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# Similarity Solutions for Generalized Variable Coefficients Zakharov–Kuznetsov Equation under Some Integrability Conditions

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***Abstract*** *In this paper, the symmetry method has been carried over to the generalized variable coefficients Zakharov–Kuznetsov equation. The infinitesimal symmetries and the optimal system are deduced and from this optimal system seven basic fields are determined, and for every vector field in the optimal system the admissible forms of the coefficients are found and this also leads us to transform the given equation into partial differential equations in two variables. After using some referenced transformations the mentioned partial differential equations eventually reduce to ordinary differential equations. The search for solutions to those equations has yielded many exact solutions in most cases.*

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## 1 Introduction

The generalized variable coefficients Zakharov–Kuznetsov (GVZK) equation is given by

$$u_t + \alpha(t)uu_x + \beta(t)u^2u_x + \rho(t)u_{xx} + \lambda(t)u_{xxx} + \gamma(t)u_{xyy} = 0\,, \tag{1}$$

where $\alpha(t)$, $\beta(t)$, $\rho(t)$, $\lambda(t)$, and $\gamma(t)$ are arbitrary functions of $t$. This equation contains many important equations for example when $\gamma(t) = 0$, this equation becomes the combined KdV-Burgers equation with variable coefficients[1] and when $\beta(t) = 0$, and $\rho(t) = 0$ Eq. (1) turns to ZK equation with variable coefficients, which describe the nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields[2–7] also when $\rho(t) = 0$, and $\lambda(t) = 1$ Eq. (1) turns to the generalized variable coefficients Zakharov–Kuznetsov equation.[8–9] For the constant coefficients version of Eq. (1) and special cases see Refs. [10–16].

## 2 Symmetry Method

We briefly outlined Steinberg's similarity method of finding explicit solutions of both linear and non-linear partial differential equations.[17] The method based on finding the symmetries of the differential equations is as follows:

Suppose that the differential operator $L$ can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u)\,, \tag{2}$$

where $u = u(t, x)$ and $H$ may depend on $t$, $x$, $u$, and any derivative of $u$ as long the derivative of $u$ does not contain more than $(p - 1)$, $t$ derivatives. Consider the symmetry operator called infintesimal symmetry, which being quasi-linear partial differential operator of first-order, has the form

$$S(u) = A(t,x,u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t,x,u)\frac{\partial u}{\partial x_i} + C(t,x,u)\,. \tag{3}$$

Define the Frèchet derivative of $L(u)$ by

$$F(L,u,v) = \frac{d}{d\varepsilon}L(u+\varepsilon v)|_{\varepsilon=0}\,. \tag{4}$$

With these definitions in the mind we need to follow the following steps: (i) Compute $F(L, u, v)$; (ii) Compute $F(L, u, S(u))$; (iii) Substitute $H(u)$ for $(\partial^p u/\partial t^p)$ in $F(L, u, S(u))$; (iv) Set this expression to zero and perform a polynomial expansion; (v) Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of $S(u)$, Eq. (2) can be used to obtain the functional form of the solutions.

## 3 Fundamental Equation

The generalized variable coefficients Zakharov–Kuznetsov equation can be expressed in the form

$$L(u) = u_t + \alpha(t)uu_x + \beta(t)u^2u_x + \rho(t)u_{xx} + \lambda(t)u_{xxx} + \gamma(t)u_{xyy} = 0\,, \tag{5}$$

where $\alpha(t)$, $\beta(t)$, $\rho(t)$, $\lambda(t)$, and $\gamma(t)$ are arbitrary functions of $t$.

## 4 Determination of Symmetries

In order to find the symmetries of Eq. (5), we set the following symmetry operator

$$S(u) = A(x,y,t,u)u_t + B(x,y,t,u)u_x + C(x,y,t,u)u_y + E(x,y,t,u)\,. \tag{6}$$

Calculating the Fréchet derivative $F(L, u, v)$ of $L(u)$ in the direction of $v$, given by Eq. (2), and replacing $v$ by $S(u)$ in $F$, we get

$$F(L,u,S(u)) = S_t + \alpha(t)[uS_x + u_xS] + \beta(t)[u^2S_x + 2uu_xS] + \rho(t)S_{xx} + \lambda(t)S_{xxx} + \gamma(t)S_{xyy}\,. \tag{7}$$

Substituting the values of different derivatives of $S(u)$ in $F$ with the aid of *Maple* program, we get a polynomial expansion in $u_x$, $u_t$, $u_y$, $u_xu_t, \ldots$, etc. On making use of Eq. (5) in the polynomial expression for $F$, rearranging terms of various powers of derivatives of $u$ and equating them to zero, we get

$$A = A(t)\,,\quad B = B(t,x)\,,\quad C = C(y)\,,$$
$$E_{xu} = E_{uu} = 0\,,\quad C_{yy} + 2E_{yu} = 0\,,$$
$$2\rho B_x - A\rho_t + 3\lambda B_{xx} - A_t\rho = 0\,,$$
$$B_t + \alpha E + \gamma E_{uyy} + \rho B_{xx} + \lambda B_{xxx} + 2\beta u E + \beta u^2 B_x + \alpha u B_x - (A\beta)_t u^2 - (A\alpha)_t u = 0\,,$$
$$3\lambda B_x - (A\lambda)_t = 0\,,\quad 2\gamma C_y - (A\gamma)_t + \gamma B_x = 0\,,$$
$$E_t + \alpha u E_x + \beta u^2 E_x + \rho E_{xx} + \lambda E_{xxx} + \gamma E_{xyy} = 0\,. \quad (8)$$

On solving system (8), we see that the infinitesimals $A$, $B$, $C$, and $E$ satisfying the above equations are:

$$A = \frac{1}{\Gamma'(t)}[(a_1 + a_2)\Gamma(t) + a_4]\,,\quad \frac{d\Gamma(t)}{dt} = \alpha(t)\,,$$
$$B = a_1 x + a_5\,,\quad C = a_3 y + a_6\,,\quad E = a_2 u\,, \quad (9)$$

where $a_i, i = 1, 2, \ldots, 6$ are arbitrary constants. The functions $\alpha(t)$, $\beta(t)$, $\rho(t)$, $\lambda(t)$, and $\gamma(t)$ are governed by the relations:

$$(a_{1+}2a_2)\beta - (A\beta)_t = 0\,,\quad 3a_1\lambda - (A\lambda)_t = 0\,,$$
$$2a_1\rho - (A\rho)_t = 0\,,\quad (a_1 + 2a_3)\gamma - (A\gamma)_t = 0\,. \quad (10)$$

The symmetries under which Eq. (5) is invariant can be spanned by the following six infinitisimal generators

$$V_1 = \frac{\Gamma(t)}{\Gamma'(t)}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\,,\quad V_2 = \frac{\Gamma(t)}{\Gamma'(t)}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}\,,$$
$$V_3 = y\frac{\partial}{\partial y}\,,\quad V_4 = \frac{1}{\Gamma'(t)}\frac{\partial}{\partial t}\,,\quad V_5 = \frac{\partial}{\partial x}\,,\quad V_6 = \frac{\partial}{\partial y}\,. \quad (11)$$

## 5 Classification of Group-Invariant Solutions

In general, to each $s$-parameter subgroup $H$ of the full symmetry group $G$ of a system of differential equations, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solution can be derived. Since elements $g \in G$ not in the subgroup $H$ will transform an $H$-invariant solution to some other group-invariant solutions, only those solutions not so related need to be listed in our optimal system.

Let $G$ be a Lie group. An optimal system of $s$-parameter subgroups is a list of conjugacy inequivalent $s$-parameter subgroups with the proberty that any other subgroup is conjugate to precisely one subgroup in the list. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation (Olver 1986),[18] where the adjoint action is given by the Lie series

$$\mathrm{Ad}\,(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_i,V_j]]\ldots\,, \quad (12)$$

where $[V_i, V_j] = V_iV_j - V_jV_i$ is the commutator for the Lie algebra, and $\varepsilon$ is a parameter. To obtain the optiaml system of the vector fields (11) we should first construct the commutator Table 1 as follows

**Table 1** The commutator table of the vector fields (11).

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ |
|---|---|---|---|---|---|---|
| $V_1$ | 0 | 0 | 0 | $-V_4$ | $-V_5$ | 0 |
| $V_2$ | 0 | 0 | 0 | $-V_4$ | 0 | 0 |
| $V_3$ | 0 | 0 | 0 | 0 | 0 | $-V_6$ |
| $V_4$ | $V_4$ | $V_4$ | 0 | 0 | 0 | 0 |
| $V_5$ | $V_5$ | 0 | 0 | 0 | 0 | 0 |
| $V_6$ | 0 | 0 | $V_6$ | 0 | 0 | 0 |

With the help of the Lie series (12) and the commutator table, the adjoint table for the Lie algebra (11) can be easily constructed as shown in Table 2.

**Table 2** The adjoint table.

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ |
|---|---|---|---|---|---|---|
| $V_1$ | $V_1$ | $V_2$ | $V_3$ | $\exp(\varepsilon)V_4$ | $\exp(\varepsilon)V_5$ | $V_6$ |
| $V_2$ | $V_1$ | $V_2$ | $V_3$ | $\exp(\varepsilon)V_4$ | $V_5$ | $V_6$ |
| $V_3$ | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $\exp(\varepsilon)V_6$ |
| $V_4$ | $V_1-\varepsilon V_4$ | $V_2-\varepsilon V_4$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ |
| $V_5$ | $V_1-\varepsilon V_5$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ |
| $V_6$ | $V_1$ | $V_2$ | $V_3-\varepsilon V_6$ | $V_4$ | $V_5$ | $V_6$ |

To obtain the optiaml system, we now take a general element

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6\,, \quad (13)$$

and subject it to various adjoint transformations to simplify it as much as possible; thus we have deduced the following basic fields which form an optimal system for the generalized variable coefficients Zakharov–Kuznetsov equation (i) $V_1 + aV_2 + bV_3$; (ii) (a) $V_2 + cV_3 + V_5$, (ii) (b) $V_2 + cV_3 - V_5$; (iii) $V_2 + wV_3$; (iv) (a) $V_3 + mV_4 + V_5$, (iv) (b) $V_3 + mV_4 - V_5$; (v) $V_3 + nV_4$; (vi) (a) $V_4 + kV_5 + V_6$, (vi) (b) $V_4 + kV_5 - V_6$; (vii) $V_4 + lV_5$; (viii) (a) $V_5 + V_6$, (viii) (b) $V_5 - V_6$; (ix) $V_5$; (x) $V_6$, where $a$, $b$, $c$, $w$, $m$, $n$, $k$, and $l$ are arbitrary constants. Because the discrete symmetry $(x, y, t, u) \to (-x, y, t, u)$ will map (ii) (b), (iv) (b), (vi) (b), (viii) (b) to (ii) (a), (iv) (a), (vi) (a), (viii) (a) respectively, also the generators (viii) (a), (ix), and (x) give trivial cases since they do not depend on $t$, and therefore, in the optimal system, we confine ourselves to seven generators only.

## 6 Similarity Reductions and Reduced Ordinary Differential Equations

In order to obtain the invariant transformation in each of the above cases we write the characteristic equation in the form

$$\frac{dt}{A(x,y,t,u)} = \frac{dx}{B(x,y,t,u)} = \frac{dy}{C(x,y,t,u)} = \frac{-du}{E(x,y,t,u)}\,. \quad (14)$$

Once this equation is solved for the above seven cases the invariant variables and the corresponding reductions to partial differential equations are obtained and by using some reference transformations those partial differential equations will be reduced to ordinary differential equations under integrability conditions between the variable coefficients of the given problem. Our results are tabulated in the following Tables 3–5.

**Table 3** The invariant variables and the corresponding forms of the coefficient functions.

| Case | The invariant variables | | | Forms of coefficient functions |
|---|---|---|---|---|
| | $\zeta$ | $\eta$ | $u$ | |
| (i) | $x\Gamma^{-1/(1+a)}(t)$ | $y\Gamma^{-b/(1+a)}(t)$ | $f(\zeta,\eta)\Gamma^{-a/(1+a)}(t)$ | $\beta(t)=k_1\Gamma'(t)\Gamma^{a/(1+a)}(t)$, $\rho(t)=k_2\Gamma'(t)\Gamma^{(1-a)/(1+a)}(t)$, $\lambda(t)=k_3\Gamma'(t)\Gamma^{(2-a)/(1+a)}(t)$, $\gamma(t)=k_4\Gamma'(t)\Gamma^{(2b-a)/(1+a)}(t)$. |
| (ii) | $\ln\Gamma(t)-x$ | $y\Gamma^{-c}(t)$ | $f(\zeta,\eta)\Gamma^{-1}(t)$ | $\beta(t)=k_5\Gamma(t)\Gamma'(t)$, $\rho(t)=k_6\Gamma^{-1}(t)\Gamma'(t)$, $\lambda(t)=k_7\Gamma^{-1}(t)\Gamma'(t)$, $\gamma(t)=k_8\Gamma^{2c-1}(t)\Gamma'(t)$. |
| (iii) | $x$ | $y\Gamma^{-w}(t)$ | $f(\zeta,\eta)\Gamma^{-1}(t)$ | $\beta(t)=k_9\Gamma(t)\Gamma'(t)$, $\rho(t)=k_{10}\Gamma^{-1}(t)\Gamma'(t)$, $\lambda(t)=k_{11}\Gamma^{-1}(t)\Gamma'(t)$, $\gamma(t)=k_{12}\Gamma^{2w-1}(t)\Gamma'(t)$. |
| (iv) | $x-(1/m)\Gamma(t)$ | $y\exp(-(1/m)\Gamma(t))$ | $f(\zeta,\eta)$ | $\beta(t)=k_{13}\Gamma'(t)$, $\rho(t)=k_{14}\Gamma'(t)$, $\lambda(t)=k_{15}\Gamma'(t)$, $\gamma(t)=k_{16}\exp((2/m)\Gamma(t))\Gamma'(t)$. |
| (v) | $x$ | $y\exp(-(1/n)\Gamma(t))$ | $f(\zeta,\eta)$ | $\beta(t)=k_{17}\Gamma'(t)$, $\rho(t)=k_{18}\Gamma'(t)$, $\lambda(t)=k_{19}\Gamma'(t)$, $\gamma(t)=k_{20}\exp((2/n)\Gamma(t))\Gamma'(t)$. |
| (vi) | $x-k\Gamma(t)$ | $y-\Gamma(t)$ | $f(\zeta,\eta)$ | $\beta(t)=k_{21}\Gamma'(t)$, $\rho(t)=k_{22}\Gamma'(t)$, $\lambda(t)=k_{23}\Gamma'(t)$, $\gamma(t)=k_{24}\Gamma'(t)$. |
| (vii) | $x-l\Gamma(t)$ | $y$ | $f(\zeta,\eta)$ | $\beta(t)=k_{25}\Gamma'(t)$, $\rho(t)=k_{26}\Gamma'(t)$, $\lambda(t)=k_{27}\Gamma'(t)$, $\gamma(t)=k_{28}\Gamma'(t)$. |

**Table 4** The corresponding reduced partial differential equation.

| |
|---|
| $k_4f_{\zeta\zeta\zeta}+k_2f_{\zeta\eta\eta}-[1/(1+a)]\zeta f_\zeta-[b/(1+a)]\eta f_\eta-[a/(1+a)]f+k_3f_{\zeta\zeta}+ff_\zeta+k_1f^2f_\zeta=0,$ |
| $k_7f_{\zeta\zeta\zeta}+k_8f_{\zeta\eta\eta}+k_5f^2f_\zeta+ff_\zeta-f_\zeta+f-k_6f_{\zeta\zeta}+c\eta f_\eta=0,$ |
| $k_{11}f_{\zeta\zeta\zeta}+k_{12}f_{\zeta\eta\eta}+k_9f^2f_\zeta+ff_\zeta-f+k_{10}f_{\zeta\zeta}-w\eta f_\eta=0,$ |
| $k_{15}f_{\zeta\zeta\zeta}+k_{16}f_{\zeta\eta\eta}+k_{13}f^2f_\zeta+ff_\zeta-(1/m)f_\zeta+k_{14}f_{\zeta\zeta}-(1/m)\eta f_\eta=0,\quad m\neq0$ |
| $k_{19}f_{\zeta\zeta\zeta}+k_{20}f_{\zeta\eta\eta}+k_{17}f^2f_\zeta+ff_\zeta+k_{18}f_{\zeta\zeta}-(1/n)\eta f_\eta=0,\quad n\neq0$ |
| $k_{23}f_{\zeta\zeta\zeta}+k_{24}f_{\zeta\eta\eta}+k_{21}f^2f_\zeta+ff_\zeta+k_{22}f_{\zeta\zeta}-f_\eta-kf_\zeta=0,$ |
| $k_{27}f_{\zeta\zeta\zeta}+k_{28}f_{\zeta\eta\eta}+k_{25}f^2f_\zeta+ff_\zeta+k_{26}f_{\zeta\zeta}-lf_\zeta=0,$ |

**Table 5** Some referenced transformations to reduce the above partial differential equations to ordenary differential equations.

| Case | The used transformation | | The ordinary differential equation |
|---|---|---|---|
| | $\theta$ | $g(\theta)$ | |
| (i) | $c_1\zeta+c_2\eta$ | $f$ | $-[1/(1+a)]\theta g'-[a/(1+a)]g+c_1^2k_2g''+c_1gg'+k_1c_1g^2g'$ $+(k_4c_1c_2^2+k_3c_1^3)g'''=0,\quad b=1$ |
| (ii) | $\zeta+\eta$ | $f$ | $(k_7+k_8)g'''+k_5g^2g'+gg'$ $-g'+g-k_6g''=0,\ c=0$ |
| (iii) | $c_1\zeta+c_2\eta$ | $f$ | $(k_{11}c_1^3+k_{12}c_1c_2^2)g'''+k_9c_1g^2g'$ $+c_1gg'-g+c_1^2k_{10}g''=0,\ w=0$ |
| (iv) | $c_1\zeta+c_2\ln\eta$ | $f$ | $k_{15}c_1^3g'''+k_{14}c_1^2g''+k_{13}c_1g^2g'+c_1gg'$ $-(1/m)(c_1+c_2)g'=0,\quad k_{16}=0$ |
| (v) | $c_1\zeta+c_2\ln\eta$ | $f$ | $k_{19}c_1^3g'''+k_{18}c_1^2g''+k_{17}c_1g^2g'$ $+c_1gg'-(c_2/n)g'=0,\quad k_{20}=0$ |
| (vi) | $\zeta+\eta$ | $f$ | $(k_{23}+k_{24})g'''+k_{21}g^2g'+gg'-(1+k)g'+k_{22}g''=0,$ |
| (vii) | $\zeta+\eta$ | $f$ | $(k_{27}+k_{28})g'''+k_{25}g^2g'+gg'-l\ g'+k_{26}g''=0.$ |

## 7 Determination of Exact Solutions

Now, we have going to our original task, finding exact solutions for the reduced ODEs, which by back substitution gives new exact solutions for the generalized variable coefficients Zakharov–Kuznetsov equation as follows:

***Case (i)*** To determine the solution for the ODE corresponding to this case, we assume that this solution takes the following form

$$g = A_0 + A_1\theta + A_2\theta^2 + \frac{B_1}{\theta} + \frac{B_2}{\theta^2}\,, \tag{15}$$

where $A_0$, $A_1$, $A_2$, $B_1$, and $B_2$ are arbitrary constant to be determined. Substituting from Eq. (15) into the reduced ODE given by case (i) and collecting the various powers of $\theta$ then equating them to zero, we get system of algebraic equations in the constants $A_0$, $A_1$, $A_2$, $B_1$, $B_2$, $a$, $c_1$, $c_2$, $k_1$, $k_2$, $k_3$, and $k_4$. Solving this system with the aid of *Maple* program, we get the following two sets of solutions

***The first set***

$$a = \frac{1}{2}\,, \quad A_1 = \frac{1}{c_1}\,, \quad B_2 = -12(k_3c_1^2 + k_4c_2^2)\,,$$
$$k_1 = k_2 = A_0 = A_2 = B_0 = B_1 = 0\,.$$

The corresponding exact solution for Eq. (1) is given by

$$u(x,y,t) = \Big(x + \frac{c_2}{c_1}y\Big)\Gamma^{-1}(t) - \frac{12[k_3 + k_4(c_2^2/c_1^2)]\Gamma(t)}{[x + (c_2/c_1)y]^2}\,. \tag{16}$$

***The second set***

$$a = 1\,, \quad A_1 = \frac{1}{c_1}\,, \quad B_1 = 2k_2c_1\,, \quad k_3 = -\frac{k_4c_2^2}{c_1^2}\,,$$
$$k_1 = A_0 = A_2 = B_0 = B_2 = 0\,.$$

Corresponding the second set we arrived at the following solution for the generalized variable coefficients Zakharov–Kuznetsov equation

$$u(x,y,t) = \Big(x + \frac{c_2}{c_1}y\Big)\Gamma^{-1}(t) + \frac{2k_2}{x + (c_2/c_1)y}\,. \tag{17}$$

***Case (ii)*** To obtain a solution for the ODE corresponding this case, we assume that $k_7 = -k_8$ and $k_5 = k_6 = 0$, then we get the following form for $g$

$$g = -\text{Lambert } W(-C_0\exp(\theta))\,,$$

where $C_0$ is an integration constant. So that Eq. (1) has the following solution

$$u(x,y,t) = -\Gamma^{-1}(t) \times \text{Lambert } W(-C_0\exp(\ln\Gamma(t) - x + y))\,. \tag{18}$$

***Case (iii)*** To solve the ODE corresponding to this case, we consider that $k_{10} = 0$, and $k_{11} = -(c_2^2/c_1^2)k_{12}$, yields

$$g = \sqrt{\frac{2}{c_1k_9}\theta} - \frac{1}{k_9}\,.$$

Therefore the corresponding solution for the generalized variable coefficients Zakharov–Kuznetsov equation

$$u(x,y,t) = \bigg[\sqrt{\frac{2}{k_9}\Big(x + \frac{c_2}{c_1}y\Big)} - \frac{1}{k_9}\bigg]\Gamma^{-1}(t)\,. \tag{19}$$

***Case (iv)*** To find travelling wave solutions for the ODE corresponding to this case first of all integrate it with respect to $\theta$ , we got

$$k_{15}c_1^3g'' + k_{14}c_1^2g' + \frac{k_{13}c_1}{3}g^3 + \frac{c_1}{2}g^2 - \frac{1}{m}(c_1 + c_2)g = 0\,, \tag{20}$$

where the integration constant equals zero. Now to solve Eq. (20) we have used the generalized tanh-function method given in Ref. [19] as follows.

Let us assume that $g$ takes the form

$$g = \sum_{i=0}^{s}\phi^i(\zeta)A_i\,, \tag{21}$$

where $\phi(\zeta)$ is a solution of the following Riccati equation

$$\phi'(\zeta) = r + \phi^2(\zeta)\,. \tag{22}$$

The previous equation has the following forms of solutions

$$\begin{aligned}
\phi(\zeta) &= -\sqrt{-r}\tanh(\sqrt{-r}\zeta)\,, & r &< 0\,,\\
\phi(\zeta) &= -\sqrt{-r}\coth(\sqrt{-r}\zeta)\,, & r &< 0\,,\\
\phi(\zeta) &= \sqrt{r}\tan(\sqrt{r}\zeta)\,, & r &> 0\,,\\
\phi(\zeta) &= -\sqrt{r}\cot(\sqrt{r}\zeta)\,, & r &> 0\,,\\
\phi(\zeta) &= -\frac{1}{\zeta}\,, & r &= 0\,.
\end{aligned} \tag{23}$$

Substituting from Eq. (21) into Eq. (20) and by making balance between the linear term $g''$ and the nonlinear term $g^3$ to determine the value of $s$, we have got that $s = 1$, so that

$$g = A_0 + A_1\phi\,, \tag{24}$$

where $A_0$ and $A_1$ are arbitrary constants. Substituting Eqs. (22) and (24) into Eq. (20) and setting to zero all coefficients of $\phi^i$ $(i = 0, 1, 2, 3)$, we have obtained a system of algebraic equations,

$$\begin{aligned}
&\frac{1}{3}k_{13}c_1A_0^3 + k_{14}c_1^2A_1r + \frac{1}{2}c_1A_0^2 - \frac{(c_1 + c_2)A_0}{m} = 0\,,\\
&k_{13}c_1A_1A_0^2 + c_1A_0A_1 + 2k_{15}c_1^3rA_1 - \frac{(c_1 + c_2)A_1}{m} = 0\,,\\
&2k_{15}c_1^3A_1 + \frac{1}{3}k_{13}c_1A_1^3 = 0\,,\\
&k_{13}c_1A_0A_1^2 + k_{14}c_1^2A_1 + \frac{1}{2}c_1A_1^2 = 0\,.
\end{aligned} \tag{25}$$

Solving the above system with the aid of *Maple* program the constants $A_0$, $A_1$, $k_{13}$, $k_{14}$, $k_{15}$, $c_1$, $c_2$, $r$, and $m$ are obtained as follows

$$A_0 = -\frac{3}{4k_{13}}, \quad A_1 = 4c_1k_{14}, \quad k_{15} = -\frac{8}{3}k_{13}k_{14}^2\,,$$
$$c_2 = -\frac{c_1(9m + 256k_{13}^2k_{14}^2c_1^2rm + 48k_{13})}{48k_{13}}\,. \tag{26}$$

By substituting from Eq. (26) into Eq. (24) and by using the different forms of solutions of the Riccati equation (22) which are given by Eq. (23), we arrive at the following exact solutions for Eq. (20)

$$g=-\frac{3}{4k_{13}}-4c_1k_{14}\sqrt{-r}\tanh[\sqrt{-r}\theta]\,,\quad r<0\,,$$
$$g=-\frac{3}{4k_{13}}-4c_1k_{14}\sqrt{-r}\coth[\sqrt{-r}\theta]\,,\quad r<0\,,$$
$$g=-\frac{3}{4k_{13}}+4c_1k_{14}\sqrt{r}\tan[\sqrt{r}\theta]\,,\quad r>0\,,$$
$$g=-\frac{3}{4k_{13}}-4c_1k_{14}\sqrt{r}\cot[\sqrt{r}\theta]\,,\quad r>0\,,$$
$$g=-\frac{3}{4k_{13}}-\frac{4c_1k_{14}}{\theta}\,,\quad r=0\,.$$

Therefore by back substitution using the similarity variables corresponding to this case, we obtain the following exact solutions for Eq. (1)

$$u(x,y,t)=-4c_1k_{14}\sqrt{-r}\tanh\Big[\sqrt{-r}\Big(c_1x+c_2\ln y-\frac{(c_1+c_2)}{m}\Gamma(t)\Big)\Big]-\frac{3}{4k_{13}}\,, \tag{27}$$

$$u(x,y,t)=-4c_1k_{14}\sqrt{-r}\coth\Big[\sqrt{-r}\Big(c_1x+c_2\ln y-\frac{(c_1+c_2)}{m}\Gamma(t)\Big)\Big]-\frac{3}{4k_{13}}\,, \tag{28}$$

$$u(x,y,t)=-\frac{3}{4k_{13}}+4c_1k_{14}\sqrt{r}\tan\Big[\sqrt{r}\Big(c_1x+c_2\ln y-\frac{(c_1+c_2)}{m}\Gamma(t)\Big)\Big]\,, \tag{29}$$

$$u(x,y,t)=-\frac{3}{4k_{13}}-4c_1k_{14}\sqrt{r}\cot\Big[\sqrt{r}\Big(c_1x+c_2\ln y-\frac{(c_1+c_2)}{m}\Gamma(t)\Big)\Big]\,, \tag{30}$$

$$u(x,y,t)=-\frac{3}{4k_{13}}-\frac{4c_1k_{14}}{c_1x+c_2\ln y-[(c_1+c_2)/m]\Gamma(t)}\,. \tag{31}$$

***Case (v)*** To find solutions for the ODE corresponding to this case we used the generalized tanh-function method as we have already done in the previous case and we have got the following new exact solutions for the generalized variable coefficients Zakharov–Kuznetsov equation

$$u(x,y,t)=-4c_1k_{18}\sqrt{-r}\tanh\Big[\sqrt{-r}\Big(c_1x+c_2\ln y-\frac{c_2}{n}\Gamma(t)\Big)\Big]-\frac{3}{4k_{17}}\,, \tag{32}$$

$$u(x,y,t)=-4c_1k_{18}\sqrt{-r}\coth\Big[\sqrt{-r}\Big(c_1x+c_2\ln y-\frac{c_2}{n}\Gamma(t)\Big)\Big]-\frac{3}{4k_{17}}\,, \tag{33}$$

$$u(x,y,t)=-\frac{3}{4k_{17}}+4c_1k_{18}\sqrt{r}\tan\Big[\sqrt{r}\Big(c_1x+c_2\ln y-\frac{c_2}{n}\Gamma(t)\Big)\Big]\,, \tag{34}$$

$$u(x,y,t)=-\frac{3}{4k_{17}}-4c_1k_{18}\sqrt{r}\cot\Big[\sqrt{r}\Big(c_1x+c_2\ln y-\frac{c_2}{n}\Gamma(t)\Big)\Big]\,, \tag{35}$$

$$u(x,y,t)=-\frac{3}{4k_{17}}-\frac{4c_1k_{18}}{c_1x+c_2\ln y-(c_2/n)\Gamma(t)}\,, \tag{36}$$

where $c_2$, $k_{19}$ are given by

$$c_2=\frac{-c_1n(9+256k_{17}^2k_{18}^2c_1^2r)}{48k_{17}}\,,\quad k_{19}=-\frac{8}{3}k_{17}k_{18}^2\,,$$

in the above solutions (32)–(36).

***Case (vi)*** Similarly exact solutions are found for the ODE corresponding this case via the generalized tanh-function method and therefore new classes of exact solutions to Eq. (1) are found to be

$$u(x,y,t)=-4k_{22}\sqrt{-r}\tanh\Big[\sqrt{-r}\Big(x+y+\frac{256k_{21}^2k_{22}^2r+9}{48k_{21}}\Gamma(t)\Big)\Big]-\frac{3}{4k_{21}}\,, \tag{37}$$

$$u(x,y,t)=-4k_{22}\sqrt{-r}\coth\Big[\sqrt{-r}\Big(x+y+\frac{256k_{21}^2k_{22}^2r+9}{48k_{21}}\Gamma(t)\Big)\Big]-\frac{3}{4k_{21}}\,, \tag{38}$$

$$u(x,y,t)=-\frac{3}{4k_{21}}+4k_{22}\sqrt{r}\tan\Big[\sqrt{r}\Big(x+y+\frac{256k_{21}^2k_{22}^2r+9}{48k_{21}}\Gamma(t)\Big)\Big]\,, \tag{39}$$

$$u(x,y,t)=-\frac{3}{4k_{21}}-4k_{22}\sqrt{r}\cot\Big[\sqrt{r}\Big(x+y+\frac{256k_{21}^2k_{22}^2r+9}{48k_{21}}\Gamma(t)\Big)\Big]\,, \tag{40}$$

$$u(x,y,t)=-\frac{3}{4k_{21}}-\frac{4k_{22}}{x+y+(3/16k_{21})\Gamma(t)}\,, \tag{41}$$

where

$$k=-\Big(1+\frac{256k_{21}^2k_{22}^2r+9}{48k_{21}}\Big)\,,\quad k_{23}=-\Big(k_{24}+\frac{8}{3}k_{21}k_{22}^2\Big)\,.$$

***Case (vii)*** Corresponding the case under consideration, we repeat the above procedure by using the generalized tanh-function method, we arrive at the following sets of travelling wave solutions

$$u(x,y,t) = -4k_{26}\sqrt{-r}\tanh\Big[\sqrt{-r}\Big(x+y+\frac{256k_{25}^2k_{26}^2r+9}{48k_{25}}\Gamma(t)\Big)\Big] - \frac{3}{4k_{25}}\,, \tag{42}$$

$$u(x,y,t) = -4k_{26}\sqrt{-r}\coth\Big[\sqrt{-r}\Big(x+y+\frac{256k_{25}^2k_{26}^2r+9}{48k_{25}}\Gamma(t)\Big)\Big] - \frac{3}{4k_{25}}\,, \tag{43}$$

$$u(x,y,t) = -\frac{3}{4k_{25}} + 4k_{26}\sqrt{r}\tan\Big[\sqrt{r}\Big(x+y+\frac{256k_{25}^2k_{26}^2r+9}{48k_{25}}\Gamma(t)\Big)\Big]\,, \tag{44}$$

$$u(x,y,t) = -\frac{3}{4k_{25}} - 4k_{26}\sqrt{r}\cot\Big[\sqrt{r}\Big(x+y+\frac{256k_{25}^2k_{26}^2r+9}{48k_{25}}\Gamma(t)\Big)\Big]\,, \tag{45}$$

$$u(x,y,t) = -\frac{3}{4k_{21}} - \frac{4k_{26}}{x+y+(3/16k_{25})\Gamma(t)}\,. \tag{46}$$

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