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A Note on Classification of Spatially Homogeneous Rotating Space-Times According to Their Teleparallel Killing Vector Fields in Teleparallel Theory of Gravitation

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Abstract *In this paper we classify spatially homogeneous rotating space-times according to their teleparallel Killing vector fields using direct integration technique. It turns out that the dimension of the teleparallel Killing vector fields is 5 or 10. In the case of 10 teleparallel Killing vector fields the space-time becomes Minkowski and all the torsion components are zero. Teleparallel Killing vector fields in this case are exactly the same as in general relativity. In the cases of 5 teleparallel Killing vector fields we get two more conservation laws in the teleparallel theory of gravitation. Here we also discuss some well-known examples of spatially homogeneous rotating space-times according to their teleparallel Killing vector fields.*

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1 Introduction

Teleparallel theory of gravitation is based on Weitzenböck geometry.[1] In this theory the presence of torsion in the space-time is responsible for the gravitational interaction. In this alternate theory of gravitation curvature of the space-time becomes zero. Thus like general relativity where gravitation is attributed to the curvature of the space-time having zero torsion, gravitation in teleparallel theory is attributed to the torsion of the space-time having zero curvature, which plays the role of a force.[2] In the next section an overview of teleparallel theory is given where one can find the process of obtaining torsion tensor.

In the literature of general relativity, a considerable attention has been given to study symmetries of the metric tensor. Certain symmetry restrictions are imposed on the space-time metric to solve the Einstein's field equations. These symmetry restrictions are well expressed in terms of Killing vector fields, which give rise to conservation laws.[3] In Ref. [4] the authors introduced the teleparallel version of the Lie derivative for Killing vector fields and used those equations to find the teleparallel Killing vector fields in Einstein universe. To understand the physical and geometrical properties of the space-time it is important to understand and explore the conservation laws in teleparallel theory of gravitation and compare these to the results available in general relativity. Keeping this point in mind, we started to investigate Killing vector fields in some well-known space-times. Previously, we have explored teleparallel Killing vector fields in Bianchi types I, II, III and Kantowski–Sachs and cylindrically symmetric static space-times.[5−8] Our results for teleparallel Killing vector fields in Bianchi types I, III and Kantowski–Sachs and cylindrically symmetric static space-times are quite consistent with the results of general relativity, while the results of teleparallel Killing vector fields in Bianchi type II space-times are different from general relativity. In our present work we focus our investigations to classify spatially homogeneous rotating space-times according to their teleparallel Killing vector fields in teleparallel theory of gravitation. This classification of the conservation laws in teleparallel theory will help in understanding the physical and geometrical properties of the space-times.

2 Overview

The teleparallel covariant derivative ∇_ρ of a covariant tensor of rank 2 is defined as[2]

$$\nabla_\rho A_{\mu\nu} = A_{\mu\nu,\rho} - \Gamma^{\theta}{}_{\rho\nu} A_{\mu\theta} - \Gamma^{\theta}{}_{\mu\rho} A_{\nu\theta}\,, \tag{1}$$

where comma denotes the partial derivative and $\Gamma^{\theta}{}_{\rho\nu}$ are Weitzenböck connections defined as[2]

$$\Gamma^{\theta}{}_{\mu\nu} = S_a{}^{\theta}\partial_\nu S^a{}_\mu\,, \tag{2}$$

where $S^a{}_\mu$ is the non-trivial tetrad field. Its inverse field is denoted by $S_a{}^\nu$ and satisfies the relations

$$S^a{}_\mu S_a{}^\nu = \delta_\mu{}^\nu\,, \quad S^a{}_\mu S_b{}^\mu = \delta_b{}^a\,. \tag{3}$$

Through out this paper $a, b, c, \ldots = 0, 1, 2, 3$ denote the tangent space indices and $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ denote the space-time indices. The Remannian metric can be generated from the tetrad field as

$$g_{\mu\nu} = \eta_{ab} S^a{}_\mu S^b{}_\nu\,, \tag{4}$$

where η_{ab} is the Minkowski metric given by $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The Weitzenböck and Levi–Civita connections

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have the relation

$$\Gamma^{\theta}{}_{\mu\nu} = \Gamma^{0^{\theta}}{}_{\mu\nu} + K^{\theta}{}_{\mu\nu}\,, \tag{5}$$

where

$$K^{\theta}{}_{\mu\nu} = \frac{1}{2}[T_{\mu}{}^{\theta}{}_{\nu} + T_{\nu}{}^{\theta}{}_{\mu} - T^{\theta}{}_{\mu\nu}] \tag{6}$$

is a tensor quantity called the contortion tensor and $\overset{\circ}{\Gamma}{}^{\theta}{}_{\mu\nu}$ is the Levi–Civita connection defined as

$$\Gamma^{0^{\theta}}{}_{\mu\nu} = \frac{1}{2}g^{\theta\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})\,. \tag{7}$$

Also the torsion of the space-time in terms of Weitzenböck connections is defined as

$$T^{\theta}{}_{\mu\nu} = \Gamma^{\theta}{}_{\nu\mu} - \Gamma^{\theta}{}_{\mu\nu}\,, \tag{8}$$

which is antisymmetric with respect to its lower indices. The Riemann curvature tensor in terms of Weitzenböck connection in teleparallel theory is given as

$$R^{\theta}{}_{\sigma\mu\nu} = \Gamma^{\theta}{}_{\sigma\nu,\mu} - \Gamma^{\theta}{}_{\sigma\mu,\nu} + \Gamma^{\theta}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\theta}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu}\,. \tag{9}$$

Now using Eq. (5) in Eq. (9) we get

$$R^{\sigma}{}_{\theta\mu\nu} = R^{0^{\sigma}}{}_{\theta\mu\nu} + Q^{\sigma}{}_{\theta\mu\nu} = 0\,, \tag{10}$$

where $R^{0^{\sigma}}{}_{\theta\mu\nu}$ represents Riemann curvature tensor in general relativity and

$$Q^{\sigma}{}_{\theta\mu\nu} = \nabla_{\mu}K^{\sigma}{}_{\theta\nu} - \nabla_{\nu}K^{\sigma}{}_{\theta\mu} - K^{\lambda}{}_{\theta\nu}K^{\sigma}{}_{\lambda\mu} + K^{\lambda}{}_{\theta\mu}K^{\sigma}{}_{\lambda\nu} \tag{11}$$

is the tensor quantity based on Weitzenböck connection only. The above details can also be found in Ref. [10]. The teleparallel Killing equation is defined as[3]

$$\begin{aligned} L_X^T g_{\mu\nu} &= g_{\mu\nu,\rho}X^{\rho} + g_{\rho\nu}X^{\rho}_{,\mu} + g_{\mu\rho}X^{\rho}_{,\nu} \\ &\quad + X^{\rho}(g_{\theta\nu}T^{\theta}{}_{\mu\rho} + g_{\mu\theta}T^{\theta}{}_{\nu\rho}) = 0\,, \end{aligned} \tag{12}$$

where L_X^T denotes the teleparallel Lie derivative with respect to the vector field X.

3 Main Results

Consider spatially homogeneous rotating space-times in usual coordinates (t, r, ϕ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element[9]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + A(r)\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2B(r)\mathrm{d}t\mathrm{d}\phi\,, \tag{13}$$

where A and B are no where zero functions of r only and $A + B^2 \neq 0$. The above space-times (13) admit at least three linearly independent Killing vector fields in general relativity, which are $\partial/\partial t$, $\partial/\partial\phi$, and $\partial/\partial z$. The tetrad components and its inverse can be obtained by using the relation (4) as

$$S^{a}{}_{\mu} = \begin{pmatrix} 1 & 0 & \sqrt{B(r)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{A(r) + B^2(r)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_{a}{}^{\mu} = \begin{pmatrix} 1 & 0 & -\dfrac{B(r)}{\sqrt{A(r) + B^2(r)}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{A(r) + B^2(r)}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

It can be verified easily that Eqs. (3) and (4) between $S^{a}{}_{\mu}$ and $S_{a}{}^{\mu}$ are satisfied. Using Eq. (2) the corresponding non-vanishing Weitzenböck connections are obtained as

$$\Gamma^{0}{}_{12} = \frac{-BA' + 2AB'}{2(A + B^2)}\,, \quad \Gamma^{2}{}_{12} = \frac{A' + 2BB'}{2(A + B^2)}\,, \tag{15}$$

where dash denotes the derivative with respect to r. Thus the non vanishing torsion components by using Eq. (8) are

$$T^{0}_{\ 21} = \frac{BA' - 2AB'}{2(A + B^2)}\,, \quad T^{2}_{\ 21} = \frac{A' + 2BB'}{2(A + B^2)}\,. \tag{16}$$

A vector field X is said to be teleparallel Killing vector field if it satisfies Eq. (12). One can write Eq. (12) explicitly and using Eqs. (13) and (16) we get

$$X^0_{,0} + B(r)X^2_{,0} = 0\,, \tag{17}$$

$$X^1_{,0} - X^0_{,1} - B(r)X^2_{,1} - B'(r)X^2 = 0\,, \tag{18}$$

$$B(r)X^0_{,0} - A(r)X^2_{,0} + X^0_{,2} + B(r)X^2_{,2} = 0\,, \tag{19}$$

$$X^3_{,0} - X^0_{,3} - B(r)X^2_{,3} = 0\,, \tag{20}$$

$$X^1_{,1} = 0\,, \tag{21}$$

$$2B(r)X^0_{,1} - 2A(r)X^2_{,1} - 2X^1_{,2} - A'(r)X^2 = 0\,, \tag{22}$$

$$X^3_{,1} + X^1_{,3} = 0\,, \tag{23}$$

$$A(r)X^2_{,2} - B(r)X^0_{,2} = 0\,, \tag{24}$$

$$X^3_{,2} - B(r)X^0_{,3} + A(r)X^2_{,3} = 0\,, \tag{25}$$

$$X^3_{,3} = 0\,. \tag{26}$$

From Eqs. (21) and (26) we get $X^1 = P^1(t, \phi, z)$ and $X^3 = P^2(t, r, \phi)$, where $P^1(t, \phi, z)$ and $P^2(t, r, \phi)$ are functions of integration which are to be determined. Now differentiating Eqs. (20) and (25) with respect to z, we get

$$X^0_{,33} + B(r)X^2_{,33} = 0\,, \tag{27}$$

$$B(r)X^0_{,33} - A(r)X^2_{,33} = 0\,, \tag{28}$$

Multiplying Eq. (27) by $A(r)$ and Eq. (28) by $B(r)$ and then adding we get $X^0_{,33} = 0$. Twice integration with respect to z we get

$$X^0 = ZP^3(t, r, \phi) + P^4(t, r, \phi)\,. \tag{29}$$

Substituting back the value of Eq. (29) in Eq. (27) we get $X^2_{,33} = 0$ twice integration with respect to z we get

$$X^2 = ZP^5(t, r, \phi) + P^6(t, r, \phi)\,, \tag{30}$$

where $P^3(t, r, \phi)$, $P^4(t, r, \phi)$, $P^5(t, r, \phi)$, and $P^6(t, r, \phi)$ are functions of integration which are to be determined. Thus we have the following system of equations

$$\begin{aligned} &X^0 = ZP^3(t, r, \phi) + P^4(t, r, \phi)\,, \quad X^1 = P^1(t, \phi, z)\,, \\ &X^2 = ZP^5(t, r, \phi) + P^6(t, r, \phi)\,, \quad X^3 = P^2(t, r, \phi)\,. \end{aligned} \tag{31}$$

Now we have to solve the system of Eq. (31) by using the remaining six equations. To avoid lengthy details here we shall present only the results which are

Case 1 In this case we have $A \neq B$, $A \neq$ constant, and $B \neq$ constant. The space-time is given in Eq. (13). Solution of Eq. (17) to Eq. (26) is given by

$$X^0 = -\frac{B}{\sqrt{A + B^2}}c_1 + c_5\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = \frac{1}{\sqrt{A+B^2}}c_1\,, \quad X^3 = -rc_2 + c_4\,, \tag{32}$$

where c_1, c_2, c_3, c_4, and $c_5 \in R$. Here the above space-time admits five linearly independent teleparallel killing vector fields which are

$$\frac{\partial}{\partial t}\,, \quad \frac{1}{\sqrt{A+B^2}}\Big(\frac{\partial}{\partial \phi} - B\frac{\partial}{\partial t}\Big)\,, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}\,, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}\,.$$

Killing vector fields in general relativity are $\partial/\partial t$, $\partial/\partial \phi$, and $\partial/\partial z$. On comparison to the killing vector fields in general relativity we see that two teleparallel killing vector fields $\partial/\partial t$ and $\partial/\partial z$ are same in both the theories while others are totally different. It is important to note that in this case we get two more conservation laws.

Case 2 In this case we have $A \neq B$, $A \neq$ constant, and $B =$ constant. In this case $B = c_4/c_7$, where c_4, $c_7 \in R\backslash\{0\}$ and the line element for spatially homogeneous rotating space-time becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + A(r)\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2\frac{c_4}{c_7}\mathrm{d}t\mathrm{d}\phi\,. \tag{33}$$

Solution of Eq. (17) to Eq. (26) is given by

$$X^0 = -\frac{c_5}{\sqrt{A+c_6}}c_1 + zc_7 + c_9\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = \frac{1}{\sqrt{A+c_6}}c_1\,, \quad X^3 = -rc_2 - \phi c_4 + tc_7 + c_8\,, \tag{34}$$

where c_1, c_2, c_3, c_4, c_8, $c_9 \in R$, $c_5 = c_4/c_7$, and $c_6 = (c_4/c_7)^2$. Here the above space-time admits five linearly independent teleparallel killing vector fields, which can be written as

$$\frac{\partial}{\partial t}, \quad \frac{1}{\sqrt{A+c_6}}\Big(\frac{\partial}{\partial \phi} - c_5\frac{\partial}{\partial t}\Big), \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}\,, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}\,.$$

Killing vector fields in general relativity are $\partial/\partial t$, $\partial/\partial \phi$, and $\partial/\partial z$. In this case two teleparallel killing vector fields $\partial/\partial t$, and $\partial/\partial z$ are same as in general relativity and others are different. It is important to note that in this case we get two more conservation laws.

Case 3 In this case we have $A \neq B$, $A = c_5$, and $B \neq$ constant, where $c_5 \in R\backslash\{0\}$. Here the line element for spatially homogeneous rotating space-time becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + c_5\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2B(r)\mathrm{d}t\mathrm{d}\phi\,. \tag{35}$$

Teleparallel killing vector fields in this case are

$$X^0 = -\frac{B}{\sqrt{c_5+B^2}}c_1 + c_6\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = \frac{1}{\sqrt{c_5+B^2}}c_1\,, \qquad X^3 = -rc_2 + c_4\,, \tag{36}$$

where c_1, c_2, c_3, c_4, $c_6 \in R$. Here the above space-time (35) admits five linearly independent teleparallel killing vector fields which can be written as

$$\frac{\partial}{\partial t}, \quad \frac{1}{\sqrt{c_5+B^2}}\Big(\frac{\partial}{\partial \phi} - B\frac{\partial}{\partial t}\Big), \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}.$$

Killing vector fields in general relativity are $\partial/\partial t$, $\partial/\partial \phi$, and $\partial/\partial z$. In this case two killing vector fields, which are $\partial/\partial t$ and $\partial/\partial z$, are same in both the theories. Other killing vector fields are totally different. It is important to note that in this case we get two more conservation laws.

Case 4 In this case $A(r) = a$ and $B(r) = b$, where a, $b \in R\backslash\{0\}$. Here the line element for spatially homogeneous rotating space-time becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + a\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2b\mathrm{d}t\mathrm{d}\phi\,. \tag{37}$$

It is important to note that in this case all the torsion components are zero. The teleparallel killing vector fields are the same as in general relativity and are given below

$$\begin{aligned} X^0 &= \frac{a}{a+b^2}c_7 r + \frac{b}{a+b^2}c_5 r - \frac{b}{a+b^2}c_9 t \\ &\quad + \frac{a}{a+b^2}c_9\phi + \frac{a}{a+b^2}c_1 z + \frac{b}{a+b^2}c_2 z + c_{10}\,, \\ X^1 &= c_7 t + c_5\phi + c_4 z + c_6\,, \\ X^2 &= \frac{b}{a+b^2}c_7 r - \frac{1}{a+b^2}c_5 r - \frac{1}{a+b^2}c_9 t \\ &\quad + \frac{b}{a+b^2}c_9\phi + \frac{b}{a+b^2}c_1 z - \frac{1}{a+b^2}c_2 z + c_8\,, \\ X^3 &= c_1 t - c_4 r + c_2\phi + c_3\,, \end{aligned} \tag{38}$$

where c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, $c_{10} \in R$. These killing vector fields can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \phi}, \\ &\frac{\partial}{\partial z}, \quad t\frac{\partial}{\partial r} + \frac{r}{a+b^2}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial \phi}\Big), \\ &\phi\frac{\partial}{\partial r} + \frac{r}{a+b^2}\Big(b\frac{\partial}{\partial t} - \frac{\partial}{\partial \phi}\Big), \\ &\frac{1}{a+b^2}\Big\{(a\phi - bt)\frac{\partial}{\partial t} + (b\phi - t)\frac{\partial}{\partial \phi}\Big\}, \\ &z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}\,, \quad t\frac{\partial}{\partial z} + \frac{z}{a+b^2}\Big(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial \phi}\Big), \\ &\phi\frac{\partial}{\partial z} + \frac{z}{a+b^2}\Big(b\frac{\partial}{\partial t} - \frac{\partial}{\partial \phi}\Big). \end{aligned}$$

4 Examples

In this section we will discuss teleparallel killing vector fields of some spatially homogeneous rotating space-times. Here we will only present the results and calculation will be omitted. One can easily reproduce the following results by using the general method, which is given in Sec. 3. These examples are as follows:

(i) ***Reboucas Space-Time***

If we choose $A(r) = -(1 + 3\cosh^2 2r)$ and $B(r) = 2\cosh 2r$, the above space-time (13) becomes Reboucas space-time and takes the form[9]

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + \mathrm{d}r^2 - (1 + 3\cosh^2 2r)\mathrm{d}\phi^2 \\ &+ \mathrm{d}z^2 - 4\cosh 2r\,\mathrm{d}t\mathrm{d}\phi\,. \end{aligned} \tag{39}$$

For the above space-time (39) the teleparallel killing vector fields are given as

$$X^0 = (-2\coth 2r)c_1 + c_5\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = (\operatorname{csch} 2r)c_1\,, \quad X^3 = -rc_2 + c_4\,, \tag{40}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. The above teleparallel killing vector fields can be written as:

$$-\,2\coth 2r\frac{\partial}{\partial t} + \operatorname{csch} 2r\frac{\partial}{\partial \phi}\,,$$

$$z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}.$$

Killing vector fields in general relativity are:

$$-\operatorname{csch} 2r\sin 2\phi\Big(2\frac{\partial}{\partial t} + \cosh 2r\frac{\partial}{\partial \phi}\Big) + \cos 2\phi\frac{\partial}{\partial r}\,,$$

$$\operatorname{csch} 2r\cos 2\phi\Big(2\frac{\partial}{\partial t} + \cosh 2r\frac{\partial}{\partial \phi}\Big) + \sin 2\phi\frac{\partial}{\partial r}\,,$$

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial z}\,.$$

Here one can see that two killing vector fields $\partial/\partial t$ and $\partial/\partial z$ are same in both the theories and others are different.

(ii) ***Som–Raychaudhuri Space-Time***

If we choose $A(r) = r^2(1-r^2)$ and $B(r) = r^2$, the above space-time (13) becomes Som–Raychaudhuri space-time and takes the form[9]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(1-r^2)\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2r^2\mathrm{d}t\mathrm{d}\phi\,. \tag{41}$$

For the above space-time (41) the teleparallel killing vector fields are given as

$$X^0 = -rc_1 + c_5\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = \frac{1}{r}c_1\,, \qquad X^3 = -rc_2 + c_4\,, \tag{42}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. One can write the above teleparallel killing vector fields as:

$$-r\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \phi}, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}\,.$$

Killing vector fields in general relativity are:

$$r\sin\phi\frac{\partial}{\partial t} + \cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\,,$$

$$-\,r\cos\phi\frac{\partial}{\partial t} + \sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial \phi}\,,$$

$$\frac{\partial}{\partial t}\,, \quad \frac{\partial}{\partial \phi}\,, \quad \frac{\partial}{\partial z}\,.$$

Here again one can see that two killing vector fields $\partial/\partial t$ and $\partial/\partial z$ are same in both the theories and others are different.

(iii) ***Hoenselaers–Vishveshwara Space-Time***

If we choose $A(r) = -(1/2)\,(\cosh r - 1)(\cosh r - 3)$ and $B(r) = (\cosh r - 1)$, the above space-time (13) becomes Hoenselaers–Vishveshwara space-time and takes the form

$$\begin{aligned}\mathrm{d}s^2 = {} & -\mathrm{d}t^2 + \mathrm{d}r^2 - \frac{1}{2}(\cosh r - 1)(\cosh r - 3)\mathrm{d}\phi^2 \\ & + \mathrm{d}z^2 - 2(\cosh r - 1)\mathrm{d}t\mathrm{d}\phi\,.\end{aligned} \tag{43}$$

For the above space-time (43) the teleparallel killing vector fields are given as

$$X^0 = -\sqrt{2}c_1 + c_5\,, \quad X^1 = zc_2 + c_3\,,$$

$$X^2 = \sqrt{2}\operatorname{csch} rc_1\,, \quad X^3 = -rc_2 + c_4\,, \tag{44}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. The above teleparallel killing vector fields can be written as:

$$-\,\sqrt{2}\frac{\partial}{\partial t} + \sqrt{2}\operatorname{csch} r\frac{\partial}{\partial \phi}\,,$$

$$z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}\,.$$

Killing vector fields in general relativity are:

$$\sqrt{2}(\coth r - \operatorname{csch} r)\sin\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial t} + \cos\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial r} - \sqrt{2}\coth r\sin\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial \phi}\,,$$

$$-\,\sqrt{2}(\coth r - \operatorname{csch} r)\cos\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial t} + \sin\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial r} + \sqrt{2}\coth r\cos\frac{\phi}{\sqrt{2}}\frac{\partial}{\partial \phi}\,, \quad \frac{\partial}{\partial t}\,, \quad \frac{\partial}{\partial \phi}\,, \quad \frac{\partial}{\partial z}\,.$$

Here two killing vector fields $\partial/\partial t$ and $\partial/\partial z$ are same in both the theories and others are different.

(iv) ***Gödel–Friedmann Space-Time***

Choosing $A(r) = \sinh^2 r(1 - \sinh^2 r)$ and $B(r) = \sqrt{2}\sinh^2 r$, the above space-time (13) becomes Gödel–Friedmann space-time and takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \sinh^2 r(1 - \sinh^2 r)\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2\sqrt{2}\sinh^2 r\,\mathrm{d}t\mathrm{d}\phi\,. \tag{45}$$

For the above space-time (45) the teleparallel killing vector fields are given as

$$X^0 = -\sqrt{2}\tanh rc_1 + c_5\,, \quad X^1 = zc_2 + c_3\,, \quad X^2 = \frac{\operatorname{csch} r}{\cosh r}c_1\,, \quad X^3 = -rc_2 + c_4\,, \tag{46}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. The above teleparallel killing vector fields can be written as:

$$-\sqrt{2}\tanh r\frac{\partial}{\partial t} + \frac{\operatorname{csch} r}{\cosh r}\frac{\partial}{\partial \phi}\,, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}\,, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}\,, \quad \frac{\partial}{\partial z}\,.$$

Killing vector fields in general relativity are:

$$\sqrt{2}(\sinh 2r - \tanh r\cosh 2r)\sin\phi\frac{\partial}{\partial t} + \cos\phi\frac{\partial}{\partial r} - 2\coth 2r\sin\phi\frac{\partial}{\partial \phi}\,,$$

$$-\,\sqrt{2}(\sinh 2r - \tanh r\cosh 2r)\cos\phi\frac{\partial}{\partial t} + \sin\phi\frac{\partial}{\partial r} + 2\coth 2r\cos\phi\frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial t}\,, \quad \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial z}\,.$$

Killing vector fields which are same in both the theories are $\partial/\partial t$ and $\partial/\partial z$. Others are different.

(iv) ***Stationary Gödel Space-Time***

Choosing $A(r) = -(1/2)\,\mathrm{e}^{2ar}$ and $B(r) = \mathrm{e}^{ar}$, where $a \in R\backslash\{0\}$. The above space-time (13) becomes stationary Gödel space-time and takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 - \frac{1}{2}\,\mathrm{e}^{2ar}\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - 2\,\mathrm{e}^{ar}\,\mathrm{d}t\mathrm{d}\phi\,. \tag{47}$$

For the above space-time (47) the teleparallel killing vector fields are given as

$$\begin{aligned} X^0 &= -\sqrt{2}c_1 + c_5\,, \quad X^1 = zc_2 + c_3\,, \\ X^2 &= \mathrm{e}^{-ar}\sqrt{2}c_1\,, \quad X^3 = -rc_2 + c_4\,, \end{aligned} \tag{48}$$

where c_1, c_2, c_3, c_4, $c_5 \in R$. The above teleparallel killing vector fields can be written as:

$$-\sqrt{2}\frac{\partial}{\partial t} + \sqrt{2}\,\mathrm{e}^{-ar}\frac{\partial}{\partial \phi}, \quad z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z}.$$

Killing vector fields in general relativity are:

$$\frac{2}{a}\,\mathrm{e}^{-ar}\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial r} - \Bigl(\frac{a}{2}\phi^2 + \frac{2}{a}\,\mathrm{e}^{-ar}\sinh ar\Bigr)\frac{\partial}{\partial \phi}\,,$$

$$\Bigl(\frac{\partial}{\partial r} - a\phi\frac{\partial}{\partial \phi}\Bigr), \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial z}\,.$$

In both the theories the two killing vector fields $\partial/\partial t$ and $\partial/\partial z$ are same. Others are different.

5 Conclusion

In this paper we classified spatially homogeneous rotating space-times according to their teleparallel killing vector fields. It is shown that the above space-times admit 5 or 10 teleparallel killing vector fields. In the case of 10 killing vector fields the space-time becomes Minkowski and all the torsion components become zero. Hence all the 10 killing vector fields are same in both the theories. In the cases of 5 teleparallel killing vector fields we get two more conservation laws in the teleparallel theory of gravitation. From the above discussion it is clear that the presence of torsion in the spatially homogeneous rotating space-time increase the number of conservation laws. If the space-time does not possess torsion like Minkowski space-time the conservation laws remain the same in both the theories of gravitation. Here, we have also discussed some well known examples of spatially homogeneous rotating space-times according to their teleparallel killing vector fields and it turns out that for these space-times the conservation laws are same in number. A brief comparison of killing vector fields in both the theories is also given. In these examples only two killing vector fields in both the theories are same and other three are totally different.

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