

## New Exact Solutions of Two Nonlinear Physical Models

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**Abstract** Abundant new exact solutions of the Schamel–Korteweg-de Vries (S-KdV) equation and modified Zakharov–Kuznetsov equation arising in plasma and dust plasma are presented by using the extended mapping method and the availability of symbolic computation. These solutions include the Jacobi elliptic function solutions, hyperbolic function solutions, rational solutions, and periodic wave solutions. In the limiting cases, the solitary wave solutions are obtained and some known solutions are also recovered.

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### 1 Introduction

Nonlinear partial differential equations (NLPDEs) are widely used as models to describe many important dynamical systems in various fields of sciences, particularly in fluid mechanics, solid state physics, plasma physics, and nonlinear optics. Exact solutions of NLPDEs of mathematical physics have attracted significant interest in the literature. The knowledge of these solutions of NLPDEs facilitates the verification of numerical solvers and aids in the stability analysis of the solutions. Traveling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. Over the last years, much work has been done on the construction of exact solitary wave solutions and periodic wave solutions of nonlinear physical equations. The investigation of new exact solutions of NLPDEs may help one to find new phenomena. Many methods have been developed by mathematicians and physicists to find special solutions of NLPDEs, such as the inverse scattering method,<sup>[1]</sup> the tanh-function method,<sup>[2]</sup> the extended tanh-function method,<sup>[3]</sup> Exp-function method,<sup>[4]</sup> sine-cosine method,<sup>[5]</sup> and the homogeneous balance method.<sup>[6]</sup> Recently, some methods were presented to construct exact solutions expressed in terms of Jacobi elliptic functions (JEFs) for nonlinear evolution equations (NLEEs). Among them the Jacobi elliptic function expansion method,<sup>[7–8]</sup> the  $F$ -expansion method,<sup>[9–10]</sup> the generalized Jacobi elliptic function method,<sup>[11]</sup> mapping method,<sup>[12]</sup> extended mapping method<sup>[13–14]</sup> and other methods.<sup>[15–18]</sup> By employing a computer package such as *Maple* or *Mathematica*, the large amount of tedious working required to verify candidate traveling wave solutions can be avoided. Hence, these methods are now much more attractive and efficient for obtaining the exact solutions of some NLEEs.

Some of the nonlinear models in fluids, plasma, and dust plasma are described by canonical models and include the Korteweg-de Vries (KdV), the modified KdV, Zakharov–Kuznetsov (ZK), and modified ZK equations.<sup>[19–25]</sup> Employing the reductive perturbation technique on the system of equations for hydrodynamics and the dynamics of plasma waves to derive the KdV equation, the modified KdV, the ZK equation, and modified ZK equation. Schamel equation<sup>[23]</sup> containing a square root nonlinearity is a very attractive model for the study of ion-acoustic waves in plasmas and dusty plasmas and a simple solitary wave solution having a  $\text{sech}^4$  profile was obtained. Hence, seeking new exact solutions of the Schamel–Korteweg-de Vries (S-KdV) equation and the modified ZK equation is important.

For a given NLPDE, say, in two independent variables

$$G(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \quad (1)$$

In general, the left hand side of Eq. (1) is a polynomial in  $u$  and its various derivatives. We seek the traveling wave solution of (1) in the form  $u(x, t) = u(\xi)$ ;  $\xi = k(x - \omega t)$ , where  $k$  and  $\omega$  are constants to be determined later. Then, Eq. (1) is transformed to the ordinary differential equation (ODE)

$$H(u, u', u'', \dots) = 0, \quad (2)$$

where  $u' = du/d\xi$  and  $H$  is a polynomial of  $u$  and its various derivatives. If  $H$  is not a polynomial of  $u$  and its various derivatives, then we may use new variables  $v = v(\xi)$  which makes  $H$  become a polynomial of  $v$  and its various derivatives.

Let us give a brief description of the extended mapping method.<sup>[13–14]</sup> We assume that the solutions of Eq. (2) can

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be expressed in the form

$$\begin{aligned}
 u(x, t) = u(\xi) = & a_0 + \sum_{i=1}^N (a_i \phi^i(\xi) + b_i \phi^{-i}(\xi)) \\
 & + \sum_{i=2}^N B_i \phi^{i-2}(\xi) \phi'(\xi) \\
 & + \sum_{i=-1}^{-N} c_i \phi^i(\xi) \phi'(\xi), \quad (3)
 \end{aligned}$$

where  $N$  in Eq. (3) is a positive integer that can be determined by balancing the nonlinear term (s) with the highest derivative term in (2) and  $a_0, a_i, b_i, B_i,$  and  $c_i$  are constants to be determined. The function  $\phi(\xi)$  satisfies the nonlinear ODE

$$(\phi'(\xi))^2 = q_0 + q_2 \phi^2(\xi) + q_4 \phi^4(\xi), \quad (4)$$

where  $q_0, q_2,$  and  $q_4$  are constants.

Substituting (3) with (4) into the ODE (2) and setting the coefficients of the different powers of  $\phi^i$  to zero yields a set of algebraic equations for  $a_0, a_i, b_i, B_i, c_i, k,$  and  $\omega$ . Solving the algebraic equations by use of *Maple* or *Mathematica*, we have  $a_0, a_i, b_i, B_i, c_i, k,$  and  $\omega$  expressed by  $q_0, q_2, q_4$ . Substituting the obtained coefficients into (3), then concentration formulas of traveling wave solutions of the NLPDE (1) can be obtained. Selecting the values of  $q_0, q_2, q_4$  and the corresponding JEFs  $\phi(\xi)$  from Appendix and substituting them into the concentration formulas of solutions to obtain the explicit and exact JEF solutions of Eq. (1). Various solutions of Eq. (4) were constructed using JEFs (see Appendix), and these results were exploited in the design of a procedure for generating solutions of NLPDEs.

The JEFs  $\text{sn}\xi = \text{sn}(\xi, m)$ ,  $\text{cn}\xi = \text{cn}(\xi, m)$ , and  $\text{dn}\xi = \text{dn}(\xi, m)$ , where  $m$  ( $0 < m < 1$ ) is the modulus of the elliptic function, are double periodic and possess the following properties:

$$\begin{aligned}
 \text{sn}^2\xi + \text{cn}^2\xi &= 1, \quad \text{dn}^2\xi + m^2\text{sn}^2\xi = 1, \\
 \frac{d}{d\xi}(\text{sn}\xi) &= \text{cn}\xi \text{dn}\xi, \quad \frac{d}{d\xi}(\text{cn}\xi) = -\text{sn}\xi \text{dn}\xi, \\
 \frac{d}{d\xi}(\text{dn}\xi) &= -m^2 \text{sn}\xi \text{cn}\xi.
 \end{aligned}$$

In addition when  $m \rightarrow 1$ , the functions  $\text{sn}\xi, \text{cn}\xi,$  and  $\text{dn}\xi$  degenerate as  $\tanh\xi, \text{sech}\xi,$  and  $\text{sech}\xi$ , respectively, while when  $m \rightarrow 0$ ,  $\text{sn}\xi, \text{cn}\xi,$  and  $\text{dn}\xi$  degenerate as  $\sin\xi, \cos\xi, 1,$  respectively. So, we can obtain hyperbolic function solutions and trigonometric function solutions in the limit cases when  $m \rightarrow 1$  and  $m \rightarrow 0$ . Some more properties of JEFs can be found in [26].

The rest of this paper is organized as follows: In Sec. 2, we apply the extended mapping method to S-KdV equation and derive the concentration formulas of solutions.

Therefore, we construct abundant new traveling wave solutions of S-KdV equation expressed by various JEFs. In Sec. 3, we find abundant exact solutions of modified ZK equation via this method. Solutions in the limiting cases are also obtained. A conclusion is given in Sec. 4.

## 2 JEF Solutions of S-KdV Equation

In this section, we consider the S-KdV equation<sup>[19–20]</sup>

$$u_t + (\alpha u^{1/2} + \beta u)u_x + \delta u_{xxx} = 0, \quad \delta\beta \neq 0, \quad (5)$$

where  $\alpha, \beta,$  and  $\delta$  are constants. We refer to this equation as the Schamel–Korteweg–de Vries (S-KdV) equation. This equation incorporates the KdV equation ( $\alpha = 0$ ) and the Schamel equation ( $\beta = 0$ ). Due to the wide range of applications of equation (5) and the KdV equation, many solutions to them have been obtained, in terms of  $\tanh\xi, \text{sech}\xi, \text{sn}\xi,$  and so on, by using different methods.<sup>[7,19,23,27]</sup> Beside these solutions, there are several interesting types of solutions expressed by various JEFs and hyperbolic functions. Khater *et al.*<sup>[27]</sup> have used the  $F$ -expansion method and obtained the periodic wave solutions of Eq. (5). More recently, Khater *et al.*<sup>[28]</sup> have obtained elliptic functions solutions of Schamel equation. Here, we obtain several classes of exact solutions of S-KdV equation expressed by various JEFs and hyperbolic functions by using the extended mapping method and the availability of symbolic computation.

In order to obtain the exact solutions of Eq. (5), we use the transformations  $u(x, t) = v^2(x, t)$ ,  $v(x, t) = V(\xi)$ ;  $\xi = k(x - \omega t)$ , to reduce Eq. (5) to the ODE

$$-\omega V V' + (\alpha V^2 + \beta V^3)V' + \delta k^2 [V V''' + 3V' V''] = 0. \quad (6)$$

The balancing procedure implies that  $N = 1$ . Therefore, we assume Eq. (5) has a solution in the form

$$V(x, t) = V(\xi) = a_0 + a_1 \phi(\xi) + \frac{b_1}{\phi(\xi)} + c_1 \frac{\phi'(\xi)}{\phi(\xi)}, \quad (7)$$

where  $a_0, a_1, b_1,$  and  $c_1$  are constants to be determined and  $\phi(\xi)$  is a solution of Eq. (4). Substituting (7) into (6) and setting each coefficient to zero, we get a system of nonlinear equations for  $a_0, a_1, b_1, c_1, k,$  and  $\omega$ . Solving this system by use of *Maple*, we get

### Case 1

$$\begin{aligned}
 a_0 &= -\frac{2\alpha}{5\beta}, \quad a_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_4}{q_2}}, \quad b_1 = 0, \\
 c_1 &= 0, \quad k = \pm \frac{2\alpha}{5\sqrt{6\delta\beta q_2}}, \quad \omega = -\frac{16\alpha^2}{75\beta}. \quad (8)
 \end{aligned}$$

### Case 2

$$\begin{aligned}
 a_0 &= -\frac{2\alpha}{5\beta}, \quad a_1 = 0, \quad b_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_0}{q_2}}, \\
 c_1 &= 0, \quad k = \pm \frac{2\alpha}{5\sqrt{6\delta\beta q_2}}, \quad \omega = -\frac{16\alpha^2}{75\beta}. \quad (9)
 \end{aligned}$$

**Case 3**

$$a_0 = -\frac{2\alpha}{5\beta}, \quad a_1 = 0, \quad b_1 = 0, \quad c_1 = \pm \frac{2\alpha}{5\beta\sqrt{q_2}}, \quad k = \pm \frac{\alpha}{5\sqrt{-3\delta\beta q_2}}, \quad \omega = -\frac{16\alpha^2}{75\beta}. \quad (10)$$

**Case 4**

$$a_0 = -\frac{2\alpha}{5\beta}, \quad a_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_4}{q_2 - 6\sqrt{q_0}\sqrt{q_4}}}, \quad b_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_0}{q_2 - 6\sqrt{q_0}\sqrt{q_4}}},$$

$$c_1 = 0, \quad k = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad \omega = -\frac{16\alpha^2}{75\beta}, \quad (11)$$

$$a_0 = -\frac{2\alpha}{5\beta}, \quad a_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_4}{q_2 + 6\sqrt{q_0}\sqrt{q_4}}}, \quad b_1 = \mp \frac{2\alpha}{5\beta} \sqrt{\frac{-2q_0}{q_2 + 6\sqrt{q_0}\sqrt{q_4}}},$$

$$c_1 = 0, \quad k = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad \omega = -\frac{16\alpha^2}{75\beta}. \quad (12)$$

**Case 5**

$$a_0 = -\frac{2\alpha}{5\beta}, \quad a_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{q_4}{q_2 + 6\sqrt{q_0}\sqrt{q_4}}}, \quad b_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{q_0}{q_2 + 6\sqrt{q_0}\sqrt{q_4}}},$$

$$c_1 = \mp \frac{2\alpha}{5\beta\sqrt{q_2 + 6\sqrt{q_0}\sqrt{q_4}}}, \quad k = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad \omega = -\frac{16\alpha^2}{75\beta},$$

$$a_0 = -\frac{2\alpha}{5\beta}, \quad a_1 = \pm \frac{2\alpha}{5\beta} \sqrt{\frac{q_4}{q_2 - 6\sqrt{q_0}\sqrt{q_4}}}, \quad b_1 = \mp \frac{2\alpha}{5\beta} \sqrt{\frac{q_0}{q_2 - 6\sqrt{q_0}\sqrt{q_4}}}, \quad (13)$$

$$c_1 = \mp \frac{2\alpha}{5\beta\sqrt{q_2 - 6\sqrt{q_0}\sqrt{q_4}}}, \quad k = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad \omega = -\frac{16\alpha^2}{75\beta}. \quad (14)$$

Substituting (8)–(10) into (7), we obtain the following concentration formulas of traveling wave solutions of Eq. (5):

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{-2q_4}{q_2}} \phi(\xi) \right]^2, \quad u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{-2q_0}{q_2}} \frac{1}{\phi(\xi)} \right]^2, \quad (15)$$

with  $\xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta q_2}} \left( x + \frac{16\alpha^2 t}{75\beta} \right)$  and

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{q_2}} \frac{\phi'(\xi)}{\phi(\xi)} \right]^2, \quad \xi = \pm \frac{\alpha}{5\sqrt{-3\delta\beta q_2}} \left( x + \frac{16\alpha^2 t}{75\beta} \right). \quad (16)$$

Moreover, substituting (11)–(14) into (7), we obtain the concentration formulas of solutions to Eq. (5) as follows:

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{-2}{q_2 - 6\sqrt{q_0}\sqrt{q_4}}} \left( \sqrt{q_4} \phi(\xi) + \sqrt{q_0} \frac{1}{\phi(\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(q_2 - 6\sqrt{q_0}\sqrt{q_4})}} \left( x + \frac{16\alpha^2 t}{75\beta} \right), \quad (17)$$

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{-2}{q_2 + 6\sqrt{q_0}\sqrt{q_4}}} \left( \sqrt{q_4} \phi(\xi) - \sqrt{q_0} \frac{1}{\phi(\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(q_2 + 6\sqrt{q_0}\sqrt{q_4})}} \left( x + \frac{16\alpha^2 t}{75\beta} \right), \quad (18)$$

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{q_2 + 6\sqrt{q_0}\sqrt{q_4}}} \left( \sqrt{q_4} \phi(\xi) + \frac{\sqrt{q_0}}{\phi(\xi)} - \frac{\phi'(\xi)}{\phi(\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(q_2 + 6\sqrt{q_0}\sqrt{q_4})}} \left( x + \frac{16\alpha^2 t}{75\beta} \right), \quad (19)$$

$$u = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{q_2 - 6\sqrt{q_0}\sqrt{q_4}}} \left( \sqrt{q_4} \phi(\xi) - \frac{\sqrt{q_0}}{\phi(\xi)} - \frac{\phi'(\xi)}{\phi(\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(q_2 - 6\sqrt{q_0}\sqrt{q_4})}} \left( x + \frac{16\alpha^2 t}{75\beta} \right). \quad (20)$$

With the aid of Appendix and formulas (15), one can get the periodic wave solutions of equation (5)

$$u_1 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{m^2 + 1}} m \operatorname{sn}\xi \right]^2, \quad u_2 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{m^2 + 1}} \operatorname{ns}\xi \right]^2, \quad (21)$$

$$u_3 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{m^2 + 1}} m \operatorname{cd}\xi \right]^2, \quad u_4 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{m^2 + 1}} \operatorname{dc}\xi \right]^2, \quad (22)$$

with  $\xi = \pm \frac{2\alpha}{5\sqrt{-6\delta\beta(m^2 + 1)}} \left( x + \frac{16\alpha^2 t}{75\beta} \right)$ ,  $\beta\delta < 0$  and

$$u_5 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{2 - m^2}} \operatorname{dn}\xi \right]^2, \quad u_6 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2(1 - m^2)}{2 - m^2}} \operatorname{nd}\xi \right]^2, \quad (23)$$

with  $\xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(2-m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right)$ ,  $\beta\delta > 0$ . We also find some exact solutions of Eq. (5) expressed by rational expressions of JEFs

$$u_7 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{m^2}{\sqrt{2-m^2}} \left( \frac{\operatorname{sn}\xi}{1 \pm \operatorname{dn}\xi} \right) \right]^2, \quad u_8 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{2-m^2}} \frac{1 \pm \operatorname{dn}\xi}{\operatorname{sn}\xi} \right]^2, \\ \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(m^2-2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta < 0, \tag{24}$$

$$u_9 = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{1-m^2}{1+m^2}} \frac{\operatorname{dn}\xi}{1 \pm \operatorname{msn}\xi} \right]^2, \quad u_{10} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{1-m^2}{1+m^2}} \frac{1 \pm \operatorname{msn}\xi}{\operatorname{dn}\xi} \right]^2, \\ \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(1+m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta > 0, \tag{25}$$

$$u_{11} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{1+m^2}} \left( \frac{1-m^2}{m \operatorname{cn}\xi \pm \operatorname{dn}\xi} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(1+m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta > 0, \\ u_{12} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{m}{\sqrt{2-m^2}} \left( \frac{1}{\operatorname{sn}\xi \pm i \operatorname{cn}\xi} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(m^2-2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta < 0, \tag{26}$$

and the combined JEF solutions

$$u_{13} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{1+m^2}} \left( m \operatorname{cn}\xi \pm \operatorname{dn}\xi \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(1+m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta > 0, \\ u_{14} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{m}{\sqrt{2-m^2}} \left( \operatorname{sn}\xi \pm i \operatorname{cn}\xi \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(m^2-2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta < 0. \tag{27}$$

From Appendix and the formulas (17) and (18), we obtain the following combined JEF solutions of Eq. (5):

$$u_{15} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{1+m^2+6m}} (m \operatorname{sn}\xi + \operatorname{ns}\xi) \right]^2, \quad u_{16} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{1+m^2+6m}} (m \operatorname{cd}\xi + \operatorname{dc}\xi) \right]^2, \\ \xi = \pm \frac{2\alpha}{5\sqrt{-6\delta\beta(1+m^2+6m)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta < 0, \tag{28}$$

$$u_{17} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{\frac{2}{2-m^2+6\sqrt{1-m^2}}} (\operatorname{dn}\xi + \sqrt{1-m^2} \operatorname{nd}\xi) \right]^2, \\ \xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(2-m^2+6\sqrt{1-m^2})}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \delta\beta > 0. \tag{29}$$

With the aid of Appendix and the formulas (17) and (18), we find the following exact solutions of Eq. (5) expressed by rational expressions of JEFs:

$$u_{18} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{2(m^2+1)}} \left( \frac{m^2 \operatorname{sn}^2 \xi + (1 \pm \operatorname{dn}\xi)^2}{\operatorname{sn}\xi(1 \pm \operatorname{dn}\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-6\delta\beta(m^2+1)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \delta\beta < 0, \tag{30}$$

$$u_{19} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{m}{\sqrt{2(m^2+1)}} \left( \frac{(\operatorname{sn}\xi \pm i \operatorname{cn}\xi)^2 + 1}{\operatorname{sn}\xi \pm i \operatorname{cn}\xi} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-6\delta\beta(m^2+1)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \delta\beta < 0, \tag{31}$$

$$u_{20} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{2(m^2+1)}} \left( \frac{\operatorname{sn}^2 \xi + (1 \pm \operatorname{cn}\xi)^2}{\operatorname{sn}\xi(1 \pm \operatorname{cn}\xi)} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-6\delta\beta(1+m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta < 0, \tag{32}$$

$$u_{21} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{4-2m^2}} \left( \frac{(m \operatorname{cn}\xi \pm \operatorname{dn}\xi)^2 + 1 - m^2}{m \operatorname{cn}\xi \pm \operatorname{dn}\xi} \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{6\delta\beta(2-m^2)}}\left(x + \frac{16\alpha^2}{75\beta}t\right), \quad \beta\delta > 0. \tag{33}$$

By using the formulas (16), (19), and (20), we can obtain new and more general types of exact solutions of (5). With the aid of Appendix and choosing  $\phi(\xi) = \operatorname{sn}\xi$  in formulas (16), (19), and (20), we can obtain the following exact

solutions of Eq. (5) expressed by JEFs:

$$\begin{aligned} u_{22} &= \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{i}{\sqrt{1+m^2}} \frac{(\operatorname{cn}\xi \operatorname{dn}\xi)}{\operatorname{sn}\xi} \right]^2, \quad \xi = \pm \frac{\alpha}{5\sqrt{3\delta\beta(1+m^2)}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad (34) \\ u_{23} &= \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{6m-m^2-1}} \left( m \operatorname{sn}\xi + \operatorname{ns}\xi - \frac{(\operatorname{cn}\xi \operatorname{dn}\xi)}{\operatorname{sn}\xi} \right) \right]^2, \\ \xi &= \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(6m-m^2-1)}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0, \\ u_{24} &= \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{i}{\sqrt{1+6m+m^2}} \left( m \operatorname{sn}\xi - \operatorname{ns}\xi - \frac{(\operatorname{cn}\xi \operatorname{dn}\xi)}{\operatorname{sn}\xi} \right) \right]^2, \\ \xi &= \pm \frac{2\alpha}{5\sqrt{3\delta\beta(1+6m+m^2)}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta > 0. \quad (35) \end{aligned}$$

With the aid of Appendix and choosing  $\phi(\xi) = \operatorname{sn}\xi/(1+\operatorname{dn}\xi)$  in formulas (19) and (20), we can obtain

$$\begin{aligned} u_{25} &= \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\sqrt{2m^2-1}} \left( \frac{m^2}{2} \frac{\operatorname{sn}\xi}{1+\operatorname{dn}\xi} + \frac{1+\operatorname{dn}\xi}{2\operatorname{sn}\xi} - \operatorname{cs}\xi \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta(2m^2-1)}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0, \\ u_{26} &= \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{i}{\sqrt{1+m^2}} \left( \frac{m^2}{2} \frac{\operatorname{sn}\xi}{1+\operatorname{dn}\xi} - \frac{1+\operatorname{dn}\xi}{2\operatorname{sn}\xi} - \operatorname{cs}\xi \right) \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{3\delta\beta(1+m^2)}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta > 0. \quad (36) \end{aligned}$$

Other JEFs are omitted here for simplicity. In the limiting case when  $m \rightarrow 1$ , the solitary wave solutions of Eq. (5) are obtained as follows:

$$u_{27} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \tanh \frac{\alpha}{5\sqrt{-3\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right) \right]^2, \quad u_{28} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \coth \frac{\alpha}{5\sqrt{-3\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right) \right]^2, \quad \beta\delta < 0, \quad (37)$$

$$u_{29} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \sqrt{2} \operatorname{sech} \frac{2\alpha}{5\sqrt{6\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right) \right]^2, \quad \beta\delta > 0, \quad (38)$$

$$u_{30} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{\tanh\xi}{1 \pm \operatorname{sech}\xi} \right]^2, \quad u_{26} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1 \pm \operatorname{sech}\xi}{\tanh\xi} \right]^2, \quad \xi = \frac{2\alpha}{5\sqrt{-3\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0, \quad (39)$$

$$u_{31} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \left( \tanh\xi + i \operatorname{sech}\xi \right) \right]^2, \quad u_{32} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{\tanh\xi + i \operatorname{sech}\xi} \right]^2, \quad \xi = \frac{2\alpha}{5\sqrt{-3\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0, \quad (40)$$

$$u_{33} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{2} (\tanh\xi + \coth\xi) \right]^2, \quad \xi = \frac{\alpha}{5\sqrt{-12\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0,$$

$$u_{34} = \frac{4\alpha^2}{25\beta^2} \left[ 1 \pm \frac{1}{2} \frac{\tanh\xi}{1 + \operatorname{sech}\xi} + \frac{1 + \operatorname{sech}\xi}{2 \tanh\xi} - \operatorname{csch}\xi \right]^2, \quad \xi = \pm \frac{2\alpha}{5\sqrt{-3\delta\beta}} \left( x + \frac{16\alpha^2}{75\beta} t \right), \quad \beta\delta < 0. \quad (41)$$

The periodic wave solutions  $u_1$ – $u_6$  were given in [27] and the solitary wave solutions  $u_{27}$  and  $u_{28}$  were found by many authors.<sup>[19,29]</sup> Compared with the results in [27], we further find some new solutions. As  $m \rightarrow 0$ , these solutions will degenerate into the corresponding trigonometric functions solutions. We do not list these solutions here for simplicity.

### 3 Solutions of Modified ZK Equation

The equation is expressed as

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} = 0, \quad (42)$$

where  $\alpha$  is a constant. The ZK equation<sup>[21]</sup> and modified ZK equation<sup>[22]</sup> are very attractive models for the study in plasma. Exact traveling wave solutions for the ZK equation and modified ZK equation have been studied in [16, 30–32].

In order to obtain the exact solutions of modified ZK equation using the extended mapping method and the

availability of symbolic computation, we use the transformation  $u(x, y, t) = u(\xi)$ ,  $\xi = k(x + ly - \omega t)$ , with constants  $k$ ,  $l$ , and  $\omega$  to be determined. Then, Eq. (42) is transformed to the following ODE:

$$-\omega u' + \alpha u^2 u' + k^2 (1 + l^2) u''' = 0. \quad (43)$$

In Eq. (43), we assume  $u(\xi)$  to be in the form

$$u(\xi) = a_0 + a_1 \phi(\xi) + \frac{b_1}{\phi(\xi)} + c_1 \frac{\phi'(\xi)}{\phi(\xi)}.$$

Substituting this form into (43) and setting each coefficient to zero, we obtain a system of nonlinear equations for  $a_0$ ,  $a_1$ ,  $b_1$ ,  $c_1$ ,  $k$ , and  $\omega$ . Solving this system with the aid of *Maple*, we get the following results:

**Case 1**

$$a_0 = b_1 = c_1 = 0, \quad a_1 = \pm \sqrt{\frac{-6\omega q_4}{\alpha q_2}},$$

$$k = \pm \sqrt{\frac{\omega}{(1+l^2)q_2}}. \quad (44)$$

**Case 2**

$$a_0 = a_1 = c_1 = 0, \quad b_1 = \pm \sqrt{\frac{-6\omega q_0}{\alpha q_2}}, \quad k = \pm \sqrt{\frac{\omega}{(1+l^2)q_2}}. \tag{45}$$

**Case 3**

$$a_0 = a_1 = b_1 = 0, \quad c_1 = \pm \sqrt{\frac{3\omega q_4}{\alpha q_2}}, \quad k = \pm \sqrt{\frac{-\omega}{2(1+l^2)q_2}}. \tag{46}$$

**Case 4**

$$a_0 = c_1 = 0, \quad a_1 = \pm \sqrt{\frac{-6\omega q_4}{\alpha(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad b_1 = \pm \sqrt{\frac{-6\omega q_0}{\alpha(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad k = \pm \sqrt{\frac{\omega}{(1+l^2)(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \tag{47}$$

$$a_0 = c_1 = 0, \quad a_1 = \pm \sqrt{\frac{-6\omega q_4}{\alpha(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad b_1 = \mp \sqrt{\frac{-6\omega q_0}{\alpha(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad k = \pm \sqrt{\frac{\omega}{(1+l^2)(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}. \tag{48}$$

**Case 5**

$$a_0 = 0, \quad a_1 = \pm \sqrt{\frac{3\omega q_4}{\alpha(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad b_1 = \mp \sqrt{\frac{3\omega q_0}{\alpha(q_2 - 6\sqrt{q_0}\sqrt{q_4})}},$$

$$c_1 = \pm \sqrt{\frac{3\omega}{\alpha(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \quad k = \pm \sqrt{\frac{-2\omega}{(1+l^2)(q_2 - 6\sqrt{q_0}\sqrt{q_4})}}, \tag{49}$$

$$a_0 = 0, \quad a_1 = \pm \sqrt{\frac{3\omega q_4}{\alpha(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad b_1 = \pm \sqrt{\frac{3\omega q_0}{\alpha(q_2 + 6\sqrt{q_0}\sqrt{q_4})}},$$

$$c_1 = \pm \sqrt{\frac{3\omega}{\alpha(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}, \quad k = \pm \sqrt{\frac{-2\omega}{(1+l^2)(q_2 + 6\sqrt{q_0}\sqrt{q_4})}}. \tag{50}$$

If  $q_0 = 1, q_2 = -(1+m^2), q_4 = m^2, \phi(\xi) = \text{sn}\xi$ , this yields the exact solutions of Eq. (42) as follows:

$$u_1(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha(1+m^2)}} m \text{sn} \sqrt{\frac{-\omega}{(1+l^2)(1+m^2)}}(x+ly-\omega t),$$

$$u_2(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha(1+m^2)}} \text{ns} \sqrt{\frac{-\omega}{(1+l^2)(1+m^2)}}(x+ly-\omega t),$$

$$u_3(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(1+m^2)}} \left( \frac{\text{cn}\xi \text{dn}\xi}{\text{sn}\xi} \right), \quad \xi = \sqrt{\frac{\omega}{2(1+l^2)(1+m^2)}}(x+ly-\omega t), \tag{51}$$

$$u_4(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha(1+6m+m^2)}} (m \text{sn}\xi + \text{ns}\xi), \quad \xi = \sqrt{\frac{-\omega}{(1+l^2)(1+6m+m^2)}}(x+ly-\omega t), \tag{52}$$

$$u_5(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(1+6m+m^2)}} \left( m \text{sn}\xi - \text{ns}\xi + \frac{\text{cn}\xi \text{dn}\xi}{\text{sn}\xi} \right), \quad \xi = \sqrt{\frac{2\omega}{(1+l^2)(1+6m+m^2)}}(x+ly-\omega t),$$

$$u_6(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(1-6m+m^2)}} \left( m \text{sn}\xi + \text{ns}\xi + \frac{\text{cn}\xi \text{dn}\xi}{\text{sn}\xi} \right), \quad \xi = \sqrt{\frac{2\omega}{(1+l^2)(1-6m+m^2)}}(x+ly-\omega t). \tag{53}$$

If  $q_0 = m^2 - 1, q_2 = 2 - m^2, q_4 = -1, \phi(\xi) = \text{dn}\xi$ , this yields the exact solutions of Eq. (42) as follows:

$$u_7(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha(2-m^2)}} \text{dn} \sqrt{\frac{\omega}{(1+l^2)(2-m^2)}}(x+ly-\omega t),$$

$$u_8(x, y, t) = \pm \sqrt{\frac{6\omega(1-m^2)}{\alpha(2-m^2)}} \text{nd} \sqrt{\frac{\omega}{(1+l^2)(2-m^2)}}(x+ly-\omega t),$$

$$u_9(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha(2-m^2)}} \left( \frac{m^2 \text{sn}\xi \text{cn}\xi}{\text{dn}\xi} \right), \quad \xi = \sqrt{\frac{-\omega}{2(1+l^2)(2-m^2)}}(x+ly-\omega t), \tag{54}$$

$$u_{10}(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha(2-m^2+6\sqrt{1-m^2})}} \left( \operatorname{dn} \xi + \sqrt{1-m^2} \operatorname{nd} \xi \right),$$

$$\xi = \sqrt{\frac{\omega}{(1+l^2)(2-m^2+6\sqrt{1-m^2})}} (x + ly - \omega t), \quad (55)$$

$$u_{11}(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(2-m^2+6\sqrt{1-m^2})}} \left( \operatorname{dn} \xi - \sqrt{1-m^2} \operatorname{nd} \xi + \frac{i m^2 \operatorname{sn} \xi \operatorname{cn} \xi}{\operatorname{dn} \xi} \right),$$

$$\xi = \sqrt{\frac{-2\omega}{(1+l^2)(2-m^2+6\sqrt{1-m^2})}} (x + ly - \omega t),$$

$$u_{12}(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(2-m^2-6\sqrt{1-m^2})}} \left( \operatorname{dn} \xi + \sqrt{1-m^2} \operatorname{nd} \xi + \frac{i m^2 \operatorname{sn} \xi \operatorname{cn} \xi}{\operatorname{dn} \xi} \right),$$

$$\xi = \sqrt{\frac{-2\omega}{(1+l^2)(2-m^2-6\sqrt{1-m^2})}} (x + ly - \omega t). \quad (56)$$

With the aid of Appendix and choosing  $\phi(\xi) = \operatorname{sn} \xi / (1 + \operatorname{dn} \xi)$  in formulas (44)–(50), we can obtain

$$u_{13}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha(2-m^2)}} \left( \frac{m^2 \operatorname{sn} \xi}{1 + \operatorname{dn} \xi} \right), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)(2-m^2)}} (x + ly - \omega t),$$

$$u_{14}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha(2-m^2)}} \left( \frac{1 + \operatorname{dn} \xi}{\operatorname{sn} \xi} \right), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)(2-m^2)}} (x + ly - \omega t),$$

$$u_{15}(x, y, t) = \pm \sqrt{\frac{-6\omega}{\alpha(2-m^2)}} \operatorname{cs} \xi, \quad \xi = \sqrt{\frac{\omega}{(1+l^2)(2-m^2)}} (x + ly - \omega t), \quad (57)$$

$$u_{16}(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha(1+m^2)}} \left( \frac{m^2}{2} \frac{\operatorname{sn} \xi}{1 + \operatorname{dn} \xi} - \frac{1 + \operatorname{dn} \xi}{2 \operatorname{sn} \xi} + \operatorname{cs} \xi \right), \quad \xi = \sqrt{\frac{2\omega}{(1+l^2)(1+m^2)}} (x + ly - \omega t),$$

$$u_{17}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha(2m^2-1)}} \left( \frac{m^2}{2} \frac{\operatorname{sn} \xi}{1 + \operatorname{dn} \xi} + \frac{1 + \operatorname{dn} \xi}{2 \operatorname{sn} \xi} + \operatorname{cs} \xi \right), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)(2m^2-1)}} (x + ly - \omega t). \quad (58)$$

Other JEFs are omitted here for simplicity. When  $m \rightarrow 1$ , the solitary wave solutions of Eq. (42) are obtained as follows:

$$u_{18}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha}} \tanh \sqrt{\frac{-\omega}{2(1+l^2)}} (x + ly - \omega t),$$

$$u_{19}(x, y, t) = \pm \frac{1}{2} \sqrt{\frac{3\omega}{\alpha}} (\tanh \xi + \operatorname{coth} \xi), \quad \xi = \sqrt{\frac{-\omega}{8(1+l^2)}} (x + ly - \omega t), \quad (59)$$

$$u_{20}(x, y, t) = \pm \sqrt{\frac{6\omega}{\alpha}} \operatorname{sech} \sqrt{\frac{\omega}{1+l^2}} (x + ly - \omega t),$$

$$u_{21}(x, y, t) = \pm \sqrt{\frac{-3\omega}{\alpha}} (\operatorname{sech} \xi + i \tanh \xi), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)}} (x + ly - \omega t), \quad (60)$$

$$u_{22}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha}} \left( \frac{\tanh \xi}{1 + \operatorname{sech} \xi} \right), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)}} (x + ly - \omega t),$$

$$u_{23}(x, y, t) = \pm \sqrt{\frac{3\omega}{\alpha}} \left( \frac{1 + \operatorname{sech} \xi}{\tanh \xi} \right), \quad \xi = \sqrt{\frac{-2\omega}{(1+l^2)}} (x + ly - \omega t), \quad (61)$$

$$u_{24}(x, y, t) = \pm \sqrt{\frac{-3\omega}{2\alpha}} \left( \frac{1}{2} \frac{\tanh \xi}{1 + \operatorname{sech} \xi} - \frac{1 + \operatorname{sech} \xi}{2 \tanh \xi} + \operatorname{csch} \xi \right), \quad \xi = \sqrt{\frac{\omega}{(1+l^2)}} (x + ly - \omega t). \quad (62)$$

**Remark 1** Compared with the work in [27], where exact solutions for the S-KdV equation are studied, we find more exact solutions in this paper. The solutions  $u_1(x, y, t)$ ,  $u_2(x, y, t)$ ,  $u_7(x, y, t)$ ,  $u_8(x, y, t)$ ,  $u_{18}(x, y, t)$ ,  $u_{20}(x, y, t)$ , and  $u_{21}(x, y, t)$  coincide with the solutions given in [16] and other solutions of Eq. (42) are new.

**Remark 2** All the solutions obtained in this paper for Eq. (5) and Eq. (42) have been checked by *Maple* software.

### 4 Conclusion

Applying the extended mapping method to the Schamel–Korteweg-de Vries ( S-KdV) equation and modified ZK equation, we obtain a rich variety of new exact traveling wave solutions expressed by JEFs. These solutions include solitary wave solutions, combined solitary wave solutions, JEF solutions, combined JEF solutions, and rational function solutions. The solutions (21)–(23), (37) and (38) are the same as the results obtained in [27] and other solutions, to the best of our knowledge, are new. Our new results ensure that the extended mapping method not only recovers all of the solutions reported in [27] but also discovers many new ones. When the modulus  $m \rightarrow 1$ , new solitary wave solutions are also given. The present exact solutions may describe various new features of waves and may help one to find applications in plasma and dust plasma physics.

### Appendix: The ODE and Jacobi Elliptic Functions

**Table** Relation between values of  $(q_0, q_2, q_4)$  and corresponding  $\phi(\xi)$  in ODE  $(\phi')^2 = q_0 + q_2\phi^2 + q_4\phi^4$ .

$q_0$	$q_2$	$q_4$	$\phi$
1	$-1 - m^2$	$m^2$	$\text{sn}\xi, \text{cd}\xi = \frac{\text{cn}\xi}{\text{dn}\xi}$
$1 - m^2$	$2m^2 - 1$	$-m^2$	$\text{cn}\xi$
$m^2 - 1$	$2 - m^2$	$-1$	$\text{dn}\xi$
$m^2$	$-1 - m^2$	1	$\text{ns}\xi = \frac{1}{\text{sn}\xi}, \text{dc}\xi = \frac{\text{dn}\xi}{\text{cn}\xi}$
$-m^2$	$2m^2 - 1$	$1 - m^2$	$\text{nc}\xi = \frac{1}{\text{cn}\xi}$
$-1$	$2 - m^2$	$m^2 - 1$	$\text{nd}\xi = \frac{1}{\text{dn}\xi}$
1	$2 - m^2$	$1 - m^2$	$\text{sc}\xi = \frac{\text{sn}\xi}{\text{cn}\xi}$
1	$2m^2 - 1$	$-m^2(1 - m^2)$	$\text{sd}\xi = \frac{\text{sn}\xi}{\text{dn}\xi}$
$1 - m^2$	$2 - m^2$	1	$\text{cs}\xi = \frac{\text{cn}\xi}{\text{sn}\xi}$
$-m^2(1 - m^2)$	$2m^2 - 1$	1	$\text{ds}\xi = \frac{\text{dn}\xi}{\text{sn}\xi}$
$\frac{1}{4}$	$\frac{m^2 - 2}{2}$	$\frac{m^4}{4}$	$\frac{\text{sn}\xi}{1 \pm \text{dn}\xi}, \frac{\text{cn}\xi}{\sqrt{1 - m^2} \pm \text{dn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2 - 2}{2}$	$\frac{m^2}{4}$	$\text{sn}\xi \pm i\text{cn}\xi, \frac{\text{dn}\xi}{i\sqrt{1 - m^2}\text{sn}\xi \pm \text{cn}\xi}, \frac{m \text{sn}\xi}{1 \pm \text{dn}\xi}$
$\frac{1}{4}$	$\frac{1 - 2m^2}{2}$	$\frac{1}{4}$	$m \text{sn}\xi + i \text{dn}\xi, \frac{\text{cn}\xi}{\sqrt{1 - m^2}\text{sn}\xi \pm \text{dn}\xi}, \frac{\text{sn}\xi}{1 \pm \text{cn}\xi}$
$\frac{m^2 - 1}{4}$	$\frac{m^2 + 1}{2}$	$\frac{m^2 - 1}{4}$	$\frac{\text{dn}\xi}{1 \pm m\text{sn}\xi}$
$\frac{1 - m^2}{4}$	$\frac{m^2 + 1}{2}$	$\frac{1 - m^2}{4}$	$\frac{\text{cn}\xi}{1 \pm \text{sn}\xi}$
$\frac{-(1 - m^2)^2}{4}$	$\frac{m^2 + 1}{2}$	$\frac{-1}{4}$	$m\text{cn}\xi \pm \text{dn}\xi$
$\frac{1}{4}$	$\frac{m^2 + 1}{2}$	$\frac{(1 - m^2)^2}{4}$	$\frac{\text{sn}\xi}{\text{cn}\xi \pm \text{dn}\xi}$

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