Commun. Theor. Phys. **55** (2011) 381–386 Vol. 55, No. 3, March 15, 2011

Traveling Wave Solutions for Generalized Bretherton Equation*

Amin Esfahani†

School of Mathematics and Computer Science, Damghan University, Damghan 36716-41167, Iran

(Received June 30, 2010; revised manuscript received August 5, 2010)

Abstract *This paper studies the Generalized Bretherton equation using trigonometric function method including the sech-function method, the sine-cosine function method, and the tanh-function method, and He's semi-inverse method (He's variational method). Various traveling wave solutions are obtained, revealing an intrinsic relationship among the amplitude, frequency, and wave speed.*

PACS numbers: 02.30.Jr, 05.45.Yv
Key words: Bretherton equation, traveling wave, trigonometric function method, variational method

1 Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as the fluid dynamics, elastic media, optical fibers, nuclear physics, high-energy physics, plasma physics, gravitation and in statistical and condensed matter physics, biology, solid state physics, chemical kinematics, chemical physics, and geochemistry, etc. Because of the increased interest in theory of solitary waves, a broad range of analytic and computational methods have been used in the analysis of these scientific models. Mathematical modeling of many physical systems leads to nonlinear evolution equations (NLEEs) in various fields of physics and engineering. Most famous model equations are the nonlinear Schrödinger equation, the Korteweg-de Vries equation, the Kadomtsev–Petviashvili equation, the Boussinesq equation, the sine-Gordon equation, the Zakharov–Kuznetsov equation and so on. An effective method is required to analyze the mathematical model, which provides solutions conforming to physical reality. Numerous analytic methods and computational techniques have been proposed for obtaining and investigating solutions of NLEEs.[1−41] Thus, seeking exact and explicit solutions (especially the traveling waves) for NLEEs, by using different methods, has long been a major concern for mathematicians, physicists, and engineers; because the explicit formulas may provide physical information and help us to understand the mechanism of related physical models; they may also facilitate the verification of numerical and analytic methods. The inverse scattering (IST) method was the pioneer dominant technique for obtaining the exact solutions.[42−43] Some of other most efficient studied techniques are Bäcklund transformation, Painlevé method, F-expansion method, adomian decomposition method, G'/G method, Cole–Hopf transformation, Stokes'expansions, Padé approximants, exponential function method, Darboux transformation, Lie symmetry method, Hirota methods, sub-ODE method, (modified) extended tanh-function method, pseudo spectral method, generalized hyperbolic-function method, Jacobi elliptic function method, Whitham's method, generalized trigonometric functions method, the imbricate-soliton series, and many more.[2−4,6−15,17−18,26−29,31−41] A feature common to some of these methods is that they reduce the equation into a more simple equation by writing the equation in terms of specific polynomials in several elementary or special functions; and then solve it.

This work studies the generalized Bretherton equation

$$u_{tt}+\alpha u_{xx}+\beta u_{xxxx}+\delta u+\gamma u^n=0\,, \tag{1}$$

where u is a real function and the constants α, β, δ, and γ are real; while for the exponent n we assume that $n\neq 1$. This equation is a generalization of the Bretherton equation

$$u_{tt}+u_{xx}+u_{xxxx}+u-u^2=0\,. \tag{2}$$

Equation (2) was first introduced by Bretherton[5] as a model of a dispersive wave system to investigate the resonant nonlinear interaction between three linear modes. When $\alpha=0$, equation (1) can be considered as a formal fourth-order extension of the classical Klein–Gordon equation, but it also inherits a Schrödinger structure; however, it can be noted that the equation satisfies neither finite speed propagation nor mass conservation. We should also note that Eq. (1) possesses the following invariants:

$$P(u)=\int_{\mathbb{R}}u_x(x,t)u_t(x,t)\,\mathrm{d}x\,, \tag{3}$$

$$\begin{aligned}E(u)=\frac{1}{2}\int_{\mathbb{R}}&[u_t^2(x,t)-\alpha u_x^2(x,t)\\&+\beta u_{xx}^2(x,t)+\delta u^2(x,t)]\mathrm{d}x\\&+\frac{\gamma}{n+1}\int_{\mathbb{R}}u^{n+1}(x,t)\,\mathrm{d}x\,.\end{aligned} \tag{4}$$

The modified Bretherton equation

$$u_{tt}+u_{xx}+u_{xxxx}+u-u^3=0\,, \tag{5}$$

---

*Supported by FAPESP/SP-Brazil under Grant No. 2008/58892-6

†Corresponding author, E-mail: amin@impa.br; esfahani@du.ac.ir

© 2011 Chinese Physical Society and IOP Publishing Ltd

was proposed by Love[25] and discussed in [2, 7, 18] to obtain periodic solutions in terms of Jacobi elliptic functions and also elementary singular solutions; see also[23] for the instability and global solutions of (5).

Equation (1) was studied by Levandosky;[20–21] who investigated stability and instability of the solitary waves of this equation by using the following Hamiltonian form of (1):

$$\boldsymbol{u}_t = B\boldsymbol{u} + \begin{pmatrix} 0 \\ -\gamma u^n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} E'(\boldsymbol{u})\,, \tag{6}$$

where $v = u_t$,

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{7}$$

$$B = -\begin{pmatrix} 0 & -I \\ \alpha\partial_x^2 + \beta\partial_x^4 + \delta I & 0 \end{pmatrix}. \tag{8}$$

Levandosky and Strauss[22] also established time decay of the solutions of equation (1). We also refer to[1,16,19,24,30] for closely related references. Recently Romeiras[33] obtained periodic and solitary traveling wave solutions by a truncated Painlevé analysis.

Our interest in the present paper is to search for the traveling wave solutions of equation (1). The first technique that will be used in Sections 2, 3, and 4 is the trigonometric function method, which is one of the most effective direct methods to construct traveling wave solutions of nonlinear evolution equations. Actually in the trigonometric function method we will use a special transformation in terms of some special functions (sech function, sine–cosine function, and tanh function) to reduce our equation into a more simple ODE. Next by equating suitably the exponents of the functions and their respective coefficients, the single soliton solutions will be derived. The trigonometric function method, in terms of sech-function, will be applied in Sec. 2 to obtain the KdV-type solitons of (1) (see for example[3–4,34] and references therein); while Sections 4 and 5 are devoted to using the sine-cosine ansätze and the tanh methods to emphasize the applicability of these methods in solving nonlinear problems.[27–29,36–37]

In Sec. 5, the He's semi-inverse method (He's variational method) is used to carry out the integration.[9–12,31–32,35,39–41] This method is one of the most newly developed techniques that is being used to solve various NLEEs. Generally, this method reduces the original equation to a stationary integral; more precisely, a specific functional similar to the lagrangian structure of the equation. Based on the obtained stationary conditions, solitary solutions in the sech-function and exponential function forms are obtained.[9–12,31–32,35,39–41]

It is worth noticing that the present solution procedures can only lead to one-soliton solution; while by a simple analysis, the used methods can be extended to search for 2-wave solution, 3-wave solution or multiple-wave solution similar to double Exp-function method, three-wave method, and Wazwaz's multiple soliton method (see for example[6,8,38]).

2 sech-Ansätze Method

As we mentioned before, the solitary wave solutions we consider for (1) are obtained using sech-ansätze method; actually our hypothesis is[3–4,34]

$$u(x,t) = A\ \mathrm{sech}^p(ax-ct) = \frac{A}{\cosh^p(ax-ct)}\,, \tag{9}$$

where A is the amplitude and a is the inverse width of the solitary wave. Also c represents the velocity of the soliton and the exponent p will be determined later. Let

$$\zeta = ax - ct.$$

Now from (9), one can obtain

$$u_{tt} = \frac{c^2Ap^2}{\cosh^p\zeta} - \frac{c^2Ap(p+1)}{\cosh^{p+2}\zeta}\,, \tag{10}$$

$$u_{xx} = \frac{a^2Ap^2}{\cosh^p\zeta} - \frac{a^2Ap(p+1)}{\cosh^{p+2}\zeta}\,, \tag{11}$$

$$u_{xxxx} = \frac{a^4Ap^4}{\cosh^p\zeta} - \frac{2a^4Ap(p+1)(p^2+2p+2)}{\cosh^{p+2}\zeta} + \frac{a^4Ap(p+1)(p+2)(p+3)}{\cosh^{p+4}\zeta}\,. \tag{12}$$

Plugging (10)–(12) into (1), there obtains

$$\frac{c^2Ap^2}{\cosh^p\zeta} - \frac{c^2Ap(p+1)}{\cosh^{p+2}\zeta} + \frac{a^2\alpha Ap^2}{\cosh^p\zeta} - \frac{a^2\alpha Ap(p+1)}{\cosh^{p+2}\zeta} + \frac{a^4\beta Ap^4}{\cosh^p\zeta} - \frac{2a^4\beta Ap(p+1)(p^2+2p+2)}{\cosh^{p+2}\zeta} + \frac{a^4\beta Ap(p+1)(p+2)(p+3)}{\cosh^{p+4}\zeta} + \frac{\delta A}{\cosh^p\zeta} + \frac{\gamma A}{\cosh^{pn}\zeta} = 0\,. \tag{13}$$

By equating the exponents $p+4$ and pn, we have

$$p = \frac{4}{n-1}\,. \tag{14}$$

Note that $\mathrm{sech}^p\zeta$, $\mathrm{sech}^{p+2}\zeta$, and $\mathrm{sech}^{p+4}\zeta$ are linearly independent functions; so that by setting their respective coefficients in (13) to zero, we obtain

$$c^2 = -\alpha a^2 - \frac{16\beta a^4}{(n-1)^2} - \frac{\delta(n-1)^2}{16}\,, \tag{15}$$

$$A = \Big[-\frac{8\beta a^4(n+1)(n+3)(3n+1)}{\gamma(n-1)^4}\Big]^{1/(n-1)}\,, \tag{16}$$

$$|a| = \frac{(n-1)^2}{8(n+1)}\sqrt{\frac{\delta}{\beta}}\,. \tag{17}$$

It should be noted that setting the coefficients of $\mathrm{sech}^{p+2}\zeta$ and $\mathrm{sech}^{p+4}\zeta$ to zero yields the consistency condition (17) between the width a of the solitary wave and the coefficients of equation (1). Thus, the solitary wave solution of the generalized Bretherton equation (1) is given by

$$u(x,t) = A\ \mathrm{sech}^{4/(n-1)}(ax-ct)\,, \tag{18}$$

where the inverse width a is given by (17), the velocity c of the solitary wave is given by (15) and the amplitude of the wave is given by (16). Finally, it is necessary to have

$$\delta\beta > 0,$$

as seen from (17); and also the coefficients α, β, and δ cannot be negative simultaneously, due to (15).

The following Fig. 1 shows the soliton profiles of (18) with $n = 3$, $\beta = a = A = 1$, $\delta = -64$, $\gamma = -120$, $\alpha = -24$, and $c = 2$ in $x - t$ plane.

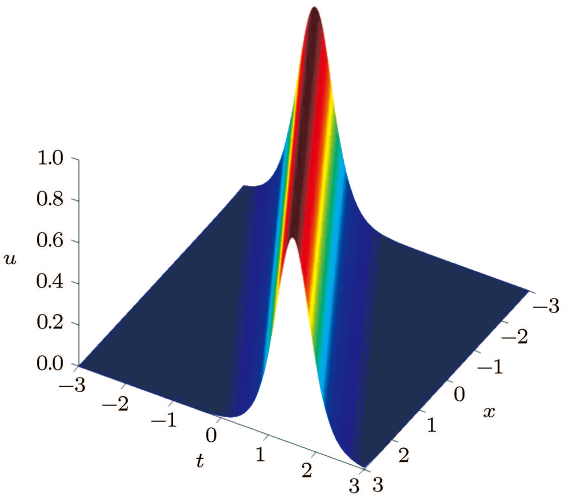

Fig. 1 The wave profile for solution (18) with $n = 3$, $\beta = a = A = 1$, $\delta = -64$, $\gamma = -120$, $\alpha = -24$, and $c = 2$ in $x - t$ plane.

Let us now address the question of the stability of the solitary wave solution of (1) with respect to small perturbations, when $a = 1$. This is essentially based on the behavior of the function

$$d(c) = E(u) - cP(u)\,. \tag{19}$$

Indeed it is expected that the solitary wave (18) is stable if $d''(c) > 0$ and unstable if $d''(c) < 0$, see [20–22]. It can be easily seen that

$$d'(c) = -P(u) = -c\int_{\mathbb{R}} |u_x|^2\,\mathrm{d}x. \tag{20}$$

3 sine-cosine-Ansätze Method

As we mentioned before, in this section sine-cosine-ansätze method will be applied to obtain new solutions of Eq. (1). The solution we looking for can be expressed in one of the following forms[27–29,36–37]

$$\begin{aligned} u(x,t) &= u(ax - ct) = u(\xi) \\ &= \lambda\cos^{\theta}(\mu\xi)\ \chi_{|\xi|\leq\frac{\pi}{2\mu}}(\xi)\,, \end{aligned} \tag{21}$$

$$\begin{aligned} u(x,t) &= u(ax - ct) = u(\xi) \\ &= \lambda\sin^{\theta}(\mu\xi)\ \chi_{|\xi|\leq\frac{\pi}{2\mu}}(\xi)\,, \end{aligned} \tag{22}$$

where λ, μ, and θ are parameters that will be determined later. So that u satisfies

$$(\alpha a^2 + c^2)u'' + \beta u'''' + \delta u + \gamma u^n = 0\,. \tag{23}$$

First, substituting the ansätze (21) into (23), we obtain

$$\begin{aligned} &\lambda[\beta\mu^4\theta^4 + \alpha a^2\mu^2\theta^2 + c^2\mu^2\theta^2 + \delta]\cos^{\theta}(\mu\xi) - \lambda\theta(\theta - 1) \\ &\times[2\beta\mu^4(\theta^2 - 2\theta + 2) + \alpha a^2\mu^2 + c^2\mu^2]\cos^{\theta-2}(\mu\xi) \\ &+\gamma\lambda^n\cos^{n\theta}(\mu\xi) + \beta\mu^4\lambda\cos^{\theta-4}(\mu\xi)\prod_{i=0}^{3}(\theta - i) = 0. \end{aligned} \tag{24}$$

Using the balance method,[27–29,36–37] by equating the exponents and the coefficients of $\cos(\cdot)-$functions and solving the resulting system, we get

$$\theta = \frac{4}{1-n}\,, \tag{25}$$

$$\mu = \frac{n-1}{2}\sqrt{\frac{\alpha a^2 + c^2}{\beta(n^2 + 2n + 5)}}\,, \tag{26}$$

$$\lambda = \Big[\frac{(c^2 + \alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2 + 2n + 5)^2}\Big]^{1/(n-1)}\,, \tag{27}$$

$$\delta = \frac{4(n+1)^2(\alpha a^2 + c^2)^2}{\beta(n^2 + 2n + 5)^2}\,. \tag{28}$$

It is necessary to have

$$\frac{\alpha a^2 + c^2}{\beta} > 0\,, \tag{29}$$

as seen from (26). One can similarly see that the same results will be obtained if one applies the sine-ansätze (22). Therefore, we obtain the following periodic solutions, when (29) holds:

$$u(x,t) = \Big[\frac{(c^2 + \alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2 + 2n + 5)^2}\Big]^{1/(n-1)}\sec^{4/(n-1)}\Big(\frac{n-1}{2}\sqrt{\frac{\alpha a^2 + c^2}{\beta(n^2 + 2n + 5)}}(ax - ct)\Big)\,, \tag{30}$$

$$u(x,t) = \Big[\frac{(c^2 + \alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2 + 2n + 5)^2}\Big]^{1/(n-1)}\csc^{4/(n-1)}\Big(\frac{n-1}{2}\sqrt{\frac{\alpha a^2 + c^2}{\beta(n^2 + 2n + 5)}}(ax - ct)\Big)\,. \tag{31}$$

The following figures 2(a), 2(b), and 2(c) show the evolution of solution (30), and Figs. 3(a), 3(b), and 3(c) present the evolution of solution (31), with $n = 3$, $\beta = 1/4$, $a = \alpha = 1$, $\delta = 16$, $\gamma = -2/\ {}^{1/}\!\sqrt[3]{30}$, $c = 2$, and $|x| \leq \pi/2$ at $t = 0, 1$ and $t = 1$ and $t = 2$ respectively.

Note that when (29) does not hold, we will obtain the following solitary wave:

$$u(x,t) = \Big[\frac{(c^2 + \alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2 + 2n + 5)^2}\Big]^{1/(n-1)}\operatorname{sech}^{4/(n-1)}\Big(\frac{n-1}{2}\sqrt{\frac{-c^2 - \alpha a^2}{\beta(n^2 + 2n + 5)}}(ax - ct)\Big)\,, \tag{32}$$

and the following traveling wave

$$u(x,t)=\Big[\frac{(c^2+\alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2+2n+5)^2}\Big]^{1/(n-1)}\operatorname{csch}^{4/(n-1)}\Big(\frac{n-1}{2}\sqrt{\frac{-c^2-\alpha a^2}{\beta(n^2+2n+5)}}(ax-ct)\Big)\,. \tag{33}$$

The wave profile for solution (32) can be imagined similar to profile of (18) (see Fig. 1); while Fig. 4 illustrates the evolution of solution (33). As seen in Fig. 4, traveling wave (33) has a singularity in $x = ct/a$.

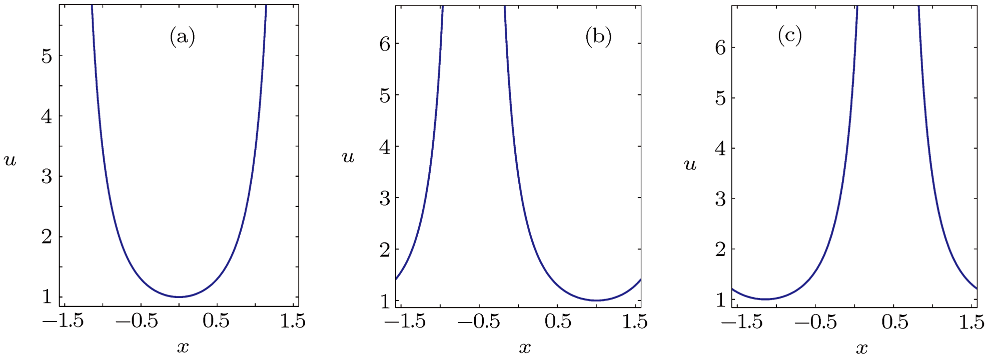

Fig. 2 The evolution of solution (30) with $n = 3$, $\beta = 1/4$, $a = \alpha = 1$, $\delta = 16$, $\gamma = -2/\ {}^{1/}\!\sqrt[3]{30}$, $c = 2$, and $|x| \leq \pi/2$ at (a) $t = 0$, (b) $t = 1$ and (c) $t = 2$.

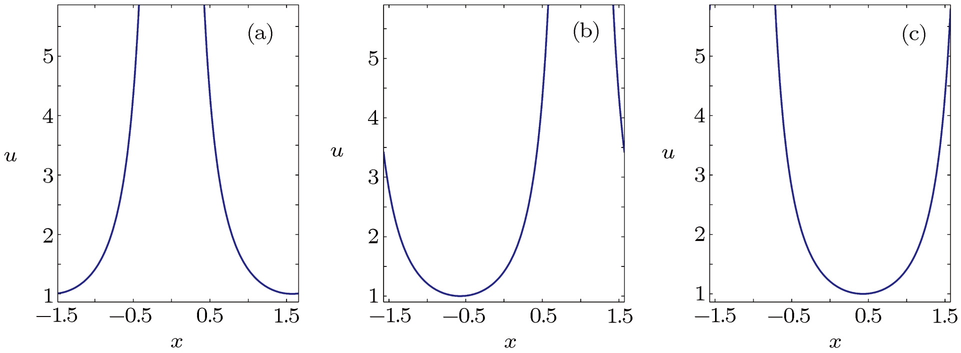

Fig. 3 The evolution of solution (31) with $n = 3$, $\beta = 1/4$, $a = \alpha = 1$, $\delta = 16$, $\gamma = -2/\ {}^{1/}\!\sqrt[3]{30}$, $c = 2$, and $|x| \leq \pi/2$ at (a) $t = 0$, (b) $t = 1$, and (c) $t = 2$.

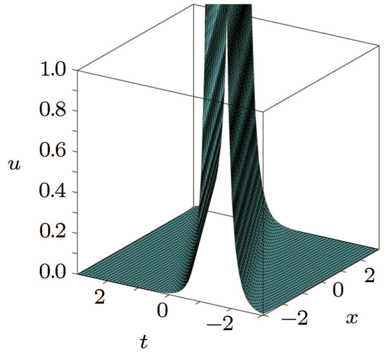

Fig. 4 The wave profile for solution (33) with $n = 3$, $\beta = 1/4$, $a = \alpha = 1$, $\delta = 16$, $\gamma = -2/\ {}^{1/}\!\sqrt[3]{30}$ and $c = 2$.

4 tanh-Ansätze Method

In this section, we will use the tanh method to handle the generalized Bretherton equation. Indeed by introducing a new independent variable[27−29,36−37]

$$\omega = \tanh(\mu\xi)\,, \tag{34}$$

there obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} =\mu^2(1-\omega^2)^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}-2\mu^2\omega(1-\omega^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\,, \tag{35}$$

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}\xi^4} =&12\mu^4\omega(1-\omega^2)\frac{\mathrm{d}}{\mathrm{d}\omega}-\mu^4(1-\omega^2)^2(8-34\omega^2)\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\\ &-12\mu^4\omega(1-\omega^2)^3\frac{\mathrm{d}^3}{\mathrm{d}\omega^3}+\mu^4(1-\omega^2)\frac{\mathrm{d}^4}{\mathrm{d}\omega^4}.\end{aligned} \tag{36}$$

By a balancing method in (24), we should use the following transformation (in order to obtain an analytic solution):

$$u = \vartheta^{2/(n-1)}\,, \tag{37}$$

so that (23) becomes

$$\begin{aligned}&\beta k\vartheta\vartheta'''' + 4\beta k\vartheta^2\vartheta'\vartheta''' + 3\beta k(k-1)\vartheta^2(\vartheta'')^2\\ &+ 6\beta k(k-1)(k-2)\vartheta(\vartheta')^2\vartheta'' + \delta\vartheta^4\\ &+ \beta k(k-1)(k-2)(k-3)(\vartheta')^4\end{aligned}$$

$$\begin{aligned}&+(\alpha a^2+c^2)k(k-1)\vartheta^2(\vartheta')^2\\&+(\alpha a^2+c^2)k\vartheta^3\vartheta''+\gamma\vartheta^6=0\,,\end{aligned}\tag{38}$$

where $k=2/(n-1)$. Using again the balance method, we need to use the following expansion

$$u=b_0+b_1\omega+b_2\omega^2\,.\tag{39}$$

Substituting (39) into (23), collecting the coefficients of ω, and solving the resulting system we find

$$b_0=-\sqrt{\frac{(c^2+\alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2+2n+5)^2}}\,,\tag{40}$$

$b_1=0$ and

$$b_2=\sqrt{\frac{(c^2+\alpha a^2)^2(n+1)(n+3)(3n+1)}{-\beta\gamma(n^2+2n+5)^2}}\,,\tag{41}$$

while

$$\mu=\frac{n-1}{2}\sqrt{\frac{-c^2-\alpha a^2}{\beta(n^2+2n+5)}}\,,\tag{42}$$

$$\delta=\frac{4(n+1)^2(\alpha a^2+c^2)^2}{\beta(n^2+2n+5)^2}\,.\tag{43}$$

Finally it is necessary to have

$$\frac{c^2+\alpha a^2}{\beta}<0\,,\tag{44}$$

as seen from (42). Using equations (40)–(43) will give the solutions (30)–(33) obtained above in Sec. 3.

5 He's Variational Method

In this section we will apply a preliminary but effective variational approach that is so-called He's Variational Method to obtain the solitary wave solutions of (1).

The starting point is the solitary wave ansätze that is given by[40]

$$u(x,t)=v(\xi),\quad \xi=ax-ct\,.\tag{45}$$

Here, c is the velocity of the solitary wave. Substituting this ansätze into Eq. (1) reduces it to the following equation

$$(\alpha a^2+c^2)v''+\beta v''''+\delta v+\gamma v^n=0\tag{46}$$

that integrates to

$$\begin{aligned}&(\alpha a^2+c^2)(v')^2-\beta(v'')^2+2\beta v'v'''\\&+\delta v^2+\frac{2\gamma}{n+1}v^{n+1}=K\,,\end{aligned}\tag{47}$$

where K is a constant. The stationary integral J is then defined as

$$\begin{aligned}J=\int_{\mathbb{R}}\Big[&(\alpha a^2+c^2)(v')^2-3\beta(v'')^2\\&+\delta v^2+\frac{2\gamma}{n+1}v^{n+1}\Big]\,\mathrm{d}\xi\,.\end{aligned}\tag{48}$$

We search for a solitary wave solution in the form of (9), viz.

$$v(\xi)=A\,\mathrm{sech}^{\,p}(\xi)\,,\tag{49}$$

where A is an unknown amplitude to be further determined. Substituting (9) into (48), we have

$$\begin{aligned}J=&\frac{(\alpha a^2+c^2)p^2A^2\kappa}{2p+1}+2\delta\kappa A^2\\&+\frac{2\gamma\kappa' A^{n+1}}{n+1}-3\beta\kappa A^2\kappa''\,,\end{aligned}\tag{50}$$

where

$$\kappa=\frac{\sqrt{\pi}\,\Gamma(p)}{\Gamma(p+1/2)},\quad \kappa'=\frac{\sqrt{\pi}\,\Gamma((np+p)/2)}{\Gamma((np+p+1)/2)},$$
$$\kappa''=p^4+\frac{4p^3(p+1)^3}{(2p+1)(2p+3)}-\frac{4p^4(p+1)}{2p+1},$$

where $\Gamma(\cdot)$ is the gamma function. Making J stationary with A results in

$$\frac{\partial J}{\partial A}=0\,,\tag{51}$$

we get[40]

$$A=\Big[\frac{3\beta\kappa\kappa''-\delta\kappa-p^2\kappa(c^2+\alpha a^2)/(2p+1)}{\gamma\kappa'(n+1)^{-1}}\Big]^{1/(n-1)}\,.\tag{52}$$

The solitary solution is, therefore, obtained as follows:

$$\begin{aligned}v(\xi)=&\Big[\frac{3\beta\kappa\kappa''-\delta\kappa-p^2\kappa(c^2+\alpha a^2)/(2p+1)}{\gamma\kappa'(n+1)^{-1}}\Big]^{1/(n-1)}\\&\times\mathrm{sech}^{\,p}(\xi)\,.\end{aligned}\tag{53}$$

6 Summary and Conclusions

This paper obtains the solitary wave solutions of the generalized Bretherton equation. The trigonometric (the sech-function, the tanh-function, the sine-cosine-function methods) and He's semi-inverse methods were successfully used to derive these solutions. Physical significance of the solutions have been presented graphically to see the propagation and asymptotic characteristics of the solitary waves (see Fig. 1). Physically the obtained solitons propagate without change of their identities and we expect to be stable against mutual collisions. Thus, these solutions will be useful in studying the soliton perturbation theory and stability studies; more precisely, because of the form of the propagation of the solitary waves, we are eager to send the information and receive them without any change; so that the soliton perturbation theory and stability analysis will be able to respond us. Various periodic traveling wave solutions have been also obtained, by the trigonometric method (see (30) and (31)). In fact, on one hand, these periodic solutions do not have compact support (compacton); and are neither solitary wave nor kink; on the other hand, they have singularity (see Figs. 2 and 3). Equation (1) also has nonperiodic traveling wave (33) with only one singularity (see Fig. 4).

We should notice that the obtained solutions in this paper may play a crucial role in applying the finite difference methods for the initial value problem associated to the generalized Bretherton equation and in numerical simulation and the understanding of solitons dynamics of the variants of (1) where they facilitate the verification of numerical solvers.

The applied methods in this paper will be used in further works to establish more entirely new solutions for other kinds of nonlinear evolution equations. In particular we will also obtain new topological solitons of the generalized Bretherton equation.

Also, some variants of the generalized Bretherton will be analyzed in viewpoint of the stochastic perturbation theory and their results will be appeared somewhere in our future studies. Moreover, existence and nonexistence of multiple soliton solutions, one-cnoidal-type and one-dnoidal-type solutions of equation (1) and solitary wave solutions of the generalized Bretherton with variable-coefficients are some of the interesting issues, which will be considered in future.

Acknowledgment

The author would also like to thank the unknown referee for many valuable suggestions and comments.

References

[1] K.M. Berger and P.A. Milewski, SIAM J. Appl. Math. **63** (2003) 1121.

[2] N.G. Berloff and L.N. Howard, Stud. Appl. Math. **100** (1998) 195.

[3] A. Biswas, Commun. Nonlinear Sci. Numer. Simulat. **14** (2009) 3226.

[4] A. Biswas, D. Milovic, and A. Ranasinghe, Commun. Nonlinear Sci. Numer. Simulat. **14** (2009) 3738.

[5] F.P. Bretherton, J. Fluid Mech. **20** (1964) 457.

[6] Z.D. Dai, C.J. Wang, S.Q. Lin, D.L. Li, and G. Mu, Nonlin. Sci. Lett. A **1** (2010) 77.

[7] A.A. Darwish and A. Ramady, Chaos, Soliton. & Fract. **33** (2007) 1263.

[8] H.M. Fu and Z.D. Dai, Int. J. Nonlin. Sci. Num. **10** (2009) 927.

[9] T.H. Hao, Int. J. Nonlin. Sci. Numer. Simul. **6** (2005) 209.

[10] J.H. He, Internat. J. Modern Phys. B **20** (2006) 1141.

[11] J.H. He, *Non-Perturbative Methods for Strongly Monlinear Problems*, dissertation. de-Verlag im Internet GmbH, Berlin (2006).

[12] J.H. He, Chaos, Solitons & Fractals **19** (2004) 847.

[13] R. Hirota and J. Satsuma, J. Phys. Soc. Jpn. **40** (1976) 611.

[14] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge (2004).

[15] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.

[16] D.D. Holm and P. Lynch, SIAM J. Appl. Dyn. Syst. **1** (2002) 44.

[17] T. Kano and T. Nishida, Osaka J. Math. **23** (1986) 389.

[18] N.A. Kudryashov, Phys. Lett. A **155** (1991) 269.

[19] A.C. Lazer and P.J. McKenna, SIAM Rev. **32** (1990) 537.

[20] S.P. Levandosky, J. Dynam. Differ. Equat. **10** (1998) 151.

[21] S.P. Levandosky, J. Differential Equations **143** (1998) 360.

[22] S.P. Levandosky and W.A. Strauss, Methods Appl. Anal. **7** (2000) 479.

[23] H.A. Levine, Trans. Amer. Math. Soc. **192** (1974) 1.

[24] J.E. Lin, Methods Appl. Anal. **11** (2004) 65.

[25] A.E.H. Love, Dover, New York (1944).

[26] V.G. Makhankov, Phys. Lett. C **35** (1978) 1.

[27] W. Malfliet, Amer. J. Phys. **60** (1992) 650.

[28] W. Malfliet, Phys. Scripta **54** (1996) 563.

[29] W. Malfliet, Phys. Scripta **54** (1996) 569.

[30] P.J. McKenna and W. Walter, Arch. Ration. Mech. Anal. **87** (1987) 167.

[31] T. Ozis and A. Yidirim, Comput. Math. Appl. **54** (2007) 1039.

[32] S. Pak, Int. J. Nonlin. Sci. Num. **10** (2009) 505.

[33] F.J. Romeiras, Appl. Math. Computat. **215** (2009) 1791.

[34] R. Sassaman and A. Biswas, Appl. Math. Computat. **215** (2009) 212.

[35] Z.L. Tao, Nonlinear Analysis: RWA **10** (2009) 1939.

[36] A.M. Wazwaz, Appl. Math. Comput. **154** (2004) 713.

[37] A.M. Wazwaz, Appl. Math. Comput. **150** (2004) 365.

[38] A.M. Wazwaz, Nonlin. Sci. Lett. A **1** (2010) 289.

[39] Y. Wu, Int. J. Nonlin. Sci. Num. **10** (2009) 1245.

[40] Y. Ye and L. Mo, Comput. Math. Appl. **58** (2009) 2420.

[41] J. Zhang, Comput. Math. Appl. **54** (2007) 1043.

[42] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, in: *London Mathematical Society Lecture Notes*, (149) Cambridge University Press, Cambridge (1991).

[43] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, Stud. Appl. Math. **53** (1974) 249.