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Some New Lie Symmetry Groups of Differential-Difference Equations Obtained from a Simple Direct Method

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Abstract *In this paper, based on the symbolic computing system Maple, the direct method for Lie symmetry groups presented by Sen-Yue Lou [J. Phys. A: Math. Gen.* **38** *(2005) L129] is extended from the continuous differential equations to the differential-difference equations. With the extended method, we study the well-known differential-difference KP equation, KZ equation and (2+1)-dimensional ANNV system, and both the Lie point symmetry groups and the non-Lie symmetry groups are obtained.*

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1 Introduction

Symmetry analysis originally developed by Sophus Lie in the nineteenth century has played an important role in the construction of exact solutions to nonlinear partial differential equations (NPDEs). To find the Lie point symmetry group of a nonlinear system, there is a standard method.[1] And the standard method has been widely used to find Lie point symmetry algebras and groups for almost all the known integrable systems. But the obtained expressions of the Lie point symmetry groups may be quite complicated and difficult for real applications especially for physicists and other non-mathematical scientists. In [2], Clarkson and Kruskal (CK) introduced the CK direct method to derive the symmetry reductions of a nonlinear system without using any group theory. In [3], based on the idea of the CK direct method, Sen-Yue Lou and Hong-Cai Ma developed a new direct method to find finite symmetry groups of NPDEs. And the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches.

Unlike for continuous differential equations, calculation of symmetries for differential-difference equations (DDEs) has to scale the difficulties set by the infinite number of variables. So the Lie approach to the DDEs is much less studied or understood. Fortunately, several different approaches to derive the symmetry groups of the DDEs have been developed.[4−11] Such as, the method[4−7] is implemented by finding symmetry vector field for a given DDE. Levi and Winternitz have also introduced the concept of conditional symmetries for the Toda lattice equation, various interesting exact solutions of the two-dimensional Toda lattice equation are obtained.[5] And the other method was presented for finding symmetries of linear difference equations.[8] All variables in the problem are assumed to be continuous, with their discrete increments. Dorodnitsyn introduced a special discretization method for difference equations so that it preserves the symmetries of the continuous differential equations.[9,10] In 2001, Kai-Seng Chou and Chang-Zheng Qu extended the generalized conditional symmetry method for NPDEs to DDEs.[11] These methods have been applied to find exact solutions and symmetry reductions for some DDEs.

The purpose of this letter is to extend the direct method[3] to tackle nonlinear differential-difference equations, involving continuous dependent variables and both continuous and discrete independent ones. Such equations have widespread applications in physics, biology as well as applied sciences.

For a given continuous PDE

$$F(x_i, u, u_{x_i}, u_{x_ix_j}, \ldots, i, j = 1, 2, \ldots, n) = 0\,, \tag{1}$$

the main idea of [3] is to seek a transformation in the form

$$u(x_1, x_2, \ldots, x_n) = W(x_1, x_2, \ldots, x_n, U(X_1, X_2, \ldots, X_n))\,, \tag{2}$$

where $X_i = X_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$. Substituting (2) into (1) and requiring $U(X_1, X_2, \ldots, X_n)$ satisfies the same PDE (1), i.e.

$$F(X_i, U, U_{X_i}, U_{X_iX_j}, \ldots, i, j = 1, 2, \ldots, n) = 0\,, \tag{3}$$

imposes conditions on W, X_i and their derivatives that enable one to solve for W and X_i.

To extend this idea into the DDEs, at first one has to change the transformation (2).

For a given DDE

$$E \equiv E(n, x_1, x_2, \ldots, x_m, \{u(k)\}_{k=n-n_1}^{n+n_2},\ u(k)_{x_i},\ u(k)_{x_ix_j}, \ldots) = 0\,, \tag{4}$$

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where n_1 and n_2 are nonnegative integers, and $u(n) = u(n, x_1, x_2, \ldots, x_m)$, i.e., the independent variables $x_1, x_2, \ldots, x_m$, are suppressed in $u(n)$. We suppose the transformation in the form

$$u(n, x_1, x_2, \ldots, x_m) = W(n, x_1, x_2, \ldots, x_m, U(n, X_1, X_2, \ldots, X_m))\,, \tag{5}$$

where X_i $(i = 1, 2, 3, \ldots, m)$ are functions of $n, x_1, x_2, \ldots, x_m$. Then

$$u(n + k, x_1, x_2, \ldots, x_m) = W(n + k, x_1, x_2, \ldots, x_m, U(n + k, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_m))\,, \tag{6}$$

where $\tilde{X}_i$ $(i = 1, 2, 3, \ldots, m)$ are functions of $n+k, x_1, x_2, \ldots, x_m$. Substituting (5) into (4) and requiring $U(n, X_1, X_2, \ldots, X_m)$ satisfies the same DDE, i.e.

$$E(n, X_1, X_2, \ldots, X_m, \{U(k)\}_{k=n-n_1}^{n+n_2},\ U(k)_{X_i},\ U(k)_{X_i X_j}, \ldots) = 0\,, \tag{7}$$

we derive the conditions on W, X_i and their derivatives that enable us to solve for W and X_i $(i = 1, 2, 3, \ldots, m)$.

In order to reduce the computational complexity, we suppose that the group transformation has some simple forms, say,

$$u(n, x_1, x_2, \ldots, x_m) = \alpha(n, x_1, x_2, \ldots, x_m) + \beta(n, x_1, x_2, \ldots, x_m) U(n, X_1, X_2, \ldots, X_m)\,, \tag{8}$$

instead of (5). It is not surprising but interesting that it is enough to use the formula (8) for various nonlinear DDEs.

In Sec. 2, we show the extend method with a proof of the generality of ansatz (5) for the differential-difference KZ equation and derive its transformation group. In Sec. 3, the method is further used for the differential-difference KP system and the ANNV system. The last section is a short summary and discussion.

2 Transformation Group of Differential-Difference KZ Equation

The differential-difference KZ equation reads:[12]

$$u(n)_{xt} + (u(n)u(n)_x)_x + u(n+1) + u(n-1) - 2u(n) = 0\,, \tag{9}$$

where $u(n) \equiv u(n, x, t)$.

First, we should prove that the assumption

$$u(n) = \alpha + \beta U(n, \xi, \tau)\,, \tag{10}$$

where α, β, ξ, τ, are functions of x, t, n, is enough instead of the most general one,

$$u(n) = W(n, x, t, U(n, \xi, \tau))\,, \tag{11}$$

for the KZ equation. Substituting (11) into (9) and requiring $U(n, \xi, \tau)$ also is a solution of the KZ equation but with independent variables ξ, τ (eliminating $U(n)_{\xi,\tau}$ and its higher order derivatives by means of the KZ equation), we have

$$\tau(n)_x W_U(\tau(n)_x W + \tau(n)_t) U(n)_{\tau\tau} + R(n, x, t, U(n), U(n+1), U(n-1), U(n)_\xi, U(n)_{\xi\xi}, U(n)_\tau) = 0\,, \tag{12}$$

where R is a complicated function independent of $U(n)_{\tau\tau}$ and $U(n)_{\xi\tau}$ and its higher order derivatives. Equation (12) is true for an arbitrary solution U only for all coefficients of the polynomials of the derivatives of U being zero. Obviously, $W(n)_U$ and $(\tau(n)_x W + \tau(n)_t)$ should not be zero, so causing the coefficients of $U(n)_{\tau\tau}$ to vanish, the only possible case is

$$\tau(n)_x = 0, \quad \text{i.e.} \quad \tau \equiv \tau(\mathrm{n}, \mathrm{t})\,. \tag{13}$$

Using condition (13), (12) is reduced to

$$\tau(n)_t (W_{xU} + \xi(n)_x W(n)_{UU} U(n)_\xi) U(n)_\tau + R_1(n, x, t, U(n), U(n+1), U(n-1), U(n)_\xi, U(n)_{\xi\xi}) = 0\,, \tag{14}$$

where R_1 is a $U(n)_\tau$ and $U(n)_\xi U(n)_\tau$ independent function. One can easily prove that there is no nontrivial solution for $\tau(n)_t \xi(n)_x = 0$, so vanishing the coefficients of $U(n)_\xi U(n)_\tau$ proves the conclusion that assumption (10) instead of the general one (13) is sufficient to find the general symmetry group of the KZ equation.

Now the substitution of (10) with (13) into the KZ equation (9) leads to

$$\begin{aligned}
&\beta(n)_x \tau(n)_t U(n)_\tau + (\beta(n)\beta(n)_x)_x U(n)^2 + \beta(n)[4\beta(n)_x \xi(n)_x + \beta(n)\xi(n)_{xx}] U(n) U(n)_\xi \\
&\quad + \beta(n)\xi(n)_x[\alpha(n)\xi(n)_x + \xi(n)_t] U(n)_{\xi\xi} + \beta(n)\xi(n)_x[\beta(n)\xi(n)_x - \tau(n)_t](U(n)U(n)_\xi)_\xi \\
&\quad + [\beta(n)_x \xi(n)_t + \alpha(n)\beta(n)\xi(n)_{xx} + \beta(n)\xi(n)_{xt} + \beta(n)_t \xi(n)_x + 2\alpha(n)_x \beta(n)\xi(n)_x + 2\alpha(n)\beta(n)_x \xi(n)_x] U(n)_\xi \\
&\quad + [\beta(n)_{xt} + \alpha(n)_{xx}\beta(n) + \alpha(n)\beta(n)_{xx} + 2\alpha(n)_x \beta(n)_x + 2\beta(n)\tau(n)_t \xi(n)_x - 2\beta(n)] U(n) \\
&\quad + [\beta(n-1) - \beta(n)\xi(n)_x \tau(n)_t] U(n-1) + [\beta(n+1) - \beta(n)\xi(n)_x \tau(n)_t] U(n+1) \\
&\quad + \alpha(n)_{xt} + \alpha(n)_x^2 + \alpha(n)\alpha(n)_{xx} + \alpha(n-1) + \alpha(n+1) - 2\alpha(n) = 0\,.
\end{aligned} \tag{15}$$

According to the above equation (15), the remained determining equation of the functions ξ, τ, α, β can be read off by vanishing the coefficients of the polynomials of $U(n)$, $U(n+1)$, $U(n-1)$ and its derivatives. Then we obtain the following differential equations of the functions ξ, τ, α, β.

$$\begin{aligned}
&\beta(n)_x\tau(n)_t = 0, \quad (\beta(n)\beta(n)_x)_x = 0, \quad \beta(n)4\beta(n)_x\xi(n)_x + \beta(n)\xi(n)_{xx} = 0\,,\\
&\beta(n)\xi(n)_x[\alpha(n)\xi(n)_x + \xi(n)_t] = 0, \quad \beta(n)\xi(n)_x[\beta(n)\xi(n)_x - \tau(n)_t] = 0\,,\\
&\beta(n)_x\xi(n)_t + \alpha(n)\beta(n)\xi(n)_{xx} + \beta(n)\xi(n)_{xt} + \beta(n)_t\xi(n)_x + 2\alpha(n)_x\beta(n)\xi(n)_x + 2\alpha(n)\beta(n)_x\xi(n)_x = 0\,,\\
&\beta(n)_{xt} + \alpha(n)_{xx}\beta(n) + \alpha(n)\beta(n)_{xx} + 2\alpha(n)_x\beta(n)_x + 2\beta(n)\tau(n)_t\xi(n)_x - 2\beta(n) = 0\,,\\
&\beta(n-1) - \beta(n)\xi(n)_x\tau(n)_t = 0, \quad \beta(n+1) - \beta(n)\xi(n)_x\tau(n)_t = 0\,,\\
&\alpha(n)_{xt} + \alpha(n)_x^2 + \alpha(n)\alpha(n)_{xx} + \alpha(n-1) + \alpha(n+1) - 2\alpha(n) = 0\,.
\end{aligned} \tag{16}$$

With the aid of *Maple*, it is straightforward to find out the general solution of the above differential system (16) of the functions ξ, τ, α, β. The solution is

$$\alpha(n,x,t) = -\frac{f(n,t)_t}{c}, \quad \beta(n,x,t) = \frac{1}{c^2}, \quad \xi(n,x,t) = cx + f(n,t), \quad \tau(n,x,t) = \frac{t}{c} + g(n)\,, \tag{17}$$

where c is a non-zero arbitrary constant and $f(n,t)$, $g(n)$ are arbitrary functions of corresponding independent variables.

From Eqs. (10) and (17) we know that if $u(n,x,t)$ is a solution of Eq. (9), then so is

$$U(n,x,t) = -\frac{f(n,t)_t}{c} + \frac{1}{c^2}u\Big(n, cx + f(n,t), \frac{t}{c} + g(n)\Big)\,, \tag{18}$$

where c is a non-zero arbitrary constant and $f(n,t)$, $g(n)$ are arbitrary functions of corresponding independent variables.

In Ref. [12], Zhu-Han Jiang obtained the intrinsic symmetry of Eq. (9) with the standard method. The symmetry is

$$(at+b)\partial_t + (h(t) - ax)\partial_x + (h'(t) - 2au(n))\partial_{u(n)}\,, \tag{19}$$

where a, b are arbitrary constants and $h(t)$ is an arbitrary function of t. According to the standard Lie approach,[1] by solving the corresponding ordinary differential equations

$$\frac{dT}{d\epsilon} = aT + b, \quad \frac{dX}{d\epsilon} = h(T) - aX, \quad \frac{dU(n)}{d\epsilon} = h'(T) - 2aU(n), \quad (T, X, U(n))|_{\epsilon=0} = (t, x, u(n))\,, \tag{20}$$

we can obtain the transformation group of Eq. (19). It is easy to check that the transformation group of Eq. (19) is consistent with (18) when $f(n,t) = f(t)$, and $g(n)$ is a constant. So the derived symmetry group (18) of Eq. (9) is more general.

3 Transformation Group of Other Differential-Difference Equations by Direct Method

In this section, we derive the transformation group of the other differential-difference equations by the presented direct method. Let us first consider the (1+1)-dimensional discretised KP equation[13,14]

$$(u(n+1) + u(n))_{xx} - 2(1 - u(n+1))u(n+1)_x + 2(1 - u(n))u(n)_x - (u(n+1) - u(n))_t = 0\,. \tag{21}$$

Substituting (10) into (21) and simplifying it with the condition

$$U(n+1)_\tau = (U(n+1) + U(n))_{\xi\xi} - 2(1 - U(n+1))U(n+1)_\xi + 2(1 - U(n))U(n)_\xi + U(n)_\tau\,,$$

we can obtain a polynomial equation of $U(n)$, $U(n+1)$ and its derivatives. Let their coefficients to be zero we get a system of functions α, β, ξ, τ. Solving the system with the aid of *Maple*, we obtain

$$\begin{aligned}
&\xi(n,x,t) = cf'^{1/2}(t)x + g(t), \quad \tau(n,x,t) = f(t)\,,\\
&\alpha(n,x,t) = \frac{f''(t)x}{4f'(t)} + \frac{cg'(t)}{2f'^{1/2}(t)} - cf'^{1/2}(t) + 1, \quad \beta(n,x,t) = cf'^{1/2}(t)\,,
\end{aligned} \tag{22}$$

where $c = \pm 1$ and $f(t)$, $g(t)$ are arbitrary functions of t. Then we can safely say that if $u(n,x,t)$ is a solution of Eq. (21), then so is

$$U(n,x,t) = \frac{f''(t)x}{4f'(t)} + \frac{cg'(t)}{2f'^{1/2}(t)} - cf'^{1/2}(t) + 1 + cf'^{1/2}(t)u(n, cf'^{1/2}(t)x + g(t), f(t))\,. \tag{23}$$

And if we set $c = 1$, $g(t) = \epsilon G(t)$, $f(t) = t + \epsilon F(t)$, we can get the Lie point symmetry $U \longrightarrow u + \epsilon\sigma(u)$, where

$$\sigma(u) = -\frac{1}{2}F'(t)u - \frac{1}{4}F''(t)x + \frac{1}{2}F'(t) - \frac{1}{2}G'(t) + F(t)u_t + \Big(\frac{1}{2}F'(t)x + G(t)\Big)u_x\,. \tag{24}$$

This is same as (4.11) presented in Ref. [13].

As a final example, we consider the (2+1)-dimensional differential-difference ANNV system[15]

$$v(n+1)_x + v(n)_x = w(n+1) - w(n), \quad v(n)_t + 3v(n)_x v(n)^2 + v(n)_{xxx} + 3(v(n)w(n))_x = 0\,. \tag{25}$$

Let

$$v(n) = a(n,x,t) + b(n,x,t)V(n,\xi,\tau), \quad w(n) = p(n,x,t) + q(n,x,t)W(n,\xi,\tau)\,, \tag{26}$$

where $\xi = \xi(n,x,t)$, $\tau = \tau(n,x,t)$, and V, W is a solution of

$$V(n+1)_\xi + V(n)_\xi = W(n+1) - W(n), \quad V(n)_\tau + 3V(n)_\xi V(n)^2 + V(n)_{\xi\xi\xi} + 3(V(n)W(n))_\xi = 0\,. \tag{27}$$

Substituting (26) into (25) and simplifying it along with (27), we obtain the result:

$$a(n,x,t) = 0, \quad b(n,x,t) = f(t), \quad p(n,x,t) = -\frac{f'(t)x + g(t)}{f(t)}\,,$$

$$q(n,x,t) = f^2(t), \quad \xi(n,x,t) = f(t)x + g(t), \quad \tau(n,x,t) = \int^t f(s)ds + c\,, \tag{28}$$

where $f(t) \neq 0$, $g(t)$ are arbitrary functions of t, and c is an arbitrary constant. Then if $v(n)$, $w(n)$ is a solution of Eq. (25), then so is

$$V(n) = f(t)v\Big(n, f(t)x + g(t), \int^t f(s)ds + c\Big), \quad W(n) = -\frac{f'(t)x + g(t)}{f(t)} + f^2(t)W\Big(n, f(t)x + g(t), \int^t f(s)ds + c\Big)\,. \tag{29}$$

It is easy to check that Eq. (25) has the following bell solitary solution:

$$v(n) = a_0 - \frac{\sinh(d)}{\cosh(d) + \cosh(\omega)}, \quad w(n) = b_0 + \frac{\cosh(d)}{\cosh(d) + \cosh(\omega)} - \frac{\sinh^2(d)}{(\cosh(d) + \cosh(\omega))^2}\,, \tag{30}$$

where $\omega = 2dn + x - (1 + 3b_0 + 3a_0^2 + 3a_0\coth(2d))t + \delta$, and a_0, b_0, d, δ are arbitrary constants. With the help of Eqs. (29),

$$\tilde{v}(n) = a_0 f(t) - \frac{\sinh(d)f(t)}{\cosh(d) + \cosh(\underline{\omega})}\,,$$

$$\tilde{w}(n) = -\frac{f'(t)x + g(t)}{f(t)} + b_0 f^2(t) + \frac{\cosh(d)f^2(t)}{\cosh(d) + \cosh(\underline{\omega})} - \frac{\sinh^2(d)f^2(t)}{(\cosh(d) + \cosh(\underline{\omega}))^2}\,, \tag{31}$$

is the bell soliton-like solution of Eq. (25), where $\underline{\omega} = 2dn + f(t)x + g(t) - (1 + 3b_0 + 3a_0^2 + 3a_0\coth(2d))(\int^t f(s)ds + c) + \delta$, and a_0, b_0, d, δ are arbitrary constants, $f(t), g(t)$ are arbitrary functions.

4 Summary and Discussions

In summary, based on the symbolic computer system *Maple*, we successfully extend the direct method for Lie symmetry groups presented by Senyue Lou from the continuous differential equations to the differential-difference equations. With the extended method we discuss the well-known differential-difference KP equation, KZ equation and (2+1)-dimensional ANNV system, both the Lie point symmetry groups and the non-Lie symmetry groups are obtained. And we derived more general symmetry groups with the direct method than these obtained by the traditional approaches. This extended method can also be used to other nonlinear DDEs.

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