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Weyl Ordering Operator Formula Derived by IWOP Technique and Its Application for Fresnel Operator*

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Abstract *Based on the technique of integration within an ordered product of operators, the Weyl ordering operator formula is derived and the Fresnel operators' Weyl ordering is also obtained, which together with the Weyl transformation can immediately lead to Fresnel transformation kernel in classical optics.*

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1 Introduction

In studying quantum mechanics and quantum field theory people always face to tackle miscellaneous operators which have different commutative relations with other operators. The operator ordering problem bothers people from time to time.[1] There are some definite operator orderings, such as normal ordering, antinormal ordering, and Weyl ordering, among them the Weyl ordering of operators is the direct result of Weyl quantization recipe[2] and is very useful to path integral theory and quantum statistics.[3–5] In Refs. [6–8] in order to establish a formalism of arranging bosonic operators into Weyl ordered forms, we have introduced the Weyl ordering symbol, denoted as $\vdots\ \vdots$, and proposed the technique of integral within Weyl ordered product (IWWOP) of operators, the single-mode Wigner operator

$$\Delta(p,q) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{ipu} \Big| q + \frac{u}{2} \Big\rangle \Big\langle q - \frac{u}{2} \Big| \,, \tag{1}$$

which is well-known in coordinate representation, and has a concise Weyl ordered form in terms of the coordinate operator Q and the momentum operator P, i.e.,

$$\begin{aligned}\Delta(p,q) &= \vdots\, \delta(p-P)\delta(q-Q) \,\vdots \\ &= \vdots\, \delta(q-Q)\delta(p-P) \,\vdots \,,\end{aligned} \tag{2}$$

this is a concise and easily remembered formula. In Ref. [8], we have derived the Weyl ordering operator formula for arranging any operator ρ (in one-mode case) into Weyl ordering,

$$\rho = 2 \vdots \int \frac{d^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \exp\left[2(a^\dagger a + a\beta^* - \beta a^\dagger)\right] \vdots \,, \tag{3}$$

where $|\beta\rangle = \exp[-(1/2)|\beta|^2 + \beta a^\dagger]|0\rangle$ is coherent state.[9–10] In the derivation of Eq. (3) the P-representation of ρ in coherent state basis has been used, which means that the P-representation should exist for operator ρ. To avoid this inconvenience, in this paper we intend to derive the Weyl ordering formula by virtue of the technique of integration within an ordered product (IWOP) of operators and Weyl correspondence. As one can see later, our derivation is neater and more concise. The work is arranged as follows. In Sec. 2 we derive the operator formula which can bring a given operator function to its Weyl ordering form. As an important application of the formula, in Sec. 3 we derive the Fresnel operators' Weyl ordering which together with the Weyl transformation can immediately lead to Fresnel transformation kernel in classical optics. Some decompositions of Fresnel operator's Weyl-ordering are shown in Sec. 4.

2 Weyl Ordering Operator Formula Derived by IWOP Technique

In coherent state representation Wigner operator takes the form[11]

$$\Delta(p,q) \to \Delta(\alpha,\alpha^*) = \int \frac{d^2 z}{\pi^2} |\alpha + z\rangle\langle \alpha - z| e^{\alpha z^* - \alpha^* z} \,,$$
$$\alpha = (q + ip)/\sqrt{2} \,. \tag{4}$$

Using the IWOP technique[12–13] and noting the vacuum state projector $|0\rangle\langle 0| =: \exp(-a^\dagger a):$ we can obtain

$$\Delta(\alpha,\alpha^*) = \frac{1}{\pi} \colon e^{-2(\alpha^* - a^\dagger)(\alpha - a)} \colon . \tag{5}$$

It is easy to see that Wigner operator $\Delta(\alpha,\alpha^*)$ is completeness, i.e.,

$$2\int d^2\alpha \Delta(\alpha,\alpha^*) = 1 \,. \tag{6}$$

Thus an arbitrary operator in single-mode case can be expanded according to the Wigner operator, i.e.,

$$H(a,a^\dagger) = 2\int d^2\alpha \Delta(\alpha,\alpha^*) h(\alpha,\alpha^*) \,, \tag{7}$$

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where $h(\alpha,\alpha^*)$ is the classical Weyl correspondence to $H(a,a^\dagger)$. The Wigner operator is also an integration kernel of Weyl correspondence,[14–15] which is a recipe of quantizing classical functions $h(\alpha,\alpha^*)$ to the corresponding quantum operators $H(a,a^\dagger)$.

We next examine the matrix element in coherent state,

$$\langle -\beta|H(a,a^\dagger)|\beta\rangle = 2\int d^2\alpha \langle -\beta|\Delta(\alpha,\alpha^*)|\beta\rangle h(\alpha,\alpha^*)\,. \quad (8)$$

Using the normal product of $\Delta(\alpha,\alpha^*)$ in Eq. (5), we have

$$\langle -\beta|\Delta(\alpha,\alpha^*)|\beta\rangle = \frac{1}{\pi}\langle -\beta|{:}\,e^{-2(\alpha^*-a^\dagger)(\alpha-a)}\,{:}|\beta\rangle = \frac{1}{\pi}\exp[-2|\alpha|^2 + 2\beta\alpha^* - 2\alpha\beta^*]\,. \quad (9)$$

Substituting Eq. (9) into Eq. (8), we can write Eq. (8) as

$$\langle -\beta|H(a,a^\dagger)|\beta\rangle = 2\int \frac{d^2\alpha}{\pi} h(\alpha,\alpha^*)e^{-2|\alpha|^2}\exp\big[2\beta\alpha^* - 2\alpha\beta^*\big]\,. \quad (10)$$

Noting that the item $2\beta\alpha^* - 2\alpha\beta^*$ is a pure imaginary. Thus Eq. (10) indicates that $(1/2\pi)\langle -\beta|H(a,a^\dagger)|\beta\rangle$ is the Fourier transformation for $h(\alpha,\alpha^*)e^{-2|\alpha|^2}$, then its inverse Fourier transformation yields

$$h(\alpha,\alpha^*) = 2\int \frac{d^2\beta}{\pi}\langle -\beta|H(a,a^\dagger)|\beta\rangle \exp\big[2(|\alpha|^2 + \alpha\beta^* - \beta\alpha^*)\big]\,. \quad (11)$$

According to the rule that classical correspondence $h(\alpha,\alpha^*)$ of a Weyl ordered operator $\vdots\, h(a,a^\dagger)\,\vdots$ can be directly obtained by simply replacing $a \to \alpha^*$, $a^\dagger \to \alpha^*$, and the reverse is true. But we should note that $h(a,a^\dagger)$ must be within the Weyl ordering $\vdots\;\vdots$. Thus we have

$$\vdots\, h(a,a^\dagger)\,\vdots = 2\,\vdots \int \frac{d^2\beta}{\pi}\langle -\beta|H(a,a^\dagger)|\beta\rangle \exp\big[2(a^\dagger a + a\beta^* - \beta a^\dagger)\big]\,\vdots\,, \quad (12)$$

which is a way to obtain the Weyl ordering of the normal ordering product of operators because the off-diagonalized matrix elements in coherent state, $\langle -\beta|H(a,a^\dagger)|\beta\rangle$, can be directly given if we know the normal ordering of $H(a,a^\dagger)$.

For example, when $H(a,a^\dagger) = a^{\dagger n}a^m$, its corresponding Weyl ordering is

$$\begin{aligned} a^{\dagger n}a^m &= 2\,\vdots \int \frac{d^2\beta}{\pi}\langle -\beta|a^{\dagger m}a^n|\beta\rangle \exp\big[2(a^\dagger a + a\beta^* - \beta a^\dagger)\big]\,\vdots \\ &= 2(-1)^n\,\vdots \int \frac{d^2\beta}{\pi}\beta^{*m}\beta^n \exp\big[-2|\beta|^2 - 2a^\dagger\beta + 2a\beta^* + 2a^\dagger a\big]\,\vdots = \big(-1/\sqrt{2}\big)^{n+m}\,\vdots\, H_{m,n}\big(\sqrt{2}a^\dagger,\sqrt{2}a\big)\,\vdots\,, \end{aligned} \quad (13)$$

where $H_{m,n}(\epsilon,\varepsilon)$ is two variables Hermite polynomial and we have used the formula

$$(-1)^n\exp(BC)\int \frac{d^2z}{\pi}\beta^{*m}\beta^n\exp[-|z|^2 + Bz - Cz^*] = H_{m,n}(B,C)\,. \quad (14)$$

For another example, $H(a,a^\dagger) = a^n a^{\dagger m}$, whose normal product is $a^n a^{\dagger m} = (-i)^{m+n}{:}\,H_{m,n}(ia^\dagger, ia)\,{:}$,[16] by using Eqs. (12) and (14), the corresponding Weyl ordering can be obtained

$$\begin{aligned} a^n a^{\dagger m} &= 2\,\vdots \int \frac{d^2\beta}{\pi}\langle -\beta|(-i)^{m+n}{:}\,H_{m,n}(ia^\dagger,ia)\,{:}|\beta\rangle \exp\big[2(a^\dagger a + a\beta^* - \beta a^\dagger)\big]\,\vdots \\ &= 2(-i)^{m+n}\,\vdots \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m!n!(-i)^{m-k}(i)^{n-k}}{k!(m-k)!(n-k)!}\int \frac{d^2\beta}{\pi}\beta^{*m-k}\beta^{n-k}\exp\big[-2|\beta|^2 - 2a^\dagger\beta + 2a\beta^* + 2a^\dagger a\big]\,\vdots \\ &= \vdots \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m!n!(1/\sqrt{2})^{n+m}}{k!(m-k)!(n-k)!} H_{m-k,n-k}(\sqrt{2}a^\dagger,\sqrt{2}a)\,\vdots\,. \end{aligned} \quad (15)$$

3 Application — Fresnel Operators' Weyl Ordering

As another important application of Eq. (12), we now derive the Fresnel operators' Weyl ordering in one-mode case. In Ref. [17], the Fresnel operator is constructed by as follows:

$$U(r,s) \equiv \sqrt{s}\int \frac{d^2z}{\pi}|sz - rz^*\rangle\langle z|\,, \quad (16)$$

where $|z\rangle = \exp[-|z|^2/2 + za^\dagger]|0\rangle$ is coherent state, and $a^\dagger$ is the Bose creation operator, $[a,a^\dagger] = 1$, s and r are complex satisfying the unimodularity condition $ss^* - rr^* = 1$. Using the IWOP technique and the normal ordering of the vacuum projector $|0\rangle\langle 0| = {:}\exp(-a^\dagger a){:}$, to perform the integration Eq. (16), we obtain

$$U(r,s) = \frac{1}{\sqrt{s^*}}{:}\exp\Big[-\frac{r}{2s^*}a^{\dagger 2} + \Big(\frac{1}{s^*} - 1\Big)a^\dagger a + \frac{r^*}{2s^*}a^2\Big]{:}\,. \quad (17)$$

Substituting Eq. (17) into Eq. (12), we can derive the Fresnel operators' Weyl ordering

$$U(r,s) = \frac{2}{\sqrt{s^*}} \vdots \int \frac{d^2\beta}{\pi} \exp\Big[-\frac{1+s^*}{s^*}|\beta|^2 - 2\beta a^\dagger + 2a\beta^* + \frac{r^*}{2s^*}\beta^2 - \frac{r}{2s^*}\beta^{*2} + 2a^\dagger a\Big] \vdots$$
$$= \frac{2}{\sqrt{2+s+s^*}} \vdots \exp\Big\{\frac{2[(s-s^*)a^\dagger a - r a^{\dagger 2} + r^* a^2]}{2+s+s^*}\Big\} \vdots \,, \tag{18}$$

where we have used the formula

$$\int \frac{d^2 z}{\pi} \exp[\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}] = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\Big[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\Big] \,, \tag{19}$$

whose convergent condition is either

$$\mathrm{Re}(\zeta + f + g) < 0, \qquad \mathrm{Re}\Big(\frac{\zeta^2 - 4fg}{\zeta + f + g}\Big) < 0 \,,$$

or

$$\mathrm{Re}(\zeta - f - g) < 0, \qquad \mathrm{Re}\Big(\frac{\zeta^2 - 4fg}{\zeta - f - g}\Big) < 0 \,. \tag{20}$$

When making the identification

$$s = \frac{1}{2}[A + D - i(B - C)], \qquad r = -\frac{1}{2}[A - D + i(B + C)] \,, \tag{21}$$

where A, B, C, D are real numbers obeying $AD - BC = 1$, we can put Eq. (18) into its canonical operator (Q, P) representation

$$U(r,s) = \frac{2}{\sqrt{2+A+D}} \vdots \exp\Big\{\frac{2i(CQ^2 - BP^2 + (D-A)PQ)}{2+A+D}\Big\} \vdots \equiv F(A,B,C) \,, \tag{22}$$

where

$$Q = \frac{a + a^\dagger}{\sqrt{2}}, \qquad P = \frac{a - a^\dagger}{\sqrt{2}i} \,. \tag{23}$$

According to the rule that classical correspondence of a Weyl ordered operator $\vdots h(P,Q) \vdots$ can be directly obtained by simply replacing $Q \to q$, $P \to p$, we can get $F(A,B,C)'$s classical correspondence

$$f(p,q) = \frac{2}{\sqrt{2+A+D}} \exp\Big\{\frac{2i(Cq^2 - Bp^2 + (D-A)qp)}{2+A+D}\Big\} \,. \tag{24}$$

In Ref. [18] we have further proved that two Fresnel operators' product is still a Fresnel operator

$$F(A,B,C)F(A',B',C') = F(A'',B'',C'') \,, \tag{25}$$

with the mapping of the matrix multiplication

$$\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \,. \tag{26}$$

From Eqs. (12) and (25) it follows that

$$\frac{2}{\sqrt{2+A+D}} \vdots \exp\Big\{\frac{2i(CQ^2 - BP^2 + (D-A)PQ)}{2+A+D}\Big\} \vdots \frac{2}{\sqrt{2+A'+D'}} \vdots \exp\Big\{\frac{2i(C'Q^2 - B'P^2 + (D'-A')PQ)}{2+A'+D'}\Big\} \vdots$$
$$= \frac{2}{\sqrt{2+A''+D''}} \vdots \exp\Big\{\frac{2i(C''Q^2 - B''P^2 + (D''-A'')PQ)}{2+A''+D''}\Big\} \vdots \,. \tag{27}$$

Equation (27) shows that two Weyl-ordered Fresnel operators product can be put into another Weyl-ordered Fresnel operator, which is characterized by the matrix $[A'', B'', C'', D'']$ related to $[A, B, C, D]$ and $[A', B', C', D']$ by Eq. (26).

As an application of Fresnel operator's Weyl ordering, we consider the matrix element, i.e.,

$$\langle q'|F(A,B,C)|q\rangle = \frac{2}{\sqrt{2+A+D}} \langle q'| \vdots \exp\Big\{\frac{2i(CQ^2 - BP^2 + (D-A)PQ)}{2+A+D}\Big\} \vdots |q\rangle \,. \tag{28}$$

Noting the Weyl transformation rule defined by[14]

$$\langle q'|H(P,Q)|q\rangle = \int^{\infty} \frac{dp}{2\pi} e^{ip(q'-q)} h\Big(p, \frac{q+q'}{2}\Big) \,, \tag{29}$$

where $h(p,q)$ is the Weyl classical correpondence of $H(P,Q)$, we can put Eq. (28) into integral form and obtain

$$\langle q'|F(A,B,C)|q\rangle = \frac{2}{\sqrt{2+A+D}} \int \frac{dp}{2\pi} e^{ip(q'-q)} \exp\Big\{\frac{-2iBp^2 + i(D-A)(q+q')p + 2iC((q+q')/2)^2}{2+A+D}\Big\}$$

$$= \frac{1}{\sqrt{i2\pi B}} \exp\Big\{\frac{i}{2B}(Aq^2 + Dq'^2 - 2qq')\Big\}, \tag{30}$$

which is just the integral kernel of the generalized Fresnel transformation kernel in classical optics.[19]

4 Decompositions of Fresnel Operator's Weyl-Ordering

On the other hand, Ref. [20] has proved that $F(A,B,C)$ in its canonical operator (Q,P) representation can be expressed as

$$F(A,B,C) = \exp\Big(\frac{iC}{2A}Q^2\Big)\exp\Big(-\frac{i}{2}(QP+PQ)\ln A\Big)\exp\Big(-\frac{iB}{2A}P^2\Big), \quad A \neq 0\,. \tag{31}$$

When decomposing the matrix as[21]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C/A & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 1 & B/A \\ 0 & 1 \end{pmatrix}, \tag{32}$$

we have

$$F(A,B,C) = F(1,0,C/A)F(A,0,0)F(1,B/A,0)\,, \tag{33}$$

where

$$F(1,0,C/A) = \exp\Big\{\frac{iC}{2A}Q^2\Big\} = \vdots\exp\Big\{\frac{iC}{2A}Q^2\Big\}\vdots, \quad F(1,B/A,0) = \exp\Big(-\frac{iB}{2A}P^2\Big) = \vdots\exp\Big(-\frac{iB}{2A}P^2\Big)\vdots,$$
$$F(A,0,0) = \exp\Big[-\frac{i}{2}(QP+PQ)\ln A\Big]. \tag{34}$$

Noting the corresponding relation

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{35}$$

then substituting Eq. (35) into Eq. (12), we have

$$\exp\Big[-\frac{i}{2}(QP+PQ)\ln A\Big] = \frac{2\sqrt{A}}{|A+1|}\vdots\exp\Big\{-2i\frac{A-1}{A+1}QP\Big\}\vdots, \tag{36}$$

which leads to an identity

$$\exp[-i\lambda(QP+PQ)] = \operatorname{sech}\lambda\,\vdots\exp\{-2iQP\tanh\lambda\}\vdots\,. \tag{37}$$

From Eqs. (12) and (33) it follows that (according to the group multiplication rule)

$$\vdots\exp\Big\{\frac{iC}{2A}Q^2\Big\}\vdots\frac{2\sqrt{A}}{|A+1|}\vdots\exp\Big\{-2i\frac{A-1}{A+1}QP\Big\}\vdots\,\vdots\exp\Big\{-\frac{iB}{2A}P^2\Big\}\vdots$$
$$= \frac{2}{\sqrt{2+A+D}}\vdots\exp\Big\{\frac{2i(CQ^2 - BP^2 + (D-A)PQ)}{2+A+D}\Big\}\vdots, \tag{38}$$

In addition, when $A=0$, the decomposition Eq. (31) is not available, instead, from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}, \tag{39}$$

we have

$$F^{-1}(A,B,C) = \exp\Big(-\frac{iC}{2D}Q^2\Big)\exp\Big(-\frac{i}{2}(QP+PQ)\ln D\Big)\exp\Big(\frac{iB}{2D}P^2\Big), \tag{40}$$

it then follows

$$F(A,B,C) = \exp\Big(-\frac{iB}{2D}P^2\Big)\exp\Big(\frac{i}{2}(QP+PQ)\ln D\Big)\exp\Big(\frac{iC}{2D}Q^2\Big), \quad D \neq 0\,. \tag{41}$$

Using the formula in Eq. (37) we have

$$\exp\Big(\frac{i}{2}(QP+PQ)\ln D\Big) = \frac{2|D|}{\sqrt{D}|1+D|}\vdots\exp\Big\{2i\frac{D-1}{D+1}QP\Big\}\vdots\,. \tag{42}$$

Thus we can also derive another formula

$$\frac{2\sqrt{D}}{|1+D|}\vdots\exp\Big(-\frac{iB}{2D}P^2\Big)\vdots\,\vdots\exp\Big\{2i\frac{D-1}{D+1}QP\Big\}\vdots\,\vdots\exp\Big(\frac{iC}{2D}Q^2\Big)\vdots$$
$$= \frac{2}{\sqrt{2+A+D}}\vdots\exp\Big\{\frac{2i(CQ^2 - BP^2 + (D-A)PQ)}{2+A+D}\Big\}\vdots, \quad D \neq 0\,. \tag{43}$$

Besides, when we notice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/B & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1/B \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A/B & 1 \end{pmatrix}, \tag{44}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & A/C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/C & 0 \\ 0 & -C \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & D/C \\ 0 & 1 \end{pmatrix}, \tag{45}$$

we have other Weyl-ordering decompositions for $F(A,B,C)$,

$$F(A,B,C) = \exp\Bigl(\frac{iD}{2B}Q^2\Bigr) \exp\Bigl(-\frac{i}{2}(QP+PQ)\ln B\Bigr) \exp\Bigl[-i\frac{\pi}{4}(Q^2+P^2)\Bigr] \exp\Bigl(\frac{iA}{2B}Q^2\Bigr), \quad B \neq 0\,, \tag{46}$$

and

$$F(A,B,C) = \exp\Bigl(-\frac{iA}{2C}P^2\Bigr) \exp\Bigl[-\frac{i}{2}(QP+PQ)\ln\Bigl(-\frac{1}{C}\Bigr)\Bigr] \exp\Bigl[-i\frac{\pi}{4}(Q^2+P^2)\Bigr] \exp\Bigl(-\frac{iD}{2C}P^2\Bigr), \quad C \neq 0\,. \tag{47}$$

According to Eq. (12), we have

$$\exp\Bigl[-i\frac{\pi}{4}(Q^2+P^2)\Bigr] = \sqrt{2} \,\vdots\, \exp[-i(Q^2+P^2)] \,\vdots\,, \quad \exp\Bigl[-\frac{i}{2}(QP+PQ)\ln B\Bigr] = \frac{2\sqrt{B}}{|B+1|} \,\vdots\, \exp\Bigl\{-2i\frac{B-1}{B+1}QP\Bigr\} \,\vdots\,,$$

$$\exp\Bigl[-\frac{i}{2}(QP+PQ)\ln\Bigl(-\frac{1}{C}\Bigr)\Bigr] = \frac{2\sqrt{-C}}{|C-1|} \,\vdots\, \exp\Bigl\{-2i\frac{1+C}{1-C}QP\Bigr\} \,\vdots\,, \tag{48}$$

then

$$F(A,B,C) = \frac{2\sqrt{2B}}{|B+1|} \exp\Bigl(\frac{iD}{2B}Q^2\Bigr) \,\vdots\, \exp\Bigl\{-2i\frac{B-1}{B+1}QP\Bigr\} \,\vdots\,\vdots\, \exp[-i(Q^2+P^2)] \,\vdots\, \exp\Bigl(\frac{iA}{2B}Q^2\Bigr), \quad B \neq 0\,, \tag{49}$$

and

$$F(A,B,C) = \frac{2\sqrt{-2C}}{|C-1|} \exp\Bigl(-\frac{iA}{2C}P^2\Bigr) \,\vdots\, \exp\Bigl\{-2i\frac{1+C}{1-C}QP\Bigr\} \,\vdots\,\vdots\, \exp[-i(Q^2+P^2)] \,\vdots\, \exp\Bigl(-\frac{iD}{2C}P^2\Bigr), \quad C \neq 0\,. \tag{50}$$

In summary, we have derived the Weyl ordering operator formula by the new method, and have derive the Fresnel operators' Weyl ordering, which together with the Weyl transformation can immediately lead to Fresnel transformation kernel in classical optics.

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