Exact Solutions of *D*-Dimensional Schrödinger Equation for an Energy-Dependent Potential by NU Method

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Abstract *We study the D-dimensional Schrödinger equation for an energy-dependent Hamiltonian that linearly depends on energy and quadraticly on the relative distance. Next, via the Nikiforov–Uvarov (NU) method, we calculate the corresponding eigenfunctions and eigenvalues.*

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1 Introduction

Among the different forms of physical potentials which appear in the Hamiltonian, those with spherical symmetry have received great attention within the recent years because of their wide applications.[1−6]

On the other hand, as the study of many physical systems corresponds to a *D*-dimensional problem, many attempts are devoted to analyze such problems.[7−22] The other key point is that we face with energy dependent potentials in many areas of physics in both relativistic and non-relativistic quantum mechanics. For example, relativistic investigation of a point charge in an external Coulomb field through the Klein–Gordon equation leads to an energy-dependent wave equation.[23−24] In some cases, even if the initial potential is not an energy-dependent one, its reduction to a wave equation leads to an effective energy-dependent potential.[25−28] A similar case arises for a fermion in the presence of scalar and vector potentials in the Dirac equation as well as its transformed form, i.e. Pauli–Schrödinger equation.[29−30] For the Schrödinger equation, the concept of energy-dependent potential leads to some interesting consequences, a typical example being the inevitable modification of the scalar product to have a conserved norm. For the acceptable energy-dependent physical potentials, the Schrödinger equation is equivalent to the one with nonlocal potential.[27] In addition, especially for the confined systems, we observe new effects for a typical potential quite different to the consequences of its normal form. For the chosen potential, however, there is no deep fundamental reasoning, with the most common one being perhaps the interesting forms of dependence for ordinary potentials. The most notable point is perhaps that the consequent phenomenology is motivating. Namely, the energy-dependent potentials have yielded acceptable results in the annals of particle and nuclear physics.[27]

To complete the history, it is notable that the besides the Schrödinger equation in *D*-dimensional space with certain central potential has been investigated,[31−38] even the quantum gravity has been studied in multi-dimensional space.[39−42]

2 Hyperradial Part in *D* Dimensions

The motion of a particle in a spherically symmetric potential in D dimensions is written as

$$\Big\{-\frac{\hbar^2}{2\mu}\Delta_D+V(r)\Big\}\psi_{n,l,m}(r,\Omega_D)=E_{n,l}\psi_{n,l,m}(r,\Omega_D)\,,\quad(1)$$

with

$$\Delta_D\equiv\nabla_D^2=\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\Big(r^{D-1}\frac{\partial}{\partial r}\Big)-\frac{\Lambda_D^2(\Omega_D)}{r^2}\,.\quad(2)$$

As hyperspherical harmonics are eigenfunctions of the operator $\Lambda_D^2(\Omega_D)$, we could write

$$\psi_{n,l,m}(r,\Omega_D)=R_{n,l}(r)\mathrm{Y}_l^m(\Omega_D)\,.\quad(3)$$

In addition, it is well-known that $\Lambda_D^2(\Omega_D)/r^2$ is a generalization of the centrifugal barrier for the case of D-dimensional space and involves the angular coordinates Ω_D and the eigenvalues of the $\Lambda_D^2(\Omega_D)$ are given by

$$\Lambda_D^2(\Omega_D)\mathrm{Y}_l^m(\Omega_D)=l(l+D-2)\mathrm{Y}_l^m(\Omega_D)\,,\quad(4)$$

where $\mathrm{Y}_l^m(\Omega_D)$, $R_{n,l}(r)$, $E_{n,l}$, and l represent the hyperspherical harmonics, the hyperradial part, the energy eigenvalues, and the orbital angular momentum, respectively. Here, we consider a quadratic dependence on the spatial coordinate and a linear one for the energy, i.e.

$$V(r,E_{n,l})=V_0(1+\eta E_{n,l})r^2\,,\quad(5)$$

where r denotes the hyperradius and the coefficients V_0 and η are constant and we know that the hyperradial in d dimensions is

$$R''_{n,l}(r)+\frac{D-1}{r}R'_{n,l}(r)-\frac{l(l+D-2)}{r^2}R_{n,l}(r)$$

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$$+\frac{2\mu}{\hbar^2}E_{n,l}R_{n,l}(r)-\frac{2\mu}{\hbar^2}V_0(1+\eta E_{n,l})r^2R_{n,l}(r)=0\,. \tag{6}$$

In the next part, we give a brief introduction to the NU method, which we use to solve the final differential equation.

3 Nikiforov–Uvarov Method

The NU method can be used to solve second order differential equations with an appropriate coordinate transformation $s=s(r)$[31]

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0\,, \tag{7}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find the particular solution of Eq. (6) by separation of variables one deals with the transformation $\psi_n(s)=\phi(s)y_n(s)$. The latter reduces to an equation of hypergeometric type

$$\sigma(s)y_n''(s)+\tau(s)y_n'(s)+\lambda y_n(s)=0\,, \tag{8}$$

where $\phi(s)$ is defined as the logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)}\,, \tag{9}$$

with $y_n(s)$ being the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)]\,, \tag{10}$$

where B_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition[23]

$$\frac{\mathrm{d}}{\mathrm{d}s}w(s)=\frac{\tau(s)}{\sigma(s)}\,,\quad w(s)=\sigma(s)\rho(s)\,. \tag{11}$$

The function $\pi(s)$ and the parameter λ entered the methodology are given by

$$\pi(s)=\frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\Big(\frac{\sigma'-\tilde{\tau}}{2}\Big)^2-\tilde{\sigma}+k\sigma}\,, \tag{12a}$$

$$\lambda=k+\pi'(s)\,. \tag{12b}$$

In order to find the value of k, the expression under the square root must be equal to the square of the polynomial. This yields in a new eigenvalue equation

$$\lambda=\lambda_n=-n\tau'-\frac{n(n-1)}{2}\sigma''\,, \tag{13}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s)$$

with its derivative being negative.

4 Exact Solution

Performing a coordinate transformation of the form $s=r^2$ in Eq. (6),

$$\frac{D-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}=2(D-1)\frac{\mathrm{d}}{\mathrm{d}s}\,,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}=4s\frac{\mathrm{d}^2}{\mathrm{d}s^2}+2\frac{\mathrm{d}}{\mathrm{d}s}\,, \tag{14}$$

we obtain

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{D/2}{s}\frac{\mathrm{d}}{\mathrm{d}s}-\frac{l(l+D-2)}{4s^2}\Big)R_{n,l}(s)=\frac{-2\mu}{4\hbar^2 s}\Big[E_{n,l}-V_0(1+\eta E_{n,l})s\Big]R_{n,l}(s)\,. \tag{15}$$

On the other hand, introduction of the following dimensionless parameters

$$\frac{l(l+D-2)}{4}=\beta_l\,,\quad \frac{2\mu}{4\hbar^2}E_{n,l}=\varepsilon_{n,l}\,,\quad V_0(1+\eta E_{n,l})\frac{2\mu}{4\hbar^2}=\alpha_{n,l}^2 \tag{16}$$

gives

$$\frac{\mathrm{d}^2R_{n,l}(s)}{\mathrm{d}s^2}+\frac{D/2}{s}\frac{\mathrm{d}R_{n,l}(s)}{\mathrm{d}s}+\frac{1}{s^2}(-\alpha_{n,l}^2s^2+\varepsilon_{n,l}s-\beta_l)R_{n,l}(s)=0\,. \tag{17}$$

Comparing the latter with Eq. (7), indicates the correspondence

$$\tilde{\tau}(s)=\frac{D}{2}\,,\quad \sigma(s)=s\,,\quad \tilde{\sigma}(s)=-\alpha_{n,l}^2s^2+\varepsilon_{n,l}s-\beta_l\,, \tag{18}$$

and from Eq. (12a), the polynomial $\pi_{n,l}(s)$ is found to be

$$\pi_{n,l}(s)=\frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\Big(\frac{\sigma'-\tilde{\tau}}{2}\Big)^2-\tilde{\sigma}+k\sigma}=\frac{2-D}{4}\pm\sqrt{\Big(\frac{2-D}{4}\Big)^2+\alpha_{n,l}^2s^2-\varepsilon_{n,l}s+\beta_l+k_{n,l}s}\,, \tag{19}$$

where $k_{n,l}$ is determined by setting the square root equal to the polynomial square.

$$\Delta=0\rightarrow(k_{n,l}-\varepsilon_{n,l})^2-4\alpha_{n,l}^2\Big[\Big(\frac{2-D}{4}\Big)^2+\beta_l\Big]=0\,. \tag{20}$$

Now, Eq. (20) gives two values for $k_{n,l}$ as

$$k_{\pm,n,l}=\varepsilon_{n,l}\pm2\alpha_{n,l}\sqrt{\Big(\frac{D-2}{4}\Big)^2+\beta_l}\,. \tag{21}$$

Thus, Eq. (19) is

$$\pi_{n,l}(s)=\Big(\frac{2-D}{4}\Big)\pm\Big[\alpha_{n,l}s+\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\Big]\quad\text{for}\quad k_{+,n,l}=\varepsilon_{n,l}+2\alpha_{n,l}\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\,, \tag{22}$$

$$\pi_{n,l}(s)=\Big(\frac{2-D}{4}\Big)\pm\Big[\alpha_{n,l}s-\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\Big]\quad\text{for}\quad k_{-,n,l}=\varepsilon_{n,l}-2\alpha_{n,l}\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\,. \tag{23}$$

Bearing in mind the relation

$$\tau_{n,l}(s)=\tilde{\tau}(s)+2\pi_{n,l}(s) \tag{24}$$

as well as using Eqs. (24), (23), and (21), result in

$$\tau_{n,l}(s)=1\pm2\Big[\alpha_{n,l}s+\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\Big]\quad\text{for}\quad k_{+,n,l}=\varepsilon_{n,l}+2\alpha_{n,l}\sqrt{\Big(\frac{2-D}{4}\Big)^2+\beta_l}\,, \tag{25}$$

$$\tau_{n,l}(s) = 1 \pm 2\Big[\alpha_{n,l} s - \sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l}\Big] \quad \text{for} \quad k_{-,n,l} = \varepsilon_{n,l} - 2\alpha_{n,l}\sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l}\,. \tag{26}$$

To have a negative derivative of $\tau_{n,l}(s)$ and a physical eigenfunction, we pick up the negative answer in Eq. (26).[31] Reminding relations

$$\lambda_{n,l} = k_{n,l} + \pi'_{n,l}(s)\,,$$

$$\lambda = \lambda_{n,l} = -n\tau'_{n,l} - \Big[\frac{n(n-1)}{2}\Big]\sigma''\,,$$

$$\sigma'' = 0\,, \quad \tau'_{n,l} = -2\alpha_{n,l}\,,$$

we obtain

$$2n\alpha_{n,l} = \varepsilon_{n,l} - 2\alpha_{n,l}\sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l} - \alpha_{n,l}\,, \tag{27}$$

or equivalently

$$\varepsilon_{n,l} = \Big[2n + 1 + 2\sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l}\,\Big]\alpha_{n,l}\,. \tag{28}$$

Substitution of Eqs. (16) into Eq. (28), gives

$$\frac{2\mu}{4\hbar^2}E_{n,l} = \Big[2n + 1 + 2\sqrt{\Big(\frac{2-D}{4}\Big)^2 + \frac{l(l+D-2)}{4}}\Big]\sqrt{\frac{3}{2}\mu\omega^2(1+\beta E_{n,l})\frac{2\mu}{4\hbar^2}}\,, \tag{29}$$

or

$$E_{n,l} = \frac{\{2n+1+2(l/2+(D-2)/4)\}^2(V_0(2\hbar^2/\mu)\eta)}{2} \pm \frac{\sqrt{[\{2n+1+2(l/2+(D-2)/4)\}^2(V_0(2\hbar^2/\mu))\eta]^2 + 4[\{2n+1+2(l/2+(D-2)/4)\}^2(V_0(2\hbar^2/\mu))]}}{2} \tag{30}$$

for the energy eigenvalues. In the case of $\eta = 0$, we have

$$E_{n,l} = \Big\{2n + 1 + 2\Big(\frac{l}{2} + \frac{D-2}{4}\Big)\Big\}\Big(V_0\frac{2\hbar^2}{\mu}\Big)^{1/2}\,. \tag{31}$$

If we put $V_0 = (1/2)\mu\omega^2$ and consider $D = 3$, i.e. the three-dimensional space, we find

$$E_{n,l} = \Big\{2n + l + \frac{3}{2}\Big\}\hbar\omega\,. \tag{32}$$

This is the famous energy eigenvalue relation for a harmonic oscillator in three dimensions. On the other hand,

$$(\sigma\rho_{n,l}(s))' = \tau_{n,l}(s)\rho_{n,l}(s)\,, \tag{33}$$

and substitution of $\tau_{n,l}(s)$ from Eq. (26) gives

$$\sigma'\rho_{n,l}(s) + \sigma\rho'_{n,l}(s) = \rho_{n,l}(s)\Big\{1 - 2\Big(\alpha_{n,l}s - \sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l}\Big)\Big\}\,, \tag{34}$$

or equivalently

$$\frac{\mathrm{d}\rho_{n,l}(s)}{\rho_{n,l}(s)} = \Big[-2\alpha_{n,l}s + 2\sqrt{\Big(\frac{2-D}{4}\Big)^2 + \beta_l}\Big]\frac{\mathrm{d}s}{s}\,, \tag{35}$$

$$\rho_{n,l}(s) = \mathrm{e}^{-2\alpha_{n,l}s}s^{2\sqrt{((2-D)/4)^2+\beta_l}}\,. \tag{36}$$

Using Eq. (10), which is the Rodrigues relation, we find the function $y_{n,l}(s)$ as

$$y_{n,l}(s) = \mathrm{e}^{-2\alpha_{n,l}s}s^{1+2\sqrt{((2-D)/4)^2+\beta_l}}\,. \tag{37}$$

Finally, the wave function is

$$R_{n,l}(s) = \mathrm{e}^{-\alpha_{n,l}s}s^{(2-D)/4+\sqrt{((2-D)/4)^2+\beta_l}} \times L_n^{2\sqrt{((2-D)/4)^2+\beta_l}}(2\alpha_{n,l}s)\,. \tag{38}$$

Setting $r = s^2$, we find

$$R_{n,l}(r) = \mathrm{e}^{-\alpha_{n,l}r^2}r^{-(2-D)/2+2\sqrt{((2-D)/4)^2+\beta_l}} \times L_n^{2\sqrt{((2-D)/4)^2+\beta_l}}(2\alpha_{n,l}r^2)\,, \tag{39}$$

where $L_n^k(x)$ is the Laguerre polynomial as

$$L_n^k(x) = \frac{e^x x^{-k}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^{n+k})\,. \tag{40}$$

5 Conclusion

We have exactly solved the Schrödinger equation for an energy-dependent potential. Our considered potential posses a good physical interpretation, i.e. linearly depends on the energy and quadraticly on the relative distance, the familiar harmonic oscillator. We have considered the general D-dimensional space for the sake of generality. Our results seem logical since they agree with the present results on three spatial dimensions. The results can be used to test the watched data in branches of physics where an energy-dependent potential is present.

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