorphic representation of differential form of $LieG$, and we call it left differential representation, denoted by D^l.

Notice that for a rotation group, in the sense of linear action via the multiplication of matrix, by the operation of transpose and conjugation, a right action leads to a left action, so a right differential representation of G corresponds naturally to a left differential representation of G, this means, if D^r is a right differential representation, then $-D^r$ is a left differential representation, *vice versa*.

If the action of G on M is $U(g)^{-1}xU(g)$ $(U(g) \in G$, $x \in M)$, we call it a right adjoint action. If the action is $U(g)xU(g)^{-1}$ $(U(g) \in G$, $x \in M)$, we call it a left adjoint action.

When the action of G on M is the right adjoint action, equation (1) becomes

$$\begin{aligned} f(v)(x) &= \lim_{t \to 0} \frac{1}{t}[\exp^{-1}(\mathrm{i} \cdot tv)x \exp(\mathrm{i} \cdot tv) - x] \\ &= x(\mathrm{i}v) - (\mathrm{i}v)x\,, \end{aligned} \tag{5}$$

and we have

$$\begin{aligned}[f(u), f(v)](x) &= (x(\mathrm{i}v) - (\mathrm{i}v)x)(\mathrm{i}u) - (\mathrm{i}u)(x(\mathrm{i}v) - (\mathrm{i}v)x) \\ &\quad - ((x(\mathrm{i}u) - (\mathrm{i}u)x)(\mathrm{i}v) \\ &\quad - (\mathrm{i}v)(x(\mathrm{i}u) - (\mathrm{i}u)x)) \\ &= x[u, v] - [u, v]x \\ &= -\mathrm{i}f([u, v])(x)\,. \end{aligned} \tag{6}$$

This implies

$$[\mathrm{i}f(u), \mathrm{i}f(v)] = \mathrm{i}f([u, v])\,. \tag{7}$$

Denote $\mathrm{i}f(v)$ by $D^{ad}_r v$, then from Eq. (6) we get

$$[D^{ad}_r u, D^{ad}_r v] = D^{ad}_r [u, v]\,, \tag{8}$$

which gives another isomorphic representation of $LieG$, and we call it right adjoint differential representation of G, denoted by D^{ad}_r.

When the action of G on M is the left adjoint action, by the same way we get

$$[f(u), f(v)](x) = \mathrm{i}f([u, v])(x)\,, \tag{9}$$

and

$$[\mathrm{i}f(u), \mathrm{i}f(v)] = -\mathrm{i}f([u, v])\,. \tag{10}$$

Denote $\mathrm{i}f(v)$ by $D^{ad}_l v$, we have

$$[D^{ad}_l u, D^{ad}_l v] = -D^{ad}_l [u, v]\,, \tag{11}$$

which gives another anti-isomorphic representation of $LieG$, and we call it the left adjoint differential representation of G, denoted by D^{ad}_l.

Similar as our previous remark, in the sense of linear action by matrix, by the operation of transpose and conjugation, a right adjoint differential representation leads to a left adjoint differential representation, *vice versa*.

For the generators $X_1, X_2, \ldots, X_r$ of G, we know that for the rotation group, the left differential representation of X_m describes the angular momentum operators acting on the intrinsic states (the states in the body-fixed frame, or the bra states $\langle\psi|$), while the right differential representation of X_m describes the angular momentum operators in the laboratory frame and acts on the ket states $|\psi\rangle$. The two representations on the total space G are independent and commute each other

$$[D^r(X_m), D^l(X_l)] = 0\,, \qquad m, l = 1, 2, \ldots, r\,. \tag{12}$$

For details, one can refer to Refs. [2] and [8].

Equations (2), (4), and (5) show that, if $M = G$, then one has

$$D^{ad}_r = -D^{ad}_l = D^r - D^l\,. \tag{13}$$

It should be pointed out that when $M \neq G$, equation (13) may be not true. This is because that firstly even though the adjoint action on M is well-defined, the right or the left action on M may be not well-defined and secondly, the right action and the left action have different orbits, the representations may have different parameters.

In general, the matrix representation of the Lie group does not have the above peculiar properties. The reason is that the differential representation defined on the group manifold preserves the group properties and acts on the whole Hilbert space, while the matrix representations defined on some finite-dimensional Hilbert subspaces will lose some information. This is the advantage of the differential representation in comparison to the matrix representation.

In many cases, one needs to consider the right and the left differential representation on different subspaces. Thus, we often choose different spaces to calculate the right and the left differential representations below.

Recall that an action of G on M is called transitive if for any $x, y \in M$, there exist some $U(g) \in G$ such that $U(g)(x) = y$. If the action of G on M is transitive, M is just an orbit of any given $x_0 \in M$ under G. Write down

$$H_{x_0} = \{U(g) \in G : U(g)(x_0) = x_0\} ,$$

we know that M can be thought as a coset space of G, one has $M = G/H_{x_0}$. In the following, we construct some explicit coset spaces to calculate the differential representations of the SO(4) group. It is noted that the number of independent variables appearing in the expressions of differential representations in the coset space depends on the dimension of M, which is a sub-manifold spanned by a reduced number of collective degrees of freedom when some collective modes are frozen. This is the advantage of the differential representation. Before we present the differential representations of SO(4) group, we discuss the explicit symbolic method to obtain these differential representations.

3 Symbolic Method to Calculate Differential Representations of a Dynamical Group

In Refs. [8] and [9] some special methods have been proposed to obtain the differential representations of a dynamical group. Here we present a general scheme, which can be applied to any dynamical group.

Let G be an r-dimensional Lie group, $g = (g^1, g^2, \ldots, g^r)$ are the parameters of G, $X_1, X_2, \ldots, X_r$ are the generators of G. From the implicit function theorem, it is not difficult to prove that in some neighborhood of the identity element of G, the group elements can be expressed as

$$\begin{aligned} & U(g^1, g^2, \ldots, g^r) \\ =\ & \exp(\mathrm{i}g^1 X_1)\exp(\mathrm{i}g^2 X_2)\ldots\exp(\mathrm{i}g^r X_r) \, . \end{aligned}$$

According to Refs. [7] and [8], the differential representation of $X_m (m = 1, 2, \ldots, r)$ has the form

$$\mathrm{i}\Big(f_1 \frac{\partial}{\partial g^1} + f_2 \frac{\partial}{\partial g^2} + \cdots + f_t \frac{\partial}{\partial g^t}\Big), \qquad (t \le r) \, ,$$

where $g^1, g^2, \ldots, g^t$ are the parameters of the coset space M, $f_1, f_2, \ldots, f_t$ are functions of $g^1, g^2, \ldots, g^t$.

Taking $M = G$, we calculate the r-parameter right differential representation of the group G. For simplicity, we denote $D_r(X_m, \partial/\partial g)$ by $D_r(X_m)$. To proceed, we assume

$$\begin{aligned} & D_r(X_m) = \mathrm{i}\Big(f_{m1}\frac{\partial}{\partial g^1} + f_{m2}\frac{\partial}{\partial g^2} + \cdots + f_{mr}\frac{\partial}{\partial g^r}\Big), \\ & \quad m = 1, 2, \ldots, r \, . \end{aligned} \tag{14}$$

For any $x \in G$ we should have

$$D_r(X_m)(x) = \mathrm{i}\Big(f_{m1}\frac{\partial x}{\partial g^1} + f_{m2}\frac{\partial x}{\partial g^2} + \cdots + f_{mr}\frac{\partial x}{\partial g^r}\Big) \tag{15}$$

$$= \mathrm{i}\Big(\lim_{t\to 0}\frac{1}{t}\big[xU(\overbrace{0, \ldots, 0, t}^{m}, \ldots, 0) - x\big]\Big) \, . \tag{16}$$

As mentioned above, since the differential representation of the group is applicable to the whole Hilbert space, it is, of course, applicable to any finite Hilbert subspace. Thus, we consider the algebraic relations of the group generators in a finite dimensional matrix representation. Taking the matrix form of $X_1, X_2, \ldots, X_r$, and calculating the matrix form of

$$\begin{aligned} x &= U(g^1, g^2, \ldots, g^r) \\ &= \exp(\mathrm{i}g^1 X_1)\exp(\mathrm{i}g^2 X_2)\cdots\exp(\mathrm{i}g^r X_r) \, , \end{aligned}$$

(this form can be treated much easier and it can be done by symbolic calculation on a PC computer). From Eq. (16), we compare the corresponding elements in the two matrices and get a set of linear equations for f_{mj} ($j = 1, 2, \ldots, r$) with only r equations independent. Solving the set of equations (it can also be done by Mathematica on a PC computer), we obtain the solutions of f_{mj} as functions of $(g^1, \ldots, g^r)$. If M is a t -dimensional coset space of G, and if we can write down the explicit form of $x \in M$, then by the same way, we can get a t-parameter right differential representation.

Taking $M = G$ as an example, the symbolic calculation method for a right differential representation can be summarized as following steps:

Step 1 Consider G as a transform group on G. First give $n \times n$ matrix form of the generators $X_1, X_2, \ldots, X_r$, and then calculate

$$\begin{aligned} U(g) &= U(g^1, g^2, \ldots, g^r) \\ &= \exp(\mathrm{i}g^1 X_1)\exp(\mathrm{i}g^2 X_2)\cdots\exp(\mathrm{i}g^r X_r) \, , \end{aligned}$$

which is also of matrix form.

Step 2 For any $U(g) \in G$, by Eq. (15), we have

$$\begin{aligned} D_r(X_m)(U(g)) = \mathrm{i}\Big(& f_{m1}(g)\frac{\partial U(g)}{\partial g^1} + f_{m2}(g)\frac{\partial U(g)}{\partial g^2} \\ & + \cdots + f_{mr}(g)\frac{\partial U(g)}{\partial g^r}\Big) \, . \end{aligned} \tag{17}$$

Calculate the right-hand side of Eq. (17), we get a matrix with r independent parameters.

Step 3 By the definition (1), calculate the limit

$$\begin{aligned} & D_r(X_m)(U(g)) \\ & = \mathrm{i}\Big(\lim_{t\to 0}\frac{1}{t}[U(g)U(\overbrace{0, \ldots, 0, t}^{m}, \ldots, 0) - U(g)]\Big) \, . \end{aligned} \tag{18}$$

The right-hand side of Eq. (18) is also a matrix with r independent parameters.

Step 4 Let the matrix in the right-hand side of Eq. (17) be equal to matrix in the right-hand side of Eq. (18), comparing the corresponding elements in the two matrices we get a set of linear equations for the undetermined functions $f_{m1}(g^1, g^2, \ldots, g^r)$, $f_{m2}(g^1, g^2, \ldots, g^r)$, $\ldots$, $f_{mr}(g^1, g^2, \ldots, g^r)$ with only r independent equations.

Step 5 Solving the above equations, we can obtain f_{mj} as functions of $(g^1, g^2, \ldots, g^r)$.

The calculation for other kinds of differential representation of G is similar. If both the right and the left actions of G on M are well-defined and have the same obit, then from Eq. (13), we can get the right and left adjoint differential representation of G. On the contrary, if the right or the left actions on M cannot be defined at the same time or they have different obits, but the adjoint action is well defined, the adjoint differential representations of G must be calculated as above through its original definition.

It should also be noted that when the differential representations of G are obtained, the differential representations of any subgroup of G can be obtained at the same time.

4 Differential Representations of SO(4) Group

As an explicit application of the above scheme for derivation of the differential representations, we consider SO(4) group. It is well known that SO(4) group can be used to describe the symmetries in the hydrogen atom. For details, one can refer to Refs. [10] ~ [12].

For $G = \mathrm{SO}(4)$, the group elements can be obtained by

$$U(g) = \exp(\mathrm{i}\alpha_1 \hat{J}_{\alpha_1}) \exp(\mathrm{i}\alpha_2 \hat{J}_{\alpha_2}) \exp(\mathrm{i}\alpha_3 \hat{J}_{\alpha_3})$$

$$\exp(\mathrm{i}\alpha_4 \hat{J}_{\alpha_4}) \exp(\mathrm{i}\alpha_5 \hat{J}_{\alpha_5}) \exp(\mathrm{i}\alpha_5 \hat{J}_{\alpha_6})\,,$$

where $\hat{J}_{\alpha_i}$ $(i = 1, 2, \ldots, 6)$ are the generators of the Lie algebra of SO(4). The matrix representation of SO(4) is given by

$$\hat{J}_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{J}_{\alpha_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{J}_{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{J}_{\alpha_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{J}_{\alpha_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{J}_{\alpha_6} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

To derive the differential representation of SO(4), we assume

$$\hat{J}_{\alpha_i} = \mathrm{i}\Big(a_i \frac{\partial}{\partial \alpha_1} + b_i \frac{\partial}{\partial \alpha_2} + c_i \frac{\partial}{\partial \alpha_3} + d_i \frac{\partial}{\partial \alpha_4} + e_i \frac{\partial}{\partial \alpha_5} + f_i \frac{\partial}{\partial \alpha_6}\Big), \qquad i = 1, 2, \ldots, 6\,.$$

4.1 *Right, Left and Adjoint Differential Representations with Six Parameters*

By using the matrix form representation of SO(4) and the definition of the right differential representation, taking $M = \mathrm{SO}(4)$, we can get the right differential representation of SO(4) with parameters α_1, α_2, α_3, α_4, α_5, and α_6 as

$$\begin{aligned}
\hat{J}_{\alpha_1} = {} & \mathrm{i}\Big(\cos\alpha_3\cos\alpha_5\sec\alpha_2\sec\alpha_4\frac{\partial}{\partial\alpha_1} - \cos\alpha_5\sec\alpha_4\sin\alpha_3\frac{\partial}{\partial\alpha_2} + \cos\alpha_3\cos\alpha_5\sec\alpha_4\tan\alpha_2\frac{\partial}{\partial\alpha_3} \\
& - \sin\alpha_5\frac{\partial}{\partial\alpha_4} + \cos\alpha_5\tan\alpha_4\frac{\partial}{\partial\alpha_5}\Big)\,, \\
\hat{J}_{\alpha_2} = {} & \mathrm{i}\Big[\sec\alpha_2\sec\alpha_4(\cos\alpha_6\sin\alpha_3 - \cos\alpha_3\sin\alpha_5\sin\alpha_6)\frac{\partial}{\partial\alpha_1} + \sec\alpha_4(\cos\alpha_3\cos\alpha_6 + \sin\alpha_3\sin\alpha_5\sin\alpha_6)\frac{\partial}{\partial\alpha_2} \\
& + (\sec\alpha_4\tan\alpha_2\cos\alpha_6\sin\alpha_3 - \sec\alpha_4\tan\alpha_2\cos\alpha_3\sin\alpha_5\sin\alpha_6 + \cos\alpha_6\tan\alpha_4\tan\alpha_5)\frac{\partial}{\partial\alpha_3} - \cos\alpha_5\sin\alpha_6\frac{\partial}{\partial\alpha_4} \\
& - \sin\alpha_5\sin\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_5} + \sec\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_6}\Big]\,, \\
\hat{J}_{\alpha_3} = {} & \mathrm{i}\Big(\sec\alpha_5\cos\alpha_6\frac{\partial}{\partial\alpha_3} - \sin\alpha_6\frac{\partial}{\partial\alpha_5} + \tan\alpha_5\cos\alpha_6\frac{\partial}{\partial\alpha_6}\Big)\,, \\
\hat{J}_{\alpha_4} = {} & \mathrm{i}\Big[\sec\alpha_2\sec\alpha_4(\sin\alpha_3\sin\alpha_6 + \cos\alpha_3\cos\alpha_6\sin\alpha_5)\frac{\partial}{\partial\alpha_1} + \sec\alpha_4(\cos\alpha_3\sin\alpha_6 - \sin\alpha_3\sin\alpha_5\cos\alpha_6)\frac{\partial}{\partial\alpha_2} \\
& + (\sec\alpha_4\tan\alpha_2\sin\alpha_6\sin\alpha_3 + \sec\alpha_4\tan\alpha_2\cos\alpha_3\sin\alpha_5\cos\alpha_6 + \sin\alpha_6\tan\alpha_4\tan\alpha_5)\frac{\partial}{\partial\alpha_3} \\
& + \cos\alpha_5\cos\alpha_6\frac{\partial}{\partial\alpha_4} + \sin\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_5} + \sin\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_6}\Big]\,, \\
\hat{J}_{\alpha_5} = {} & \mathrm{i}\Big(\sec\alpha_5\sin\alpha_6\frac{\partial}{\partial\alpha_3} + \cos\alpha_6\frac{\partial}{\partial\alpha_5} + \tan\alpha_5\sin\alpha_6\frac{\partial}{\partial\alpha_6}\Big)\,,
\end{aligned}$$

$$\hat{J}_{\alpha_6} = \mathrm{i}\frac{\partial}{\partial\alpha_6}\,.$$

Similarly, the left differential representation of SO(4) with six parameters α_1, α_2, α_3, α_4, α_5, α_6 is

$$\begin{aligned}
\hat{\hat{J}}_{\alpha_1} &= \mathrm{i}\frac{\partial}{\partial\alpha_1}\,,\\
\hat{\hat{J}}_{\alpha_2} &= \mathrm{i}\Big(\sin\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\frac{\partial}{\partial\alpha_2} + \sec\alpha_2\sin\alpha_1\frac{\partial}{\partial\alpha_3}\Big)\,,\\
\hat{\hat{J}}_{\alpha_3} &= \mathrm{i}\Big(\cos\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\frac{\partial}{\partial\alpha_2} + \sec\alpha_2\cos\alpha_1\frac{\partial}{\partial\alpha_3}\Big)\,,\\
\hat{\hat{J}}_{\alpha_4} &= \mathrm{i}\Big[\sec\alpha_2\sin\alpha_1\tan\alpha_4\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2} + (\sin\alpha_1\tan\alpha_2\tan\alpha_4 + \sec\alpha_4\tan\alpha_5\cos\alpha_1\cos\alpha_3\sin\alpha_2\\
&\quad + \sec\alpha_4\tan\alpha_5\sin\alpha_1\sin\alpha_3)\frac{\partial}{\partial\alpha_3} + \cos\alpha_1\cos\alpha_2\frac{\partial}{\partial\alpha_4} + (\sec\alpha_4\cos\alpha_3\sin\alpha_1 - \sec\alpha_4\cos\alpha_1\sin\alpha_2\sin\alpha_3)\frac{\partial}{\partial\alpha_5}\\
&\quad + (\sec\alpha_4\sec\alpha_5\cos\alpha_1\cos\alpha_3\sin\alpha_2 + \sec\alpha_4\sec\alpha_5\sin\alpha_1\sin\alpha_3)\frac{\partial}{\partial\alpha_6}\Big]\,,\\
\hat{\hat{J}}_{\alpha_5} &= \mathrm{i}\Big[\sec\alpha_2\cos\alpha_1\tan\alpha_4\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2}\\
&\quad + (\cos\alpha_1\tan\alpha_2\tan\alpha_4 - \sec\alpha_4\tan\alpha_5(\cos\alpha_3\sin\alpha_1\sin\alpha_2 - \cos\alpha_1\sin\alpha_3))\,\frac{\partial}{\partial\alpha_3} - \cos\alpha_2\sin\alpha_1\frac{\partial}{\partial\alpha_4}\\
&\quad + \sec\alpha_4(\cos\alpha_1\cos\alpha_3 + \sin\alpha_1\sin\alpha_2\sin\alpha_3)\frac{\partial}{\partial\alpha_5} + \sec\alpha_4\sec\alpha_5(-\sin\alpha_1\sin\alpha_2\cos\alpha_3 + \cos\alpha_1\sin\alpha_3)\frac{\partial}{\partial\alpha_6}\Big]\,,\\
\hat{\hat{J}}_{\alpha_6} &= \mathrm{i}\Big(\cos\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2} + \cos\alpha_2\cos\alpha_3\sec\alpha_4\tan\alpha_5\frac{\partial}{\partial\alpha_3} - \sin\alpha_2\frac{\partial}{\partial\alpha_4} - \cos\alpha_2\sin\alpha_3\sec\alpha_4\frac{\partial}{\partial\alpha_5}\\
&\quad + \cos\alpha_2\cos\alpha_3\sec\alpha_4\sec\alpha_5\frac{\partial}{\partial\alpha_6}\Big)\,.
\end{aligned}$$

From Eq. (13), we also obtain the adjoint differential representations of SO(4) with six parameters.

4.2 *Right and Left Differential Representations with Three Parameters*

For the right representation, taking

$$M = \{(0\quad 0\quad 0\quad 1)U(g) : U(g)\in \mathrm{SO}(4)\}\,,$$

we obtain a three-parameter right differential representation of SO(4):

$$\begin{aligned}
\hat{J}_{\alpha_1} &= \mathrm{i}\Big(-\sin\alpha_5\frac{\partial}{\partial\alpha_4} + \cos\alpha_5\tan\alpha_4\frac{\partial}{\partial\alpha_5}\Big),\\
\hat{J}_{\alpha_2} &= \mathrm{i}\Big(-\cos\alpha_5\sin\alpha_6\frac{\partial}{\partial\alpha_4} - \sin\alpha_5\sin\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_5}\\
&\quad + \sec\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_6}\Big)\,,\\
\hat{J}_{\alpha_3} &= \mathrm{i}\Big(-\sin\alpha_6\frac{\partial}{\partial\alpha_5} + \tan\alpha_5\cos\alpha_6\frac{\partial}{\partial\alpha_6}\Big),\\
\hat{J}_{\alpha_4} &= \mathrm{i}\Big(\cos\alpha_5\cos\alpha_6\frac{\partial}{\partial\alpha_4} + \sin\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_5}\\
&\quad + \sin\alpha_5\cos\alpha_6\tan\alpha_4\frac{\partial}{\partial\alpha_6}\Big),\\
\hat{J}_{\alpha_5} &= \mathrm{i}\Big(\cos\alpha_6\frac{\partial}{\partial\alpha_5} + \tan\alpha_5\sin\alpha_6\frac{\partial}{\partial\alpha_6}\Big),\\
\hat{J}_{\alpha_6} &= \mathrm{i}\frac{\partial}{\partial\alpha_6}\,.
\end{aligned}$$

Similarly, for the left representation, taking

$$M = \{U(g)\,(1\quad 0\quad 0\quad 0)^{\mathrm{T}} : U(g)\in \mathrm{SO}(4)\}\,,$$

we obtain a three-parameter left differential representation as follows:

$$\begin{aligned}
\hat{\hat{J}}_{\alpha_1} &= \mathrm{i}\frac{\partial}{\partial\alpha_1}\,,\\
\hat{\hat{J}}_{\alpha_2} &= \mathrm{i}\Big(\sin\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\frac{\partial}{\partial\alpha_2}\Big),\\
\hat{\hat{J}}_{\alpha_3} &= \mathrm{i}\Big(\cos\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\frac{\partial}{\partial\alpha_2}\Big),\\
\hat{\hat{J}}_{\alpha_4} &= \mathrm{i}\Big(\sec\alpha_2\sin\alpha_1\tan\alpha_4\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2}\\
&\quad + \cos\alpha_1\cos\alpha_2\frac{\partial}{\partial\alpha_4}\Big),\\
\hat{\hat{J}}_{\alpha_5} &= \mathrm{i}\Big(\sec\alpha_2\cos\alpha_1\tan\alpha_4\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2}\\
&\quad - \cos\alpha_2\sin\alpha_1\frac{\partial}{\partial\alpha_4}\Big),\\
\hat{\hat{J}}_{\alpha_6} &= \mathrm{i}\Big(\cos\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2} - \sin\alpha_2\frac{\partial}{\partial\alpha_4}\Big).
\end{aligned}$$

4.3 *Right and Left Adjoint Differential Representations with Three Parameters*

Now we give the right and the left adjoint differential representations with three parameters.

Let SO(4) act on the manifold

$$M_R = \left\{U^{-1}(g)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}U(g) : U(g)\in \mathrm{SO}(4)\right\}\,,$$

by right adjoint action, we can get a three-parameter left adjoint differential representation as follows:

$$\hat{J}_{\alpha_1} = \mathrm{i}\Big[-\sin(\alpha_5)\frac{\partial}{\partial\alpha_4} + \cos(\alpha_5)\tan(\alpha_4)\frac{\partial}{\partial\alpha_5}\Big],$$

$$\begin{aligned}\hat{J}_{\alpha_2} = \mathrm{i}\Big[&-\cos(\alpha_5)\sin(\alpha_6)\frac{\partial}{\partial\alpha_4}\\ &-\sin(\alpha_5)\sin(\alpha_6)\tan(\alpha_4)\frac{\partial}{\partial\alpha_5}\\ &+\cos(\alpha_6)\sec(\alpha_5)\ \tan(\alpha_4)\frac{\partial}{\partial\alpha_6}\Big],\end{aligned}$$

$$\hat{J}_{\alpha_3} = \mathrm{i}\Big[-\sin(\alpha_6)\frac{\partial}{\partial\alpha_5} + \cos(\alpha_6)\tan(\alpha_5)\frac{\partial}{\partial\alpha_6}\Big],$$

$$\begin{aligned}\hat{J}_{\alpha_4} = \mathrm{i}\Big[&\cos(\alpha_5)\cos(\alpha_6)\frac{\partial}{\partial\alpha_4}\\ &+\cos(\alpha_6)\sin(\alpha_5)\tan(\alpha_4)\frac{\partial}{\partial\alpha_5}\\ &+\sec(\alpha_5)\ \sin(\alpha_6)\ \tan(\alpha_4)\frac{\partial}{\partial\alpha_6}\Big],\end{aligned}$$

$$\hat{J}_{\alpha_5} = \mathrm{i}\Big[\cos(\alpha_6)\frac{\partial}{\partial\alpha_5} + \sin(\alpha_6)\tan(\alpha_5)\frac{\partial}{\partial\alpha_6}\Big],$$

$$\hat{J}_{\alpha_6} = \mathrm{i}\frac{\partial}{\partial\alpha_4}\,.$$

Let SO(4) act on the manifold

$$M_L = \left\{U(g)\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}U^{-1}(g) : U(g)\in \mathrm{SO}(4)\right\}\,,$$

by the left adjoint action, we can get a three-parameter left adjoint differential representation as follows:

$$\hat{J}_{\alpha_1} = \mathrm{i}\frac{\partial}{\partial\alpha_1}\,,$$

$$\hat{J}_{\alpha_2} = \mathrm{i}\Big(-\sin\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\frac{\partial}{\partial\alpha_2}\Big),$$

$$\hat{J}_{\alpha_3} = \mathrm{i}\Big(\cos\alpha_1\tan\alpha_2\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\frac{\partial}{\partial\alpha_2}\Big),$$

$$\begin{aligned}\hat{J}_{\alpha_4} = \mathrm{i}\Big(&\sec\alpha_2\sin\alpha_1\tan\alpha_4\frac{\partial}{\partial\alpha_1} + \cos\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2}\\ &+\cos\alpha_1\cos\alpha_2\frac{\partial}{\partial\alpha_4}\Big),\end{aligned}$$

$$\begin{aligned}\hat{J}_{\alpha_5} = \mathrm{i}\Big(&\cos\alpha_1\sec\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_1} - \sin\alpha_1\sin\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2}\\ &-\cos\alpha_2\sin\alpha_1\frac{\partial}{\partial\alpha_4}\Big),\end{aligned}$$

$$\hat{J}_{\alpha_6} = \mathrm{i}\Big(\cos\alpha_2\tan\alpha_4\frac{\partial}{\partial\alpha_2} - \sin\alpha_2\frac{\partial}{\partial\alpha_4}\Big)$$

Note that we cannot get a three-parameter left (right) adjoint differential representation on $M_R(M_L)$, because the left (right) adjoint action on $M_R(M_L)$ is not closed.

Finally we point out that the above representations also give the corresponding differential representations of the group SO(3), since SO(3) is a subgroup of SO(4) and $\hat{J}_{\alpha_1}$, $\hat{J}_{\alpha_2}$, $\hat{J}_{\alpha_3}$ generate SO(3). Particularly we have obtained several six-parameter representations of SO(3).

5 Summary

In this paper we present a general scheme to derive the differential representations of a dynamical group. Using this scheme, we derive explicitly the left and the right differential representations and the corresponding adjoint differential representations in six- and three-parameter spaces. These representations may find their applications in the study of physical systems with SO(4) and SO(3) dynamical symmetry, such as the Hubbard model,[12,13] the Euler or Euler–Manakov rigid body or top,[14] and the nonlinear equations with relevant dynamical symmetry groups.

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