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Classification of Teleparallel Homothetic Vector Fields in Cylindrically Symmetric Static Space-Times in Teleparallel Theory of Gravitation

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Abstract *In this paper we classify cylindrically symmetric static space-times according to their teleparallel homothetic vector fields using direct integration technique. It turns out that the dimensions of the teleparallel homothetic vector fields are 4, 5, 7 or 11, which are the same in numbers as in general relativity. In case of 4, 5 or 7 proper teleparallel homothetic vector fields exist for the special choice to the space-times. In the case of 11 teleparallel homothetic vector fields the space-time becomes Minkowski with all the zero torsion components. Teleparallel homothetic vector fields in this case are exactly the same as in general relativity. It is important to note that this classification also covers the plane symmetric static space-times.*

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1 Introduction

Symmetries play a vital role in the theory of general relativity. Over the past few decades the researchers are studying different kind of symmetries such as Killing, homothetic, and conformal vector fields. We impose these symmetry restrictions on the space-time metric to solve the gravitational field equations, which are highly nonlinear. Laws of conservation of the matter in the space-time can be studied with the help of these symmetry restrictions. These symmetry restrictions not only give us the laws of conservation[1] but provide some geometrical features and physical information about the space-time. For instance, in general relativity self-similarity solutions are extensively used for cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids, and cosmic censorship.[2] Despite the theory of general relativity there are other theories of gravitation where researchers are working to find some reasonably good results for gravitational behaviour. Teleparallel is one such theory where gravitation is attributed only to the torsion of the space-time by considering the space-time curvature as zero and this torsion plays the role of a force.[3] Teleparallel theory of gravity is based on Weitzenböck geometry.[4] A natural question in one's mind arises about the conservation laws in this description of gravitation. Working in this field Sharif and Amir[5] introduced the teleparallel version of the Lie derivative for Killing vector fields and used those equations to find the teleparallel Killing vector fields in Einstein universe. In Ref. [6] teleparallel Killing vector fields for spherically symmetric static space-times were studied. Previously we have also discussed conservation laws in terms of Killing vector fields in Bianchi type I,[7] cylindrically symmetric static,[8] and Kantowski–Sachs and Bianchi type III space-times.[9]

In general relativity homothetic vector fields give one extra conservation law. It will be interesting to explore homothetic vector fields in teleparallel theory of gravity and to see if the results of teleparallel and general relativity coincide. The current study will not only help to understand the geometrical and physical properties of the space-time but to find the effect of torsion on the laws of gravitation. These results may give us interesting information about the compatibility of both the theories. The teleparallel Killing equation is defined as[5]

$$\underset{X}{L}^{T} g_{\mu\nu} = g_{\mu\nu,\rho}X^{\rho} + g_{\rho\nu}X^{\rho}_{,\mu} + g_{\mu\rho}X^{\rho}_{,\nu} + X^{\rho}(g_{\theta\nu}T^{\theta}_{\mu\rho} + g_{\mu\theta}T^{\theta}_{\nu\rho}) = 0\,,$$

where $\underset{X}{L}^{T}$ denotes the teleparallel Lie derivative with respect to the vector field X and $T^{\theta}_{\mu\nu}$ denotes the torsion tensor, which is anti-symmetric with respect to its lower two indices. We shall use this definition in the extended form for homothetic vector fields as:

$$\underset{X}{L}^{T} g_{\mu\nu} = 2\,\alpha\, g_{\mu\nu}, \quad \alpha \in R\,. \tag{1}$$

A vector field X is said to be a teleparallel homothetic vector field if it satisfies Eq. (1). X is called a proper teleparallel homothetic vector field if $\alpha \neq 0$. Otherwise X is called a teleparallel Killing vector field.

2 Main Results

Consider cylindrically symmetric static space-times in usual coordinates (t, r, θ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element

$$ds^2 = -e^{A(r)}\,dt^2 + dr^2 + e^{B(r)}\,d\theta^2 + e^{C(r)}\,dz^2\,, \tag{2}$$

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where A, B, and C are functions of r only. The above space-time admits three teleparallel Killing vector fields which are[8]

$$e^{-A(r)/2}\,\frac{\partial}{\partial t},\quad e^{-B(r)/2}\,\frac{\partial}{\partial \theta},\quad e^{-C(r)/2}\,\frac{\partial}{\partial z}\,. \tag{3}$$

The case when $B(r)=C(r)$, the above space-times (2) become plane symmetric static space-times. The tetrad components S^a_μ and the inverse tetrad components S^μ_a for the space-time (2) can be obtained as:

$$\begin{aligned} S^a_\mu &= \operatorname{diag}\big(e^{A(r)/2},\,1,\,e^{B(r)/2},e^{C(r)/2}\big)\,,\\ S^\mu_a &= \operatorname{diag}\big(e^{-A(r)/2},\,1,\,e^{-B(r)/2},\,e^{-C(r)/2}\big)\,. \end{aligned} \tag{4}$$

The corresponding non-vanishing Weitzenböck connections for (4) are obtained as

$$\Gamma^0_{01}=\frac{A'}{2},\quad \Gamma^2_{21}=\frac{B'}{2},\quad \Gamma^3_{31}=\frac{C'}{2},\quad \Gamma^1_{22}=-e^{B/2}\,, \tag{5}$$

where "prim" denotes the derivative with respect to r. Thus the non-vanishing torsion components are obtained as[8]

$$T^0_{10}=\frac{A'}{2},\quad T^2_{12}=\frac{B'}{2}\,,\quad T^3_{13}=\frac{C'}{2}\,. \tag{6}$$

A vector field X is said to be teleparallel homothetic vector field if it satisfies Eq. (1). Using Eq. (1) explicitly and using (2) and (6) we get the following equations

$$X^0_{,0}=\alpha,\quad X^1_{,1}=\alpha,\quad X^2_{,2}=\alpha,\quad X^3_{,3}=\alpha\,, \tag{7}$$

$$e^{B(r)}\,X^2_{,0}-e^{A(r)}\,X^0_{,2}=0\,, \tag{8}$$

$$e^{C(r)}\,X^3_{,0}-e^{A(r)}\,X^0_{,3}=0\,, \tag{9}$$

$$e^{C(r)}\,X^3_{,2}+e^{B(r)}\,X^2_{,3}=0\,, \tag{10}$$

$$2\,X^1_{,0}-2e^{A(r)}\,X^0_{,1}-A'\,e^{A(r)}\,X^0=0\,, \tag{11}$$

$$2\,e^{B(r)}\,X^2_{,1}+2X^1_{,2}+B'e^{B(r)}X^2=0\,, \tag{12}$$

$$2\,e^{C(r)}\,X^3_{,1}+2X^1_{,3}+C'\,e^{C(r)}\,X^3=0\,. \tag{13}$$

Solving Eq. (7) by simple integration we get

$$\begin{aligned} X^0&=\alpha\,t+P^1(r,\theta,z)\,,\\ X^1&=\alpha\,r+P^2(t,\theta,z)\,,\\ X^2&=\alpha\,\theta+P^3(t,r,z)\,,\\ X^3&=\alpha\,z+P^4(t,r,\theta)\,, \end{aligned} \tag{14}$$

where $P^1(r,\theta,z)$, $P^2(t,\theta,z)$, $P^3(t,r,z)$, and $P^4(t,r,\theta)$ are functions of integration and we need to determine these functions. Solution of the above Eqs. (7)–(13) using (14) involves lengthy and tedious calculations. To avoid all the lengthy details, direct results are given in the following for different possibilities:

Case 1 In this case we have $A\neq B$, $A\neq C$, and $B\neq C$. The line element for (2) becomes

$$\begin{aligned} ds^2=&-\Big(\frac{2}{\alpha}d_4\,r+d_5\Big)\,dt^2+dr^2+\Big(\frac{2}{\alpha}d_6\,r+d_7\Big)\,d\theta^2\\ &+\Big(\frac{2}{\alpha}d_8\,r+d_9\Big)\,dz^2\,, \end{aligned} \tag{15}$$

where $d_4,d_5,d_6,d_7,d_8,d_9\in R$ ($d_4\neq 0$, $d_6\neq 0$, $d_8\neq 0$, $d_4\neq d_6$, $d_4\neq d_8$, $d_6\neq d_8$). Solution of Eqs. (7) to (13) is given by

$$\begin{aligned} X^0&=\alpha\,t+e^{-A(r)/2}d_1\,,\\ X^1&=\alpha\,r+\frac{t^2}{2}d_4+\frac{\theta^2}{2}d_6+\frac{z^2}{2}d_8\,,\\ X^2&=\alpha\,\theta+e^{-B(r)/2}d_2\,,\\ X^3&=\alpha\,z+e^{-A(r)/2}d_3\,, \end{aligned} \tag{16}$$

where $d_1,d_2,d_3\in R$. The above space-time (15) admits four linearly independent teleparallel homothetic vector fields in which three are teleparallel Killing vector fields which are given in (3) and one is proper teleparallel homothetic vector field. Proper teleparallel homothetic vector field after subtracting Killing vector fields from equation (16) is

$$\begin{aligned} &X^0=\alpha\,t,\quad X^1=\alpha\,r+\frac{t^2}{2}d_4+\frac{\theta^2}{2}d_6+\frac{z^2}{2}d_8\,,\\ &X^2=\alpha\,\theta,\quad X^3=\alpha\,z\,. \end{aligned} \tag{17}$$

Case 2 In this case there exist the following three possibilities which are

(a) $A=\text{constant}$, $B=B(r)$, and $C=C(r)$,
(b) $A=A(r)$, $B=\text{constant}$, and $C=C(r)$,
(c) $A=A(r)$, $B=B(r)$, and $C=\text{constant}$.

In (a) the space-time (2) can, after a suitable rescaling of t, be written in the form

$$\begin{aligned} ds^2=&-\,dt^2+dr^2+\Big(\frac{2}{\alpha}d_6\,r+d_7\Big)\,d\theta^2\\ &+\Big(\frac{2}{\alpha}d_8\,r+d_9\Big)\,dz^2\,, \end{aligned} \tag{18}$$

where $d_6,d_7,d_8,d_9\in R$ ($d_6\neq 0$, $d_8\neq 0$, $d_6\neq d_8$). It follows from [8] that the above space-times (18) admit three linear independent teleparallel Killing vector fields which are $\partial/\partial\,t$, $e^{-B(r)/2}\,(\partial/\partial\,\theta)$, and $e^{-C(r)/2}\,(\partial/\partial\,z)$. Teleparallel homothetic vector fields in this case are

$$\begin{aligned} X^0&=\alpha\,t+d_1\,,\\ X^1&=\alpha\,r+\frac{\theta^2}{2}d_6+\frac{z^2}{2}d_8\,,\\ X^2&=\alpha\,\theta+e^{-B(r)/2}d_2\,,\\ X^3&=\alpha\,z+e^{-C(r)/2}d_3\,, \end{aligned} \tag{19}$$

where $d_1,d_2,d_3\in R$. The above space-time (18) admits four linearly independent teleparallel homothetic vector fields in which three are teleparallel Killing vector fields and one is proper teleparallel homothetic vector field. Proper teleparallel homothetic vector field after subtracting Killing vector fields from Eq. (19) is

$$\begin{aligned} &X^0=\alpha\,t,\quad X^1=\alpha\,r+\frac{\theta^2}{2}d_6+\frac{z^2}{2}d_8\,,\\ &X^2=\alpha\,\theta,\quad X^3=\alpha\,z\,. \end{aligned} \tag{20}$$

Cases (b) and (c) are exactly the same.

Case 3 In this case the following three possibilities exist, which are

(d) $A = A(r)$, and $B(r) = C(r)$,
(e) $B = B(r)$, and $A(r) = C(r)$,
(f) $C = C(r)$, and $A(r) = B(r)$.

In (d) the space-time (2) takes the form

$$\begin{aligned} ds^2 = &-\Big(\frac{2}{\alpha}d_4\,r + d_5\Big)dt^2 + dr^2 \\ &+ \Big(\frac{2}{\alpha}d_6\,r\, +\, d_7\Big)(\,d\,\theta^2 +\, dz^2)\,, \end{aligned} \tag{21}$$

where $d_4, d_5, d_6, d_7 \in R$ $(d_4 \neq 0, d_6 \neq 0, d_4 \neq d_6)$. It follows from [8] that the above space-times (21) admit four linearly independent teleparallel Killing vector fields which are $e^{-A(r)/2}(\partial/\partial\, t)$, $e^{-B(r)/2}(\partial/\partial\,\theta)$, $e^{-B(r)/2}(\partial/\partial\, z)$, and $e^{-B(r)/2}\,(\theta\partial/\partial\, z\, -\, z\,\partial/\partial\theta\,)$. Here, teleparallel homothetic vector fields in this case are

$$\begin{aligned} &X^0 = \alpha\, t + e^{-A(r)/2}d_1\,, \\ &X^1 = \alpha\, r + \frac{t^2}{2}d_4 + \frac{1}{2}(\theta^2 + z^2)\, d_6\,, \\ &X^2 = \alpha\,\theta - ze^{-B(r)/2}d_8 + e^{-B(r)/2}d_2\,, \\ &X^3 = \alpha\, z + \theta\, e^{-B(r)/2}d_8 + e^{-B(r)/2}d_3\,, \end{aligned} \tag{22}$$

where $d_1, d_2, d_3, d_8 \in R$. The above space-time (21) admits five linearly independent teleparallel homothetic vector fields in which four are teleparallel Killing vector fields and one is proper homothetic vector field. Proper teleparallel homothetic vector fields after subtracting Killing vector fields from Eq. (22) is

$$\begin{aligned} &X^0 = \alpha\, t, \quad X^1 = \alpha\, r + \frac{t^2}{2}d_4 + \frac{1}{2}(\theta^2 + z^2)\, d_6\,, \\ &X^2 = \alpha\,\theta, \quad X^3 = \alpha\, z\,. \end{aligned} \tag{23}$$

Cases (e) and (f) are exactly the same. Here, the above space-times (21) become special class of plane symmetric static space-times.

Case 4 In this case there exist the following three possibilities, which are

(g) $A =$ constant and $B(r) = C(r)$,
(h) $B =$ constant and $A(r) = C(r)$,
(i) $C =$ constant and $A(r) = B(r)$.

In (g) the space-time (2) can, after a suitable rescaling of t, be written in the form

$$ds^2 = -dt^2 +\, dr^2 + \Big(\frac{2}{\alpha}d_6\, r + d_7\Big)(\,d\,\theta^2 +\, dz^2)\,. \tag{24}$$

where $d_6, d_7 \in R$ $(d_6 \neq 0)$. It follows from [8] that the above space-times (24) admit four linearly independent teleparallel Killing vector fields which are $\partial/\partial\, t$, $e^{-\,B(r)/2}\,(\partial/\partial\,\theta)$, $e^{-\,B(r)/2}\,(\partial/\partial\, z)$, and $e^{-\,B(r)/2}\,(\theta\partial/\partial\, z - z\partial/\partial\theta\,)$. Teleparallel homothetic vector fields in this case are

$$\begin{aligned} &X^0 = \alpha\, t + d_1, \quad X^1 = \alpha\, r + \frac{1}{2}(\theta^2 + z^2)\, d_6\,, \\ &X^2 = \alpha\,\theta\, - ze^{-B(r)/2}d_8 + e^{-B(r)/2}d_2\,, \\ &X^3 = \alpha\, z + \theta\, e^{-B(r)/2}d_8 + e^{-B(r)/2}d_3\,, \end{aligned} \tag{25}$$

where $d_1, d_2, d_3, d_8 \in R$. Here the above space-time (24) admits five linearly independent teleparallel homothetic vector fields in which four are teleparallel Killing vector fields and one is proper homothetic vector field. Proper teleparallel homothetic vector fields after subtracting Killing vector fields from Eq. (25) is

$$\begin{aligned} &X^0 = \alpha\, t, \quad X^1 = \alpha\, r + \frac{1}{2}(\theta^2 + z^2)\, d_6\,, \\ &X^2 = \alpha\,\theta, \quad X^3 = \alpha\, z\,. \end{aligned} \tag{26}$$

Cases (h) and (i) are exactly the same. It is important to note that in this case the above space-times (24) become special class of plane symmetric static space-times.

Case 5 In this case there exist the following three possibilities, which are

(j) $A = A(r)$ and $B(r) = C(r) =$ constant,
(k) $B = B(r)$ and $A(r) = C(r) =$ constant,
(l) $C = C(r)$ and $A(r) = B(r) =$ constant.

In (j) the space-time (2) can, after a suitable rescaling of θ and z, be written in the form

$$ds^2 = -\Big(\frac{2}{\alpha}d_4\, r + d_5\Big)\, dt^2 +\, dr^2 + (\,d\,\theta^2 +\, dz^2)\,, \tag{27}$$

where $d_4, d_5 \in R$ $(d_4 \neq 0)$. It follows from [8] that the above space-times (27) admit four linearly independent teleparallel Killing vector fields which are $e^{-A(r)/2}\,(\partial/\partial\, t)$, $\partial/\partial\,\theta$, $\partial/\partial\, z$, and $(\theta\partial/\partial\, z - z\,\partial/\partial\theta\,)$. Teleparallel homothetic vector fields in this case are

$$\begin{aligned} &X^0 = \alpha\, t + e^{-A(r)/2}d_1, \quad X^1 = \alpha\, r + \frac{t^2}{2}d_4\,, \\ &X^2 = \alpha\,\theta - zd_8 + d_2, \quad X^3 = \alpha\, z + \theta\, d_8 + d_3\,, \end{aligned} \tag{28}$$

where $d_1, d_2, d_3, d_8 \in R$. Here the above space-time (27) admits five linearly independent teleparallel homothetic vector fields in which four are teleparallel Killing vector fields and one is proper homothetic vector field. Proper teleparallel homothetic vector fields after subtracting Killing vector fields from Eq. (28) is

$$\begin{aligned} &X^0 = \alpha\, t, \quad X^1 = \alpha\, r + \frac{t^2}{2}d_4\,, \\ &X^2 = \alpha\,\theta, \quad X^3 = \alpha\, z\,. \end{aligned} \tag{29}$$

Cases (k) and (l) are exactly the same. Here again the above space-times (27) become special class of plane symmetric static space-times.

Case 6 In this case we have $A(r) = B(r) = C(r)$. The line element takes the form

$$ds^2 =\, dr^2 + \Big(\frac{2}{\alpha}d_4\, r\, +\, d_5\Big)(-\, dt^2 + d\,\theta^2 +\, dz^2)\,, \tag{30}$$

where $d_4, d_5 \in R$. It follows from [8] that the above space-times (30) admit six linearly independent teleparallel Killing vector fields which are $e^{-A(r)/2}\,(\partial/\partial\, t)$, $e^{-A(r)/2}(\partial/\partial\,\theta)$, $e^{-A(r)/2}\,(\partial/\partial\, z)$, $e^{-\,A(r)/2}(\theta\partial/\partial z\; -\; z\partial/\partial\theta)$, $e^{-\,A(r)/2}(z\partial/\partial\, t + t\partial/\partial\, z)$, and $e^{-\,A(r)/2}(\theta\partial/\partial\, t +$

$t\partial/\partial\theta$). It is important to note that the above space-time (30) becomes space-like version of the Friedmann–Robertson Walker $k=0$ model. Teleparallel homothetic vector fields in this case are

$$\begin{aligned}
&X^0 = \alpha\, t + z\, e^{-A(r)/2} d_6 + \theta\, e^{-A(r)/2} d_7 + e^{-A(r)/2} d_1\,,\\
&X^1 = \alpha\, r + \frac{1}{2}(t^2+\theta^2+z^2)\, d_4\,,\\
&X^2 = \alpha\,\theta\, - z\, e^{-A(r)/2} d_8 + t\, e^{-A(r)/2}\, d_7 + e^{-A(r)/2} d_2\,,\\
&X^3 = \alpha\, z + \theta\, e^{-A(r)/2} d_8' + t\, e^{-A(r)/2} d_6 + e^{-A(r)/2} d_3\,, \quad (31)
\end{aligned}$$

where $d_1, d_2, d_3, d_6, d_7, d_8 \in R$. Here the above space-time (30) admits seven linearly independent teleparallel homothetic vector fields in which six are teleparallel Killing vector fields and one is proper teleparallel homothetic vector field. Proper teleparallel homothetic vector fields after subtracting Killing vector fields from Eq. (31) is

$$\begin{aligned}
&X^0 = \alpha\, t, \quad X^1 = \alpha\, r + \frac{1}{2}(t^2+\theta^2+z^2)\, d_4\,,\\
&X^2 = \alpha\,\theta, \quad X^3 = \alpha\, z\,. \quad (32)
\end{aligned}$$

Case 7 In this case we have $A = B = C = \lambda$, where $\lambda \in R$. The space-time (2) can, after a suitable rescaling of t, θ, and z, be written in the form

$$ds^2 = -dt^2 + dr^2 + d\theta^2 + dz^2\,. \quad (33)$$

The above space-time (33) is Minkowski and its all torsion components are zero. The teleparallel homothetic vector fields are the same as in general relativity and are given below[10]

$$\begin{aligned}
&X^0 = \alpha\, t + z\, d_4 + \theta\, d_5 + r\, d_7 + d_1,\\
&X^1 = \alpha\, r + t\, d_7 + z\, d_8 + \theta\, d_9 + d_0\,,\\
&X^2 = \alpha\,\theta + t\, d_5\, - z\, d_6 - r\, d_9 + d_2\,,\\
&X^3 = \alpha\, z + t\, d_4 + \theta\, d_6 - r\, d_8 + d_3\,, \quad (34)
\end{aligned}$$

where $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9 \in R$. The above space-time (33) admits eleven teleparallel homothetic vector fields in which ten are teleparallel Killing vector fields. Proper teleparallel homothetic vector fields after subtracting teleparallel Killing vector fields is

$$X^0 = \alpha\, t, \quad X^1 = \alpha\, r, \quad X^2 = \alpha\,\theta, \quad X^3 = \alpha\, z\,. \quad (35)$$

Here, the proper homothetic vector field is the same as in general relativity.

3 Conclusion

In this paper we classified cylindrically symmetric static space-times according to their teleparallel homothetic vector fields. From the above study it is shown that the above space-times admit 4, 5, 7 or 11 teleparallel homothetic vector fields, which are same in number as in general relativity. Only in the case of 11 teleparallel homothetic vector fields the space-time become Minkowski space-time with all the torsion components zero. Hence in this case all these 11 teleparallel homothetic vector fields are same as in general relativity. From the above discussion it is clear that the presence of the torsion in the space-time for cylindrically symmetric static space-times do not increase the number of teleparallel homothetic vector field. Many theories are introduced to explore the gravitational interactions between universal objects. It is still a puzzle that the gravitational interaction requires curvature, torsion or any other connection in the space-time. This paper will help in studying properties of space-times such as anisotropy and red shifts in this alternate description of gravity.

References

[1] A.Z. Petrov, *Phys. Einstein Spaces*, Oxford University Press, Pergamon (1969).

[2] T. Harada and H. Maeda, arxiv: gr-qc/0311014v3.

[3] R. Aldrovandi and J.G. Pereira, *An Introduction to Geometrical Physics*, World Scientific, Singapore (1995).

[4] R. Weitzenböck, *Invarianten Theorie*, Gronningen, Noordhoft (1923).

[5] M. Sharif and M.J. Amir, Mod. Phys. Lett. A **23** (2008) 963.

[6] M. Sharif and Bushra Majeed, Commun. Theor. Phys. **52** (2009) 435.

[7] G. Shabbir and Suhail Khan, Mod. Phys. Lett. A **25** (2010) 55.

[8] G. Shabbir and Suhail Khan, Mod. Phys. Lett. A **25** (2010) 525.

[9] G. Shabbir and Suhail Khan, Commun. Theor. Phys. **54** (2010) 469.

[10] G. Shabbir and M. Ramzan, Applied Sciences **9** (2007) 148.