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Grammian Determinant Solution and Pfaffianization for a (3+1)-Dimensional Soliton Equation

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Abstract *Based on the Pfaffian derivative formulae, a Grammian determinant solution for a (3+1)-dimensional soliton equation is obtained. Moreover, the Pfaffianization procedure is applied for the equation to generate a new coupled system. At last, a Gram-type Pfaffian solution to the new coupled system is given.*

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1 Introduction

The Wronskian technique has been proven to be an efficient and direct approach to construct soliton solutions of nonlinear evolution equations possessing Hirota bilinear forms. A known fact is that the soliton solutions of the famous Kadomtsev–Petviashvili (KP) equation can be represented in terms of a Wronskian determinant. Besides the Wronskian determinant, Grammian determinant provides another expression of soliton solutions for the KP equation.[1] Recently, a procedure called Pfaffianization was developed by Hirota and Ohta to produce new coupled systems of equations from known soliton equations.[2−3] These Pfaffianized equations appear as coupled systems of the original equations and have solutions expressed in terms of Pfaffians. In the past decades, remarkable progress has been made in this area. Many continuous and discrete soliton equations were investigated, such as the nonisospectral KP equation,[4] the discrete KP equation,[5] the two-dimensional Toda lattice,[6] the semi-discrete Toda equation,[7] and others.

In this paper, our aim is to investigate the (3+1)-dimensional soliton equation

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x\partial_x^{-1}w_y)_x = 0\,, \quad (1)$$

which was derived in Ref. [8]. So far as we know, a lot of research on Eq. (1) has been conducted. In Ref. [8], Eq. (1) was decomposed into systems of solvable ordinary differential equations with the help of the (1+1)-dimensional AKNS equations and algebraic-geometrical solutions for it were explicitly given in terms of the Riemann theta functions. In Ref. [9], an N-soliton solution for Eq. (1) and its Wronskian form were derived using the Hirota method and Wronskian technique. Recently, Wu[10] has given a bilinear Bäcklund transformation and some soliton solutions for Eq. (1). However, the Grammian determinant solution and the Pfaffianization of the (3+1)-dimensional soliton equation have not been revealed in previous articles.

The present paper is organized as follows. In Sec. 2, we derive a Grammian determinant solution for Eq. (1). In Sec. 3, we apply the Pfaffianization procedure to Eq. (1) to generate a new coupled system and give a Gram-type Pfaffian solution for the coupled system. Section 4 is our conclusions.

2 Grammian Determinant Solution

Through the dependent variable transformation

$$w = -3(\ln f)_{xx}\,, \quad (2)$$

the (3+1)-dimensional soliton equation (1) can be written in the bilinear form[8]

$$(3D_xD_z - 2D_yD_t - D_yD_x^3)f\cdot f = 0\,. \quad (3)$$

Besides the Wronskian determinant solution in Ref. [9], we might expect Eq. (1) possesses an N-soliton solution in terms of a Grammian determinant

$$\begin{aligned} &f_N = \det(a_{ij})_{1\le i,j\le 2N}\,,\\ &a_{ij} = c_{ij} + \int^x f_i g_j dx, \quad c_{ij} = \text{const.}\,, \end{aligned} \quad (4)$$

with the elements $f_i = f_i(x,y,z,t)$ and $g_j = g_j(x,y,z,t)$ satisfying

$$\begin{aligned} &f_{i,y} = f_{i,xx}, \quad g_{j,y} = -g_{j,xx},\\ &f_{i,t} = f_{i,xxx}, \quad g_{j,t} = g_{j,xxx},\\ &f_{i,z} = f_{i,xxxx}, \quad g_{j,z} = -g_{j,xxxx},\\ &(i,j = 1,2,\ldots,2N)\,. \end{aligned} \quad (5)$$

In order to prove the Grammian determinant f_N satisfies the bilinear equation (3), we rewrite f_N as a Pfaffian

$$f_N = (1,2,\ldots,N,N^*,\ldots,2^*,1^*)\,, \quad (6)$$

$$(i,j^*) = c_{ij} + \int^x f_i g_j dx\,, \quad (i,j) = (i^*,j^*) = 0\,. \quad (7)$$

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Next, let us introduce Pfaffian entries $(m,n=0,1,2,\dots)$

$$(d_n,j^*)=\frac{\partial^n}{\partial x^n}g_j\,,\quad (d_m,d_n^*)=0\,,$$
$$(d_n^*,i)=\frac{\partial^n}{\partial x^n}f_i\,,\quad (d_m^*,i^*)=(d_n,i)=0\,. \tag{8}$$

In terms of these, derivatives of the elements a_{ij} of f_N are given by

$$\frac{\partial}{\partial x}a_{ij}=(d_0,d_0^*,i,j^*)\,, \tag{9}$$
$$\frac{\partial}{\partial y}a_{ij}=-(d_1,d_0^*,i,j^*)+(d_0,d_1^*,i,j^*)\,, \tag{10}$$
$$\frac{\partial}{\partial t}a_{ij}=(d_2,d_0^*,i,j^*)-(d_1,d_1^*,i,j^*)+(d_0,d_2^*,i,j^*)\,, \tag{11}$$
$$\frac{\partial}{\partial z}a_{ij}=(d_0,d_3^*,i,j^*)-(d_3,d_0^*,i,j^*)+(d_2,d_1^*,i,j^*)-(d_1,d_2^*,i,j^*)\,. \tag{12}$$

Therefore, from the above results (9)–(12), we have the following differential formulae for f_N

$$\begin{aligned}
&f_{N,x}=(d_0,d_0^*,\bullet)\,,\quad f_{N,y}=-(d_1,d_0^*,\bullet)+(d_0,d_1^*,\bullet)\,,\quad f_{N,t}=(d_2,d_0^*,\bullet)-(d_1,d_1^*,\bullet)+(d_0,d_2^*,\bullet)\,,\\
&f_{N,z}=(d_0,d_3^*,\bullet)-(d_3,d_0^*,\bullet)+(d_2,d_1^*,\bullet)-(d_1,d_2^*,\bullet)\,,\quad f_{N,xx}=(d_1,d_0^*,\bullet)+(d_0,d_1^*,\bullet)\,,\\
&f_{N,xy}=(d_0,d_2^*,\bullet)-(d_2,d_0^*,\bullet)\,,\quad f_{N,xxx}=(d_2,d_0^*,\bullet)+2(d_1,d_1^*,\bullet)+(d_0,d_2^*,\bullet)\,,\\
&f_{N,xxy}=(d_0,d_3^*,\bullet)-(d_3,d_0^*,\bullet)+(d_1,d_2^*,\bullet)-(d_2,d_1^*,\bullet)\,,\\
&f_{N,xz}=(d_0,d_4^*,\bullet)-(d_4,d_0^*,\bullet)+(d_2,d_1^*,d_0,d_0^*,\bullet)-(d_1,d_2^*,d_0,d_0^*,\bullet),\\
&f_{N,yt}=(d_0,d_4^*,\bullet)-(d_4,d_0^*,\bullet)+(d_3,d_1^*,\bullet)-(d_1,d_3^*,\bullet)+(d_2,d_0^*,d_0,d_1^*,\bullet)-(d_0,d_2^*,d_1,d_0^*,\bullet),\\
&f_{N,xxxy}=(d_0,d_4^*,\bullet)-(d_4,d_0^*,\bullet)-2(d_3,d_1^*,\bullet)+2(d_1,d_3^*,\bullet)-(d_0,d_2^*,d_1,d_0^*,\bullet)+(d_2,d_0^*,d_0,d_1^*,\bullet)\,,
\end{aligned} \tag{13}$$

where we have used the abbreviated notation $f_N=(1,2,\dots,N,N^*,\dots,2^*,1^*)=(\bullet)$.

Substituting the above derivatives of f_N into the LHS of Eq. (3), we arrive at

$$\begin{aligned}
(3D_xD_z-2D_yD_t-D_yD_x^3)f_N\cdot f_N&=[(d_0,d_0^*,d_2,d_1^*,\bullet)(\bullet)-(d_0,d_0^*,\bullet)(d_2,d_1^*,\bullet)-(d_0,d_1^*,\bullet)(d_0^*,d_2,\bullet)]\\
&-[(d_0,d_0^*,d_1,d_2^*,\bullet)(\bullet)-(d_0,d_0^*,\bullet)(d_1,d_2^*,\bullet)-(d_0,d_2^*,\bullet)(d_0^*,d_1,\bullet)]=0\,,
\end{aligned}$$

utilizing the known Jacobi identities. Thus we have proved the function f_N given by (4) is a Grammian determinant solution for the bilinear equation (3).

3 Pfaffianization

In what follows, we shall apply the Pfaffianization procedure to the bilinear equation (3) to generate a new coupled system. In a way similar to Ref. [1], we take a Wronski-type Pfaffian $f=(1,2,\dots,2N)$ with its elements satisfy

$$\begin{aligned}
&\partial_x(i,j)=(i+1,j)+(i,j+1),\quad \partial_y(i,j)=(i+2,j)+(i,j+2),\\
&\partial_t(i,j)=(i+3,j)+(i,j+3),\quad \partial_z(i,j)=(i+4,j)+(i,j+4),\quad (i,j=1,2,\dots,2N).
\end{aligned}$$

Taking the above assumption into account, we tediously calculate the differentials of f, that is

$$\begin{aligned}
f_x&=(1,2,\dots,2N-1,2N+1)\,,\\
f_y&=(1,2,\dots,2N-1,2N+2)-(1,2,\dots,2N-2,2N,2N+1)\,,\\
f_t&=(1,2,\dots,2N-3,2N-1,2N,2N+1)-(1,2,\dots,2N-2,2N,2N+2)+(1,2,\dots,2N-1,2N+3)\,,\\
f_z&=-(1,2,\dots,2N-4,2N-2,2N-1,2N,2N+1)+(1,2,\dots,2N-3,2N-1,2N,2N+2)\\
&\quad-(1,2,\dots,2N-2,2N,2N+3)+(1,2,\dots,2N-1,2N+4)\,,\\
f_{xx}&=(1,2,\dots,2N-1,2N+2)+(1,2,\dots,2N-2,2N,2N+1)\,,\\
f_{xy}&=(1,2,\dots,2N-1,2N+3)-(1,2,\dots,2N-3,2N-1,2N,2N+1)\,,\\
f_{xz}&=-(1,2,\dots,2N-5,2N-3,2N-2,2N-1,2N,2N+1)+(1,2,\dots,2N-3,2N-1,,2N+1,2N+2)\\
&\quad-(1,2,\dots,2N-2,2N+1,2N+3)+(1,2,\dots,2N-1,2N+5)\,,\\
f_{xxx}&=(1,2,\dots,2N-1,2N+3)+2(1,2,\dots,2N-2,2N,2N+2)+(1,2,\dots,2N-3,2N-1,2N,2N+1)\,,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
f_{yt}&=(1,2,\dots,2N-4,2N-2,2N-1,2N,2N+2)-(1,2,\dots,2N-3,2N-1,2N+1,2N+2)\\
&\quad+(1,2,\dots,2N-1,2N+5)+(1,2,\dots,2N-2,2N+1,2N+3)\\
&\quad-(1,2,\dots,2N-2,2N,2N+4)-(1,2,\dots,2N-5,2N-3,\dots,2N+1),\\
f_{xxy}&=(1,2,\dots,2N-2,2N,2N+3)+(1,2,\dots,2N-1,2N+4)
\end{aligned}$$

$$
\begin{aligned}
&- (1,2,\ldots,2N-4,2N-2,2N-1,2N,2N+1) - (1,2,\ldots,2N-3,2N-1,2N,2N+2)\,,\\
f_{xxxy} &= (1,2,\ldots,2N-5,2N-3,\ldots,2N+1) + (1,2,\ldots,2N-2,2N+1,2N+3)\\
&\quad + (1,2,\ldots,2N-1,2N+5) - (1,2,\ldots,2N-3,2N-1,2N+1,2N+2)\\
&\quad - 2(1,2,\ldots,2N-4,2N-2,2N-1,2N,2N+2)\,. \qquad (15)
\end{aligned}
$$

Then substituting the above derivatives (14) and (15) into the LHS of the bilinear equation (3), we obtain by some calculation that

$$
\begin{aligned}
(3D_xD_z - 2D_yD_t - D_yD_x^3)f\cdot f &= 12[(1,2,\ldots,2N-3,2N-1,2N+1,2N+2)(1,2,\ldots,2N)\\
&- (1,2,\ldots,2N-3,2N-1,2N,2N+2)(1,2,\ldots,2N-1,2N+1)\\
&+ (1,2,\ldots,2N-3,2N-1,2N,2N+1)(1,2,\ldots,2N-1,2N+2)]\\
&- 12[(1,2,\ldots,2N-2,2N+1,2N+3)(1,2,\ldots,2N)\\
&- (1,2,\ldots,2N-2,2N,2N+3)(1,2,\ldots,2N-1,2N+1)\\
&+ (1,2,\ldots,2N-1,2N+3)(1,2,\ldots,2N-2,2N,2N+1)]\,. \qquad (16)
\end{aligned}
$$

Following the Hirota–Ohta Pfaffianization procedure, we now introduce four new variables $g,\ h,\ \hat{g},\ \hat{h}$ defined by

$$
\begin{aligned}
g &= (1,2,\ldots,2N-2)\,,\\
h &= (1,2,\ldots,2N+1,2N+3)\,,\\
\hat{g} &= (1,2,\ldots,2N-3,2N-1)\,,\\
\hat{h} &= (1,2,\ldots,2N+2)\,. \qquad (17)
\end{aligned}
$$

Then employing the Pfaffian identities, we obtain three bilinear equations from Eqs. (16) and (17)

$$
\begin{aligned}
&(3D_xD_z - 2D_yD_t - D_yD_x^3)f\cdot f - 12(\hat{g}\hat{h} - gh) = 0\,, \qquad (18)\\
&(D_x^3 + 3D_xD_y + 2D_t)g\cdot f = 0\,,\\
&(D_x^3 - 3D_xD_y + 2D_t)\hat{h}\cdot f = 0\,. \qquad (19)
\end{aligned}
$$

The procedures for deducing the two expressions of (19) are similar to that of (18), which are omitted here. We call Eqs. (18)–(19) the coupled bilinear (3+1)-dimensional nonlinear soliton equation.

It is quite interesting that when setting

$$
w = -3(\ln f)_{xx}, \quad u = \frac{\hat{g}}{f}, \quad \hat{u} = \frac{\hat{h}}{f}, \quad v = \frac{g}{f}, \quad \hat{v} = \frac{h}{f}\,, \qquad (20)
$$

the coupled bilinear system (18) and (19) is changed into a coupled (3+1)-dimensional nonlinear evolution equation

$$
\begin{aligned}
&3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x\partial_x^{-1}w_y)_x\\
&\quad +18(u\hat{u} - v\hat{v})_{xx} = 0,\\
&2v_t - 2wv_x + v_{xxx} + 3v_{xy} - 2v\partial_x^{-1}w_y = 0,\\
&2\hat{u}_t - 2w\hat{u}_x + \hat{u}_{xxx} - 3\hat{u}_{xy} + 2\hat{u}\partial_x^{-1}w_y = 0\,. \qquad (21)
\end{aligned}
$$

It is easy to see that the coupled system (21) can be reduced to the (3+1)-dimensional soliton equation (1) when $u = \hat{u} = v = \hat{v} = 0$.

In what follows, we shall give a solution for the coupled system (18)–(19) expressed as Gram-type Pfaffians

$$
\begin{aligned}
&f = (1,2,\ldots,2N), \quad g = (c_1,c_0,1,2,\ldots,2N),\\
&h = (d_0,d_2,1,2,\ldots,2N), \quad \hat{g} = (c_2,c_0,1,2,\ldots,2N),\\
&\hat{h} = (d_0,d_1,1,2,\ldots,2N)\,, \qquad (22)
\end{aligned}
$$

where the different types of Pfaffian entries are defined by

$$
\begin{aligned}
&(i,j) = c_{ij} + \int^x (f_ig_j - f_jg_i)dx, \quad c_{ij} = -c_{ji},\\
&(d_n,i) = \frac{\partial^n}{\partial x^n}f_i, \quad (c_n,i) = \frac{\partial^n}{\partial x^n}g_i,\\
&(d_m,d_n) = (c_m,c_n) = (c_m,d_n) = 0\,, \qquad (23)
\end{aligned}
$$

and f_i, g_j $(i,j = 1,2,\ldots,2N)$ satisfy the differential equations

$$
\begin{aligned}
&f_{i,y} = f_{i,xx}, \quad g_{j,y} = -g_{j,xx},\\
&f_{i,t} = f_{i,xxx}, \quad g_{j,t} = g_{j,xxx},\\
&f_{i,z} = f_{i,xxxx}, \quad g_{j,z} = -g_{j,xxxx}\,. \qquad (24)
\end{aligned}
$$

Next we have to prove Eq. (22) defined by Eqs. (23)–(24) is indeed a Gram-type Pfaffian solution of the new coupled system (18)–(19). Noting the formulae

$$
\begin{aligned}
&\frac{\partial}{\partial x}(i,j) = (c_0,d_0,i,j)\,, \qquad (25)\\
&\frac{\partial}{\partial y}(i,j) = (c_0,d_1,i,j) - (c_1,d_0,i,j)\,, \qquad (26)\\
&\frac{\partial}{\partial t}(i,j) = (c_0,d_2,i,j) - (c_1,d_1,i,j) + (c_2,d_0,i,j)\,, \qquad (27)\\
&\frac{\partial}{\partial z}(i,j) = (c_0,d_3,i,j) - (c_3,d_0,i,j) + (c_2,d_1,i,j)\\
&\qquad - (c_1,d_2,i,j)\,, \qquad (28)
\end{aligned}
$$

we obtain expression for the derivatives of f

$$
\begin{aligned}
&f_x = (c_0,d_0,\bullet)\,, \quad f_y = -(c_1,d_0,\bullet) + (c_0,d_1,\bullet)\,, \quad f_t = (c_2,d_0,\bullet) - (c_1,d_1,\bullet) + (c_0,d_2,\bullet)\,,\\
&f_z = (c_0,d_3,\bullet) - (c_3,d_0,\bullet) + (c_2,d_1,\bullet) - (c_1,d_2,\bullet)\,, \quad f_{xx} = (c_1,d_0,\bullet) + (c_0,d_1,\bullet)\,,\\
&f_{xxx} = (c_2,d_0,\bullet) + 2(c_1,d_1,\bullet) + (c_0,d_2,\bullet)\,, \quad f_{xy} = (c_0,d_2,\bullet) - (c_2,d_0,\bullet)\,,
\end{aligned}
$$

$$\begin{aligned}
&f_{xz} = (c_0, d_4, \bullet) - (c_4, d_0, \bullet) + (c_2, d_1, c_0, d_0, \bullet) - (c_1, d_2, c_0, d_0, \bullet),\\
&f_{xxy} = (c_0, d_3, \bullet) - (c_3, d_0, \bullet) + (c_1, d_2, \bullet) - (c_2, d_1, \bullet),\\
&f_{yt} = (c_0, d_4, \bullet) - (c_4, d_0, \bullet) + (c_3, d_1, \bullet) - (c_1, d_3, \bullet) + (c_2, d_0, c_0, d_1 \bullet) - (c_0, d_2, c_1, d_0, \bullet),\\
&f_{xxxy} = (c_0, d_4, \bullet) - (c_4, d_0, \bullet) - 2(c_3, d_1, \bullet) + 2(c_1, d_3, \bullet) - (c_0, d_2, c_1, d_0, \bullet) + (c_2, d_0, c_0, d_1, \bullet)\,,
\end{aligned} \tag{29}$$

where we have denoted $f = (1, 2, \ldots, 2N) = (\bullet)$. Substituting the above derivatives (29) into the LHS of (18) by employing Eq. (22), we obtain by some calculation that

$$\begin{aligned}
&(3D_xD_z - 2D_yD_t - D_yD_x^3)f \cdot f - 12(\hat{g}\hat{h} - gh)\\
&= [(c_0, d_0, c_2, d_1, \bullet)(\bullet) - (c_0, d_0, \bullet)(c_2, d_1, \bullet) + (c_0, c_2, \bullet)(d_0, d_1, \bullet) - (c_0, d_1, \bullet)(d_0, c_2, \bullet)]\\
&\quad - [(c_0, d_0, c_1, d_2, \bullet)(\bullet) - (c_0, d_0, \bullet)(c_1, d_2, \bullet) + (c_0, c_1, \bullet)(d_0, d_2, \bullet) - (c_0, d_2, \bullet)(d_0, c_1, \bullet)]\,.
\end{aligned} \tag{30}$$

Equation (30) is equal to zero for it is nothing but the difference between two Pfaffian identities. Similarly, the two bilinear equations of (19) can also be reduced to Pfaffian identities. This means that f, g, h, $\hat{g}$, $\hat{h}$ defined by (22) is a Gram-type Pfaffian solution of the coupled system (18) and (19).

4 Conclusions

In summary, we have presented a Grammian determinant solution for the (3+1)-dimensional soliton equation (1). In addition, we have applied the Pfaffianization procedure to derive a new coupled system. Further, a Gram-type Pfaffian solution to the new coupled system is given by employing the Pfaffian identities.

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