A Categorification of Quantum $\widehat{sl_2}$*

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Abstract *In this paper, we categorify the algebra $\boldsymbol{U}_q(\widehat{sl_2})$ with the same approach as in [A. Lauda, Adv. Math. (2010), arXiv:math.QA/0803.3652; M. Khovanov, Comm. Algebra* **11** *(2001) 5033]. The algebra $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\widehat{sl_2})$ is obtained from $\boldsymbol{U}_q(\widehat{sl_2})$ by adjoining a collection of orthogonal idempotents 1_λ, $\lambda \in P$, in which P is the weight lattice of $\boldsymbol{U}_q(\widehat{sl_2})$. Under such construction the algebra $\boldsymbol{U}$ is decomposed into a direct sum $\bigoplus_{\lambda\in P} 1_{\lambda'}U1_\lambda$. We set the collection of $\lambda \in P$ as the objects of the category $\mathcal{U}$, 1-morphisms from λ to λ' are given by $1_{\lambda'}\boldsymbol{U}1_\lambda$, and 2-morphisms are constructed by some semilinear form defined on $\boldsymbol{U}$. Hence we get a 2-category $\mathcal{U}$ from the algebra $\boldsymbol{U}_q(\widehat{sl_2})$.*

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1 Introduction

Categorification is a process of replacing set-theoretic theorems by category-theoretic analogues. It replaces sets with categories, functions with functors, and equations between functions by natural transformations between functors, which in turn should satisfy certain equations of their own called the coherence law.[1] Most progress towards a categorification of a quantum group has been achieved by categorifying their representations.[2−3] In this paper, we take a different approach to categorify quantum affine algebra $\boldsymbol{U}_q(\widehat{sl_2})$, which is similar to categorifying the quantum group $\boldsymbol{U}_q(sl_2)$ in [4]. This is possible for Lusztig's important discovery of canonical bases, which have surprising positivity and integrality propreties.[5] The existence of these bases suggests that the representation theory of quantized enveloping algebras, and even the algebras themselves, can be realized as Grothendieck rings of some higher categorical structure where every object decomposes into a direct sum of objects lifting Lusztig's canonical basis.

Similar to [4,6], we get a 2-category from categorifying the algebra $\dot{\boldsymbol{U}}$, since Lusztig's version of $\boldsymbol{U}_q(\widehat{sl_2})$ is naturally a category. $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\widehat{sl_2})$ is obtained from $\boldsymbol{U}_q(\widehat{sl_2})$ by adjoining a collection of orthogonal idempotents 1_λ, $\lambda \in P$, in which P is the weight lattice of $\boldsymbol{U}_q(\widehat{sl_2})$. This decomposes the algebra $\dot{\boldsymbol{U}}$ into a direct sum $\oplus 1_{\lambda'}\dot{\boldsymbol{U}}1_\lambda$. We set the collection of $\lambda \in P$ as the objects of the category $\mathcal{U}$, the hom sets from λ to λ' are given by $1_{\lambda'}\boldsymbol{U}1_\lambda$, composition is given by multiplication, the identity morphisms are the idempotents 1_λ, thus, it is natural to expect that a categorification of $\dot{\boldsymbol{U}}$ would have the structure of a 2-category.

In this paper, we recall the definition of quantum affine algebra and some important maps in Sec. 2, Lusztig's quantum $\widehat{sl_2}$ by adjoining a collection of orthogonal idempotents and a semilinear form on $\boldsymbol{U}$ in Sec. 3; we construct the 2-category $\mathcal{U}$ in Sec. 4.

2 Quantum Affine Algebra $\boldsymbol{U_q(\widehat{sl_2})}$

2.1 *Definition of Quantum Affine Algebra* $\boldsymbol{U_q(\widehat{sl_2})}$

For a positive integer a, we define the quantum integer

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$$

with $[0] = 1$ by convention. The quantum factorial is then $[a]! = [a][a-1]\cdots[1]$, and the quantum binomial coefficient $\left[{a \atop b}\right] = [a]!/[b]![a-b]!$, for $0 \le b \le a$.

Consider the free abelian group[7−8] on the letters $\Lambda_0, \Lambda_1, \delta$:

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta\,.$$

We call P the weight lattice, Λ_i $(i = 0, 1)$ the fundamental weights and δ the null root. Define the simple roots α_i $(i = 0, 1)$ and the element ρ by

$$\alpha_0 + \alpha_1 = \delta, \quad \Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}, \quad \rho = \Lambda_0 + \Lambda_1\,.$$

Let (h_0, h_1, d) be an ordered basis of $P^* = \mathrm{Hom}(P, \mathbb{Z})$ dual to $(\Lambda_0, \Lambda_1, \delta)$. We define the symmetric bilinear form $(,) : P \times P \to (1/2)\mathbb{Z}$ by

$$(\Lambda_0, \Lambda_1) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1\,,$$
$$(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0\,.$$

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Regarding $P^* \subset P$ via this bilinear form, we have the identifications

$$h_0 = \alpha_0, \quad h_1 = \alpha_1, \quad d = \Lambda_0 .$$

In the following definition of $\boldsymbol{U}_q(\widehat{sl_2})$ we take q to be a real variable[9−10] and $-1 < q < 0$, though in the most cases, it is sufficient to assume that $q^n \neq 1$ for $n = 1, 2, \ldots$

Definition 1 The quantum affine algebra $\boldsymbol{U} = \boldsymbol{U}_q(\widehat{sl_2})$ is an algebra with unit 1 over $\mathbb{Q}(q)$, defined on the generators e_i, f_i $(i = 0, 1)$ and q^h $(h \in P^*)$ and through the defining relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} ,$$
$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i ,$$
$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}} ,$$
$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0, \ (i \neq j) ,$$
$$f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0, \ (i \neq j) ,$$

here $t_i = q^{h_i} (i = 0, 1)$, $t_2 = q^d$.

For $a \geq 0$, define the divided powers $e_i^{(a)} = e_i^a/[a]!$ and $f_i^{(a)} = f_i^a/[a]!$. All products of elements in the set $\{e_i^{(a)}, f_i^{(a)}, t_i^{\pm 1} (i = 0, 1), t_2^{\pm 1}\}$ span an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ algebra, which is a subalgebra of $\boldsymbol{U}$ and is denoted by ${}_{\mathcal{A}}\boldsymbol{U}$.

2.2 *Some Important Algebra Maps of U*

Let $^-$ be the $\mathbb{Q}$ linear involution of $\mathbb{Q}(q)$ which maps q to q^{-1}.[4,6] We have some important algebra maps of $\boldsymbol{U}$ as follows.

The $\mathbb{Q}(q)$ antilinear algebra isomorphism $\psi : \boldsymbol{U} \to \boldsymbol{U}$ is defined as

$$\psi(e_i) = e_i, \ \psi(f_i) = f_i, \ \psi(t_\mu) = t_\mu^{-1}, \ i = 0, 1, \ \mu = 0, 1, 2 ,$$
$$\psi(fx) = \bar{f} \psi(x) \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U} .$$

The $\mathbb{Q}(q)$ linear algebra automorphism $\omega : \boldsymbol{U} \to \boldsymbol{U}$ is defined by

$$\omega(e_i) = f_i, \ \omega(f_i) = e_i, \ \omega(t_\mu) = t_\mu^{-1}, \ i = 0, 1, \ \mu = 0, 1, 2 ,$$
$$\omega(fx) = f \omega(x) \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U} ,$$
$$\omega(xy) = \omega(x) \omega(y) \text{ for } x, y \in \boldsymbol{U} ,$$

and $\omega^2 = 1$.

The $\mathbb{Q}(q)$ linear algebra antiautomorphism $\sigma : \boldsymbol{U} \to \boldsymbol{U}$ is defined by

$$\sigma(e_i) = e_i, \ \sigma(f_i) = f_i, \ \sigma(t_\mu) = t_\mu^{-1}, \ i = 0, 1, \ \mu = 0, 1, 2 ,$$
$$\sigma(fx) = f \sigma(x) \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U} ,$$
$$\sigma(xy) = \sigma(y) \sigma(x) \text{ for } x, y \in \boldsymbol{U} .$$

The $\mathbb{Q}(q)$ linear algebra antiautomorphism $\rho : \boldsymbol{U} \to \boldsymbol{U}$ is defined by

$$\rho(e_i) = q t_i f_i, \ \rho(f_i) = q t_i^{-1} e_i, \ \rho(t_\mu) = t_\mu ,$$
$$i = 0, 1, \ \mu = 0, 1, 2 ,$$
$$\rho(fx) = f \rho(x) \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U} ,$$
$$\rho(xy) = \rho(y) \rho(x) \text{ for } x, y \in \boldsymbol{U} .$$

It is clear that $\rho^2 = 1$.

We set $\tau = \psi \rho$. Then τ is a $\mathbb{Q}(q)$ linear algebra antiautomorphism $\tau : \boldsymbol{U} \to \boldsymbol{U}$, which satisfies the following equations:

$$\tau(e_i) = q^{-1} t_i^{-1} f_i = q f_i t_i^{-1}, \ \tau(f_i) = q^{-1} t_i e_i = q e_i t_i ,$$
$$\tau(t_\mu) = t_\mu^{-1}, \ i = 0, 1, \ \mu = 0, 1, 2 ,$$
$$\tau(fx) = \bar{f} \tau(x) \text{ for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U} ,$$
$$\tau(xy) = \tau(y) \tau(x) \text{ for } x, y \in \boldsymbol{U} .$$

3 Lusztig's Quantum $\widehat{sl_2}$

3.1 *Definition of Lusztig's Quantum $\widehat{sl_2}$*

The $\mathbb{Q}(q)$ algebra $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\widehat{sl_2})$ is a version of $\boldsymbol{U}_q(sl_2)$ best suited for studying representations that admit decompositions into weight spaces. It was first introduced by Beilinson, Lusztig, and MacPherson,[3] and was later generalized by Lusztig.[6] $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\widehat{sl_2})$ is a modified form of $\boldsymbol{U}$ obtained by adjoining a collection of orthogonal idempotents 1_λ for $\lambda \in P$. We recall that $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$, so we can write a λ as a triplet (m, n, l). We use λ_0 and λ_1 to denote $(2, -2, 1)$ and $(-2, 2, 0)$, respectively. The idempotents 1_λ satisfy the relation:

$$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda , \tag{1}$$

and the following relations:

$$t_0 1_\lambda = q^m 1_\lambda, \ t_1 1_\lambda = q^n 1_\lambda, \ t_2 1_\lambda = q^l 1_\lambda ,$$
$$e_0 1_\lambda = 1_{\lambda+\lambda_0} e_0 = 1_{\lambda+\lambda_0} e_0 1_\lambda ,$$
$$f_0 1_\lambda = 1_{\lambda-\lambda_0} f_0 = 1_{\lambda-\lambda_0} f_0 1_\lambda ,$$
$$e_1 1_\lambda = 1_{\lambda+\lambda_1} e_1 = 1_{\lambda+\lambda_1} e_1 1_\lambda ,$$
$$f_1 1_\lambda = 1_{\lambda-\lambda_1} f_1 = 1_{\lambda-\lambda_1} f_1 1_\lambda ,$$
$$e_0 f_0 1_\lambda - f_0 e_0 1_\lambda = [m] 1_\lambda ,$$
$$e_1 f_1 1_\lambda - f_1 e_1 1_\lambda = [n] 1_\lambda , \tag{2}$$

so that $\dot{\boldsymbol{U}}$ is spanned by the products of $e_i, f_i, 1_\lambda$, for $i = 0, 1$ and $\lambda = (m, n, l), m, n, l \in \mathbb{Z}$. Similarly, the $\mathbb{Z}[q, q^{-1}]$ subalgebra ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ is obtained by adjoining the collection of orthogonal idempotents 1_λ and satisfy

$$t_0 1_\lambda = q^m 1_\lambda, \ t_1 1_\lambda = q^n 1_\lambda, \ t_2 1_\lambda = q^l 1_\lambda ,$$
$$e_0^{(a)} 1_\lambda = 1_{\lambda+\lambda_0 a} e_0^{(a)}, \ f_0^{(a)} 1_\lambda = 1_{\lambda-\lambda_0 a} f_0^{(a)} ,$$
$$e_1^{(a)} 1_\lambda = 1_{\lambda+\lambda_1 a} e_1^{(a)}, \ f_1^{(a)} 1_\lambda = 1_{\lambda-\lambda_1 a} f_1^{(a)} , \tag{3}$$

so that ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ is spanned by the products of $e_i^{(a)}, f_i^{(a)}, 1_\lambda$, for $i = 0, 1$ and $\lambda = (m, n, l)$, $m, n, l \in \mathbb{Z}$. There are direct sum decompositions of algebras

$$\dot{\boldsymbol{U}} = \bigoplus_{\lambda,\lambda' \in P} 1_\lambda \dot{\boldsymbol{U}} 1_{\lambda'}, \qquad {}_{\mathcal{A}}\dot{\boldsymbol{U}} = \bigoplus_{\lambda,\lambda' \in P} 1_\lambda ({}_{\mathcal{A}}\dot{\boldsymbol{U}}) 1_{\lambda'} .$$

$\dot{\boldsymbol{U}}$ is a $\mathbb{Q}(q)$ algebra without unit since the infinite sum $\sum_{\lambda \in P} 1_\lambda$ is not an element in $\dot{\boldsymbol{U}}$; however, the system of idempotents $\{1_\lambda | \lambda \in P\}$ in some sense serves as a substitute for a unit. Algebras with systems of idempotents have a natural interpretation as categories. In this interpretation, $\dot{\boldsymbol{U}}$ is a category with one object λ for each

$\lambda \in P$ with homs from λ to λ' given by the abelian group $1_{\lambda'}\dot{\boldsymbol{U}}1_\lambda$. The idempotents 1_λ are the identity morphisms for this category and the composition is given by the algebra structure of $\dot{\boldsymbol{U}}$.

According to [Lusztig,[6] 25.2], there is a canonical basis $\dot{B}$ of $\dot{\boldsymbol{U}}$. For any triplet $a, b, c \in \dot{B}$, ab is the product of a and b in $\dot{\boldsymbol{U}}$, using the relations in $\dot{\boldsymbol{U}}$, we can write $ab = \sum_c m_{ab}^c c$, in which m_{ab}^c is called structure constants of $\dot{\boldsymbol{U}}$, and $m_{ab}^c \in \mathbb{N}(q, q^{-1})$.

We set

$$\psi(1_\lambda) = 1_\lambda\,, \quad \omega(1_\lambda) = 1_{-\lambda}\,, \quad \sigma(1_\lambda) = 1_{-\lambda}\,,$$
$$\rho(1_\lambda) = 1_\lambda\,, \quad \tau(1_\lambda) = 1_\lambda\,,$$

then all the algebra maps can be extended to $\dot{\boldsymbol{U}}$.

3.2 *Semilinear form on* $\dot{\boldsymbol{U}}$

Proposition 1 (Lusztig,[6] 26.1.1). There exists a $\mathbb{Q}(q)$ bilinear pairing $(,) : \dot{\boldsymbol{U}} \times \dot{\boldsymbol{U}} \to \mathbb{Q}(q)$ with the properties:

(i) $(1_{\lambda_1} x 1_{\lambda_1'}, 1_{\lambda_2} y 1_{\lambda_2'}) = 0$ for all $x, y \in \dot{\boldsymbol{U}}$ unless $\lambda_1 = \lambda_2$ and $\lambda_1' = \lambda_2'$;

(ii) $(ux, y) = (x, \rho(u)y)$ for $u \in U$ and $x, y \in \dot{\boldsymbol{U}}$;

(iii) $(1_\lambda, 1_\lambda) = 1$;

(iv) $(f_0 1_\lambda, f_0 1_\lambda) = (f_1 1_\lambda, f_1 1_\lambda) = 1/(1 - q^{-2})$;

(v) $(x, y) = (y, x)$ for all $x, y \in \dot{\boldsymbol{U}}$.

From the proposition 1, we can obtain that:

Corollary 1

$$(e_0 1_\lambda, e_0 1_\lambda) = (e_1 1_\lambda, e_1 1_\lambda) = \frac{1}{1 - q^{-2}}\,. \tag{4}$$

Proof

$$\begin{aligned}(e_0 1_\lambda, e_0 1_\lambda) &= (1_\lambda, \rho(e_0) e_0 1_\lambda) = (1_\lambda, q t_0 f_0 e_0 1_\lambda) \\ &= q^{m+1}(1_\lambda, f_0 e_0 1_\lambda) \\ &= q^{m+1}(1_\lambda, e_0 f_0 1_\lambda - [m] 1_\lambda) \\ &= q^{m+1}((1_\lambda, e_0 f_0 1_\lambda) - [m]) \\ &= q^{m+1}((\rho(e_0) 1_\lambda, f_0 1_\lambda) - [m]) \\ &= q^{2m}(f_0 1_\lambda, f_0 1_\lambda) - q^{m+1}[m] \\ &= \frac{1}{1 - q^{-2}}\,.\end{aligned}$$

Similarly to $(e_1 1_\lambda, e_1 1_\lambda)$. □

According to [Lusztig,[6] 1.4.4], we get

Corollary 2

$$\begin{aligned}(e_0^{(a)} 1_\lambda, e_0^{(a)} 1_\lambda) &= (e_1^{(a)} 1_\lambda, e_1^{(a)} 1_\lambda) \\ &= (f_0^{(a)} 1_\lambda, f_0^{(a)} 1_\lambda) \\ &= (f_1^{(a)} 1_\lambda, f_1^{(a)} 1_\lambda) \\ &= \prod_{s=1}^{a} \frac{1}{1 - q^{-2s}}\,.\end{aligned} \tag{5}$$

Definition 2 We define a semilinear form $\langle , \rangle\ \dot{\boldsymbol{U}} \times \dot{\boldsymbol{U}} \to \mathbb{Q}(q)$ by

$$\langle x, y \rangle = \overline{(x, \psi(y))} \text{ for all } x, y \in \dot{\boldsymbol{U}}\,.$$

From the above proposition this semilinear form has the following properties:[4,6]

Proposition 2 The map $\langle , \rangle : \dot{\boldsymbol{U}} \times \dot{\boldsymbol{U}} \to \mathbb{Q}(q)$ has the following properties

(i) $\langle , \rangle$ is semilinear, i.e., $\langle fx, y \rangle = \bar{f}\langle x, y \rangle$, $\langle x, fy \rangle = f\langle x, y \rangle$, for $f \in \mathbb{Q}(q)$ and $x, y \in \dot{\boldsymbol{U}}$;

(ii) Hom property: $\langle 1_\lambda x 1_\mu, 1_{\lambda'} y 1_{\mu'} \rangle = 0$ for all $x, y \in \dot{\boldsymbol{U}}$ unless $\lambda = \lambda'$ and $\mu = \mu'$;

(iii) Adjoint property: $\langle ux, y \rangle = \langle x, \tau(u) y \rangle$ for $u \in \boldsymbol{U}$ and $x, y \in \dot{\boldsymbol{U}}$;

(iv) Grassmannian property: $\langle e_i^{(a)} 1_\lambda, e_i^{(a)} 1_\lambda \rangle = \langle f_i^{(a)} 1_\lambda, f_i^{(a)} 1_\lambda \rangle = \prod_{j=1}^{a} [1/(1 - q^{2j})]$, for $i = 0, 1$;

(v) We have $\langle x, y \rangle = \langle \psi(y), \psi(x) \rangle$ for all $x, y \in \dot{\boldsymbol{U}}$.

We denote by ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ the $\mathbb{Z}[q, q^{-1}]$ subalgebra of $\dot{\boldsymbol{U}}$ spanned by products of elements in the set

$$\{e_0^{(a)} 1_\lambda, e_1^{(a)} 1_\lambda, f_0^{(a)} 1_\lambda, f_1^{(a)} 1_\lambda | a \in \mathbb{Z}_+, \lambda \in P\}\,.$$

4 The 2-Category $\mathcal{U}$

We recall some categorical preliminaries first.

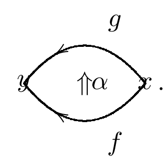
4.1 *Some Categorical Preliminaries*

Definition 3 A strict 2-category consists of

(i) objects $x, y, z, \ldots$;

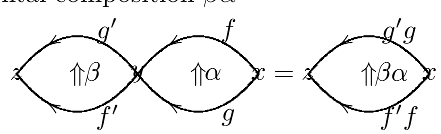
(ii) 1-morphisms between objects, $f : y \leftarrow x$;

(iii) 2-morphisms between 1-morphisms, $\alpha : f \Rightarrow g$

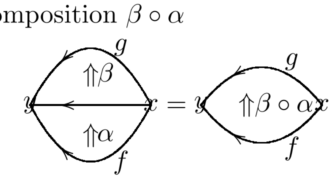


They satisfy

- horizontal composition $\beta\alpha$



- vertical composition $\beta \circ \alpha$



and all types of compositions satisfy associativity and identity axioms.

Definition 4[11] A 2-category $\mathcal{A}$ consists of

(i) a class $|\mathcal{A}|$;

(ii) for each pair x, y of elements of $|\mathcal{A}|$, a small category $\mathrm{Hom}_{\mathcal{A}}(x, y)$;

(iii) for each tripe x, y, z of elements of $|\mathcal{A}|$, a bifunctor

$$c_{xyz} : \mathrm{Hom}_{\mathcal{A}}(x, y) \times \mathrm{Hom}_{\mathcal{A}}(y, z) \to \mathrm{Hom}_{\mathcal{A}}(x, z)\,;$$

(iv) for each element $x \in |\mathcal{A}|$, a functor

$$\mu_x : \mathbf{1} \to \mathrm{Hom}_{\mathcal{A}}(x, x)\,,$$

where **1** is the terminal object of the category of small categories.

These data are required to satisfy the following axioms.

- Associativity axiom: given four elements $w, x, y, z \in |\mathcal{A}|$, the following equality holds:
$$c_{wyz} \circ (c_{wxy} \times 1) = c_{wxz} \circ (1 \times c_{xyz})\,;$$
- Unit axiom: given two elements $x, y \in |\mathcal{A}|$, the following equalities hold:
$$c_{xxy} \circ (\mu_x \times 1) \cong 1 \cong c_{xyy} \circ (1 \times \mu_y)\,.$$

In definition 4, the unit axioms require that for $\forall\, x, y \in |\mathcal{A}|$, and $\forall f : x \to y$ in $\mathrm{Hom}_{\mathcal{A}}(x, y)$, there exist isomorphisms
$$1_y \circ f \cong f, \quad \text{and} \quad f \circ 1_x \cong f\,,$$
where $1_x = \mu_x(\mathbf{1})$, $1_y = \mu_y(\mathbf{1})$. If the above two isomorphisms are replaced by identities in definition 4 and everything else is kept unchanged, then we will equivalently get definition 3.

We also recall some definitions of graded additive 2-categories[1] here.

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A $\mathcal{V}$-category $\mathcal{A}$ is a category, in which the hom sets $\mathrm{Hom}_{\mathcal{A}}(x, y)$ are objects in $\mathcal{V}$, compositions and units are given by
$$\mathrm{Hom}_{\mathcal{A}}(x, y) \otimes \mathrm{Hom}_{\mathcal{A}}(y, z) \to \mathrm{Hom}_{\mathcal{A}}(x, z)\,,$$
$$1_x : I \to \mathrm{Hom}_{\mathcal{A}}(x, x)\,.$$

A graded preadditive category $\mathcal{A}$ is a category, in which the hom set $\mathrm{Hom}(x, y)$ between any two objects $x, y \in \mathcal{A}$ is a graded abelian group, i.e.,
$$\mathrm{Hom}_{\mathcal{A}}(x, y) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_s(x, y)\,,$$
where $\mathrm{Hom}_s(x, y)$ is the abelian group of all morphisms of degree s. The composition map is degree homogeneous
$$\mathrm{Hom}_s(x, y) \otimes \mathrm{Hom}_{s'}(y, z) \to \mathrm{Hom}_{s+s'}(x, z)\,.$$

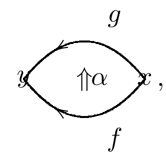
A graded additive category is a graded preadditive category with a zero object and direct sums. Here the zero object and direct sums are defined as in the ungraded case, with the additional condition that the injections and projections are homogeneous. Graded additive functors are defined similarly. We can form the monoidal category **Gr-Add-Cat** of graded additive categories, and graded additive functors. A graded additive category is said to admit translation if for any object x and integer m there is an object $x\{m\}$ with an isomorphism $x \to x\{m\}$ of degree m. Let **GAT** denote the monoidal category of graded additive categories with translations, together with graded additive functors.

Definition 5

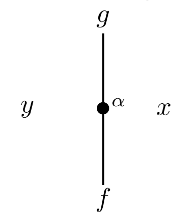
- A graded additive 2-category is a 2-category enriched in the monoidal category **Gr-Add-Cat** of graded additive categories.
- A graded additive 2-category is said to admit translation if it is a category enriched over the monoidal category **GAT** of graded additive categories that admit translation.

4.2 *String Diagrams for 2-Categories*

We use string diagrams to represent 2-morphisms between 1-morphisms.[12−15] For example, on the 2-morphism $\alpha :\ f \Rightarrow g$ in the above definition of 2-category,

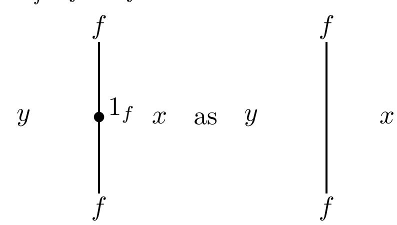


we can use the following string diagram to represent all the information in the above diagram,

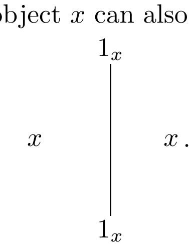


which is read from bottom to top and from right to left. Then all the information can be read immediately. Two regions labeled x, y represent two objects. Going from right to left, we pass from a region labeled x to a region labeled y, this indicates that there are two 1-morphisms $f, g : x \to y$. Reading from bottom to top, the bullet labeled by α divides the line labeled by f and g, this indicates that the 2-morphism α is a map from the 1-morphism f to the 1-morphism g.

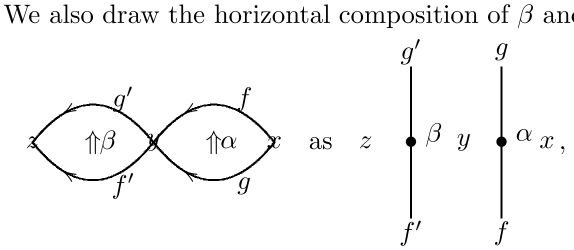
We draw the 1-morphism $f : x \to y$ and the identity 2-morphism $1_f : f \Rightarrow f$



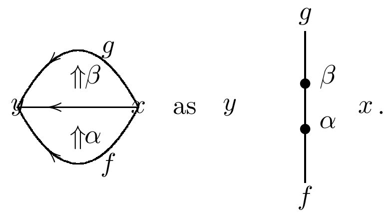
for simplicity. An object x can also be drawn as:



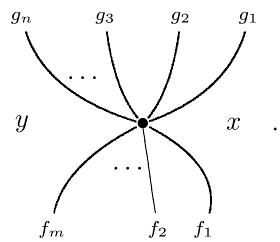
We also draw the horizontal composition of β and α



and the vertical composition of β and α



In general, a 2-morphism from the 1-morphism $f_m \cdots f_2 f_1 : x \to y$ to the 1-morphism $g_n \cdots g_2 g_1 : x \to y$ can be drawn as



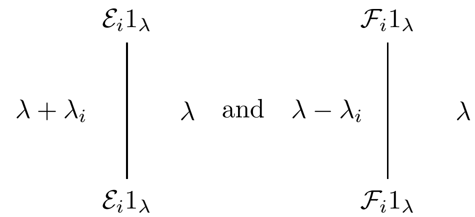
4.3 *The 2-Category $\mathcal{U}$*

$\mathcal{U}$ is a graded additive 2-category with translations. The 2-category $\mathcal{U}$ has one object λ for each $\lambda \in P$. The 1-morphisms of $\mathcal{U}$ are formal direct sums of composites of the morphisms

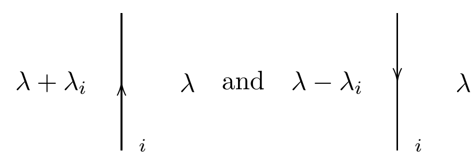
$$\begin{aligned} 1_\lambda : \quad & \lambda \to \lambda \,, \\ 1_{\lambda+\lambda_i}\mathcal{E}_i 1_\lambda : \quad & \lambda \to \lambda + \lambda_i \,, \\ 1_{\lambda-\lambda_i}\mathcal{F}_i 1_\lambda : \quad & \lambda \to \lambda - \lambda_i \,, \end{aligned} \tag{6}$$

for $i = 0, 1$, together with their shifts $\{s\}$ for $s \in \mathbb{Z}$. They correspond to 1_λ, $e_i 1_\lambda$, $f_i 1_\lambda$, $q^s 1_\lambda$, $q^s e_i 1_\lambda$, $q^s f_i 1_\lambda$ respectively. The morphisms 1_λ are the identity 1-morphisms. The morphism $1_{\lambda+\lambda_i}\mathcal{E}_i 1_\lambda$ maps λ to $\lambda + \lambda_i$. So we often simplify the notation by writing only $\mathcal{E}_i 1_\lambda$, or generically as $\mathcal{E}_i$, with it understood that $\mathcal{E}_i$ increases the subscript by λ_i, passing from right to left. Similarly, the morphism $1_{\lambda-\lambda_i}\mathcal{F}_i 1_\lambda$ maps λ to $\lambda - \lambda_i$, so we often write this morphism as $\mathcal{F}_i 1_\lambda$ or $\mathcal{F}_i$. This simplification is extended to composites as well, so that $\mathcal{E}_i\mathcal{F}_i 1_\lambda$ represents the composite $1_\lambda \mathcal{E}_i 1_{\lambda-\lambda_i} \circ 1_{\lambda-\lambda_i}\mathcal{F}_i 1_\lambda$. When no confusion is likely to arise we simplify our notation even further and simply write $\mathcal{E}_i\mathcal{F}_i$ for $\mathcal{E}_i\mathcal{F}_i 1_\lambda$.

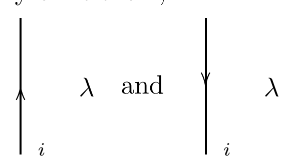
Using the string diagram calculus, we depict an object λ as the region labeled by λ. The 1-morphisms $\mathcal{E}_i 1_\lambda$ and $\mathcal{F}_i 1$ are defined as



We omit the labels of $\mathcal{E}_i 1_\lambda$ and $\mathcal{F}_i 1_\lambda$ by introducing the convention that the $\mathcal{E}_i$'s are depicted by an upward pointing arrow with i in the bottom right corner of the vertical line, the $\mathcal{F}_i$'s are depicted by a downward pointing arrow with i in the bottom right corner of the vertical line,



which are read from bottom to up and from right to left. $\lambda \pm \lambda_i$ in the diagram can be uniquely determined by the rest, so we usually omit them,



Therefore, every 1-morphism can be written as a formal direct sum of

$$1_{\lambda'}\mathcal{E}_0^{\alpha_{10}}\mathcal{F}_0^{\beta_{10}}\mathcal{E}_1^{\alpha_{11}}\mathcal{F}_1^{\beta_{11}}\cdots\mathcal{E}_0^{\alpha_{m0}}\mathcal{F}_0^{\beta_{m0}}\mathcal{E}_1^{\alpha_{m1}}\mathcal{F}_1^{\beta_{m1}}1_\lambda\{s\}\,,$$

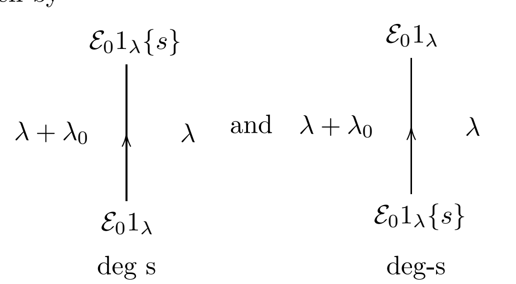
where $\lambda' = \lambda + \sum_{i=1}^m(\lambda_0(\alpha_{i0} - \beta_{i0}) + \lambda_1(\alpha_{i1} - \beta_{i1}))$ and $s \in \mathbb{Z}$. The compositions of 1-morphisms are given by

$$\begin{aligned} & 1_{\mu'}\mathcal{E}_0^{\alpha'_{10}}\mathcal{F}_0^{\beta'_{10}}\mathcal{E}_1^{\alpha'_{11}}\mathcal{F}_1^{\beta'_{11}}\cdots\mathcal{E}_0^{\alpha'_{m0}}\mathcal{F}_0^{\beta'_{m0}}\mathcal{E}_1^{\alpha'_{m1}}\mathcal{F}_1^{\beta'_{m1}}1_\mu\{s'\} \circ 1_{\lambda'}\mathcal{E}_0^{\alpha_{10}}\mathcal{F}_0^{\beta_{10}}\mathcal{E}_1^{\alpha_{11}}\mathcal{F}_1^{\beta_{11}}\cdots\mathcal{E}_0^{\alpha_{m0}}\mathcal{F}_0^{\beta_{m0}}\mathcal{E}_1^{\alpha_{m1}}\mathcal{F}_1^{\beta_{m1}}1_\lambda\{s\} \\ & \quad = \delta_{\mu,\lambda'}1_{\mu'}\mathcal{E}_0^{\alpha'_{10}}\mathcal{F}_0^{\beta'_{10}}\mathcal{E}_1^{\alpha'_{11}}\mathcal{F}_1^{\beta'_{11}}\cdots\mathcal{E}_0^{\alpha'_{m0}}\mathcal{F}_0^{\beta'_{m0}}\mathcal{E}_1^{\alpha'_{m1}}\mathcal{F}_1^{\beta'_{m1}}\,. \end{aligned}$$

We construct 2-morphisms as follows.

For each 1-morphism $x \in \mathcal{U}$, there is a degree zero identity 2-morphism $1_x : x \Rightarrow x$.

For each 1-morphism x, the isomorphism $x \simeq x\{s\}$ is given by 2-morphisms $x \Rightarrow x\{s\}$ and $x\{s\} \Rightarrow x$ of degree s and $-s$, respectively. These are represented by the identity 2-morphism together with a shift on the source or target. For example, the isomorphism $\mathcal{E}_0 1_\lambda \simeq \mathcal{E}_0 1_\lambda\{s\}$ is given by



Let x, y be two 1-morphisms in $\mathcal{U}$, which correspond to elements $[x],[y]\in\dot{\boldsymbol{U}}$. We have known that there is a semi-linear form $\langle,\rangle:\dot{\boldsymbol{U}}\times\dot{\boldsymbol{U}}\to\mathbb{Q}(q)$, with $\langle[x],[y]\rangle\in\mathbb{Q}(q)$. If $\langle[x],[y]\rangle$ is in $\mathbb{Q}(q)-\mathbb{N}[q^{-1},q]]$ or equals zero, then the set of 2-morphisms from x to y is set to $\{0\}$; if $\langle[x],[y]\rangle\neq 0$ is in $\mathbb{Q}(q)\cap\mathbb{N}[q^{-1},q]]$, then there are non-zero 2-morphisms from x to y. Assume

$$\langle[x],[y]\rangle=\sum_{j=n}^{m}a_jq^j,\ a_j\in\mathbb{N}\,.$$

Then all 2-morphisms of degree j constitute a vector space of dimension a_j. The construction is as follows.

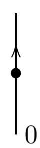
(i) Let $x=y=\mathcal{E}_0 1_\lambda$, then $[x]=[y]=e_0 1_\lambda$,

$$\langle[x],[y]\rangle=\langle e_0 1_\lambda,e_0 1_\lambda\rangle=\frac{1}{1-q^2}=1+q^2+q^4+\cdots \quad (7)$$

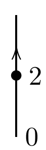
The constant term 1 in the last equation of (7) represents that the degree zero 2-morphisms from x to y span a 1-dimensional vector space. We put the basis of this vector space to be the identity 2-morphism $1_{\mathcal{E}_0 1_\lambda}:\mathcal{E}_0 1_\lambda\Rightarrow\mathcal{E}_0 1_\lambda$. We depict it by the string diagram


λ with $\deg=0$;

q^2: this corresponds to a new 2-morphism $\mathcal{E}_0 1_\lambda\Rightarrow\mathcal{E}_0 1_\lambda$,


λ with $\deg=2$;

$q^4=q^2q^2$: vertical composition of the dot with itself once,


λ with $\deg=4$;

q^{2n}, $n\in\mathbb{N}^+$:

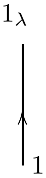

λ with $\deg=2n$.

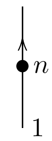
We make similar constructions to $x=y=\mathcal{E}_1 1_\lambda,\ \mathcal{F}_0 1_\lambda,\ \mathcal{F}_1 1_\lambda$, since

$$\langle e_0 1_\lambda,e_0 1_\lambda\rangle=\langle f_0 1_\lambda,f_0 1_\lambda\rangle=\langle f_1 1_\lambda,f_1 1_\lambda\rangle=\frac{1}{1-q^2}\,.$$

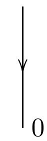
These 2-morphisms are:

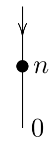
(a) $\mathcal{E}_1 1_\lambda\Rightarrow\mathcal{E}_1 1_\lambda$


λ with $\deg=0$,

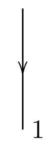

λ with $\deg=2n$;

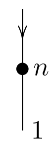
(b) $\mathcal{F}_0 1_\lambda\Rightarrow\mathcal{F}_0 1_\lambda$


λ with $\deg=0$,


λ with $\deg=2n$;

(c) $\mathcal{F}_1 1_\lambda\Rightarrow\mathcal{F}_1 1_\lambda$

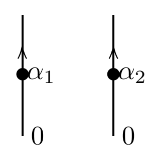

λ with $\deg=0$,


λ with $\deg=2n$.

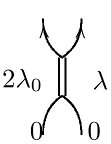
(ii) Let $x=y=\mathcal{E}_0\mathcal{E}_0 1_\lambda$,

$$\begin{aligned}\langle e_0^2 1_\lambda,e_0^2 1_\lambda\rangle&=[2]![2]!\langle e_0^{(2)}1_\lambda,e_0^{(2)}1_\lambda\rangle\\&=\frac{q^2-q^{-2}}{q-q^{-1}}\frac{q^2-q^{-2}}{q-q^{-1}}\frac{1}{1-q^2}\frac{1}{1-q^4}\\&=(1+q^{-2})\Big(\frac{1}{1-q^2}\Big)^2\\&=(1+q^{-2})(1+q^2+q^4+\cdots)\\&\quad\times(1+q^2+q^4+\cdots)\,,\end{aligned}\quad (8)$$

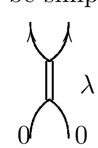
$q^{2\alpha_1}q^{2\alpha_2}$ ($\alpha_1,\ \alpha_2\in\mathbb{N}$): horizontal composition of the dot α_1 and the dot α_2, we depict it by


λ with $\deg=2\alpha_1+2\alpha_2\,.$

We introduce an additional generating 2-morphism of degree -2

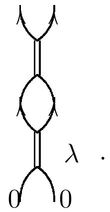
$\lambda+2\lambda_0$  λ with $\deg=-2\,,$

which can also be simplified as


λ with $\deg=-2\,.$ (9)

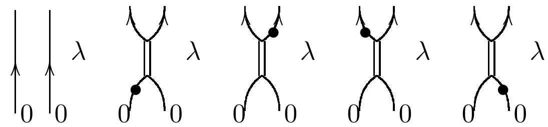
The compositions of above two kinds of 2-morphisms are all 2-morphisms $\mathcal{E}_0\mathcal{E}_0 1_\lambda\Rightarrow\mathcal{E}_0\mathcal{E}_0 1_\lambda$, since there are no 2-morphisms of degree -4, the vertical compositions of 2-

morphism in Eq. (9) with itself are zero, i.e.



$$. \qquad (10)$$

The vector space $\mathrm{Mor}(\mathcal{E}_0\mathcal{E}_0 1_\lambda, \mathcal{E}_0\mathcal{E}_0 1_\lambda)_0$ generated by 2-morphisms $\mathcal{E}_0\mathcal{E}_0 1_\lambda \Rightarrow \mathcal{E}_0\mathcal{E}_0 1_\lambda$ of degree zero is three-dimensional, therefore

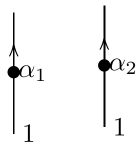


(11)

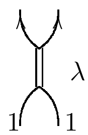
are not linearly independent.

The constructions to $x = y = \mathcal{E}_1\mathcal{E}_1 1_\lambda,\ \mathcal{F}_0\mathcal{F}_0 1_\lambda,\ \mathcal{F}_1\mathcal{F}_1 1_\lambda$ are similar.

(a) $\mathcal{E}_1\mathcal{E}_1 1_\lambda \Rightarrow \mathcal{E}_1\mathcal{E}_1 1_\lambda$,

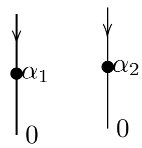


with $\deg = 2\alpha_1 + 2\alpha_2, \alpha_1, \alpha_2 \in \mathbb{N}$,

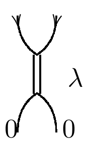


with $\deg = -2$; (12)

(b) $\mathcal{F}_0\mathcal{F}_0 1_\lambda \Rightarrow \mathcal{F}_0\mathcal{F}_0 1_\lambda$,

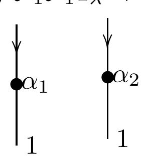


with $\deg = 2\alpha_1 + 2\alpha_2, \alpha_1, \alpha_2 \in \mathbb{N}$,

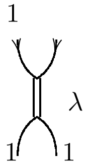


with $\deg = -2$; (13)

(c) $\mathcal{F}_1\mathcal{F}_1 1_\lambda \Rightarrow \mathcal{F}_1\mathcal{F}_1 1_\lambda$,



with $\deg = 2\alpha_1 + 2\alpha_2, \alpha_1, \alpha_2 \in \mathbb{N}$,

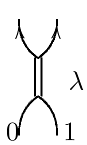


with $\deg = -2$. (14)

(iii) (a) Let $x = \mathcal{E}_0\mathcal{E}_1 1_\lambda,\ y = \mathcal{E}_1\mathcal{E}_0 1_\lambda$,

$$\langle [x],[y]\rangle = q^2(1+q^2+q^4+\cdots)(1+q^2+q^4+\cdots)$$

in which the term q^2 gives rise to a new 2-morphism, we depict it by

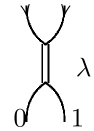


with $\deg = 2$. (15)

(ii) Let $x = \mathcal{F}_0\mathcal{F}_1 1_\lambda$, $y = \mathcal{F}_1\mathcal{F}_0 1_\lambda$,

$$\langle [x],[y]\rangle = q^2(1+q^2+q^4+\cdots)(1+q^2+q^4+\cdots)\,,$$

in which q^2 denotes a new 2-morphism, we depict it by

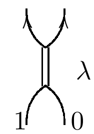


with $\deg = 2$. (16)

(c) Let $x = \mathcal{E}_1\mathcal{E}_0 1_\lambda$, $y = \mathcal{E}_0\mathcal{E}_1 1_\lambda$,

$$\langle [x],[y]\rangle = q^2(1+q^2+q^4+\cdots)(1+q^2+q^4+\cdots)\,,$$

in which q^2 denotes a new 2-morphism, we depict it by

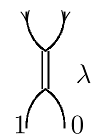


with $\deg = 2$. (17)

(d) Let $x = \mathcal{F}_1\mathcal{F}_0 1_\lambda$, $y = \mathcal{F}_0\mathcal{F}_1 1_\lambda$,

$$\langle [x],[y]\rangle = q^2(1+q^2+q^4+\cdots)(1+q^2+q^4+\cdots)\,,$$

in which q^2 denotes a new 2-morphism, we depict it by

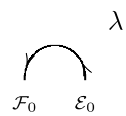


with $\deg = 2$. (18)

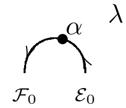
(iv) Let $x = \mathcal{F}_0\mathcal{E}_0 1_\lambda$, $y = 1_\lambda$, since

$$\begin{aligned}\langle [x],[y]\rangle &= \langle f_0 e_0 1_\lambda, 1_\lambda\rangle = \langle e_0 1_\lambda, \tau(f_0) 1_\lambda\rangle \\ &= q^{m+1}\langle e_0 1_\lambda, e_0 1_\lambda\rangle = \frac{q^{1+m}}{1-q^2} \\ &= q^{1+m}(1+q^2+q^4+\cdots)\,,\end{aligned}$$

in which q^{1+m} denotes a new 2-morphism, we depict by



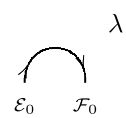
with $\deg = 1+m$,



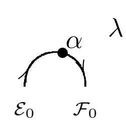
with $\deg = 1+m+2\alpha,\ \alpha \in \mathbb{N}^+$.

Similarly, we depict the following 2-morphisms:

(a) $\mathcal{E}_0\mathcal{F}_0 1_\lambda \Rightarrow 1_\lambda$,

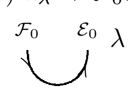


with $\deg = 1-m$,

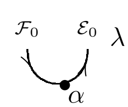


with $\deg = 1-m+2\alpha,\ \alpha \in \mathbb{N}^+$;

(b) $1_\lambda \Rightarrow \mathcal{F}_0\mathcal{E}_0 1_\lambda$,

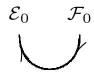


with $\deg = 1+m$,

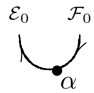


with $\deg = 1+m+2\alpha,\ \alpha \in \mathbb{N}^+$;

(c) $1_\lambda \Rightarrow \mathcal{E}_0\mathcal{F}_0 1_\lambda$,



λ with $\deg = 1-m\,,$



λ with $\deg = 1-m+2\alpha,\ \alpha\in\mathbb{N}^+\,.$

Replacing all subscript 0 in (iv) by 1, we can obtain similar 2-morphisms, the only thing we must do is to change all m to n.

Proposition 3 Let

$$A = e_0^{\alpha_{10}} f_0^{\beta_{10}} e_1^{\alpha_{11}} f_1^{\beta_{11}} \cdots e_0^{\alpha_{m0}} f_0^{\beta_{m0}} e_1^{\alpha_{m1}} f_1^{\beta_{m1}}\,,$$
$$B = e_0^{\alpha'_{10}} f_0^{\beta'_{10}} e_1^{\alpha'_{11}} f_1^{\beta'_{11}} \cdots e_0^{\alpha'_{n0}} f_0^{\beta'_{n0}} e_1^{\alpha'_{n1}} f_1^{\beta'_{n1}}\,,$$

we have

$$\langle A1_\lambda, B1_\lambda\rangle = \sum_s a_s \Pi_{i=1}^l \langle A_{is}1_{\lambda_{is}}, B_{is}1_{\lambda_{is}}\rangle\,,$$

in which A_{is}, B_{is} are all possible parts that compose A and B respectively, a_s's are the coefficients.

We take some examples to explain what the above proposition says.

Example 1 Consider

$$\langle e_1 f_1 e_0^2 1_\lambda, e_0^2 1_\lambda\rangle = q^{5-n}(1+q^{-2}+q^4)[1/(1-q^2)]^3 = \langle e_1 f_1 1_{\lambda+2\lambda_0}, 1_{\lambda+2\lambda_0}\rangle\langle e_0^2 1_\lambda, e_0^2 1_\lambda\rangle\,.$$

The above computation implies that the 2-morphism $\mathcal{E}_1\mathcal{F}_1\mathcal{E}_0^2 1_\lambda \Rightarrow \mathcal{E}_0 1_\lambda$ can be considered as the horizontal composition of $\mathcal{E}_1\mathcal{F}_1 1_{\lambda+2\lambda_0} \Rightarrow 1_{\lambda+2\lambda_0}$ and $\mathcal{E}^2 1_\lambda \Rightarrow \mathcal{E}^2 1_\lambda$.

Example 2 Consider

$$\begin{aligned}\langle e_0 f_0 e_0 f_0 e_0 1_\lambda, e_0 1_\lambda\rangle &= \langle e_0 f_0 e_0 f_0 1_{\lambda+\lambda_0}, 1_{\lambda+\lambda_0}\rangle\langle e_0 1_\lambda, e_0 1_\lambda\rangle \\ &\quad + \langle e_0 f_0 1_{\lambda+\lambda_0}, 1_{\lambda+\lambda_0}\rangle\langle e_0 1_\lambda, e_0 1_\lambda\rangle\langle f_0 e_0 1_\lambda, 1_\lambda\rangle + \langle e_0 1_\lambda, e_0 1_\lambda\rangle\langle f_0 e_0 f_0 e_0 1_\lambda, 1_\lambda\rangle\,.\end{aligned}$$

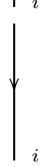
The above equation implies that the set of 2-morphisms $\mathcal{E}_0\mathcal{F}_0\mathcal{E}_0\mathcal{F}_0\mathcal{E}_0 1_\lambda \Rightarrow \mathcal{E}_0 1_\lambda$ contains three types of horizontal composition. They are the composition of $\mathcal{E}_0\mathcal{F}_0\mathcal{E}_0\mathcal{F}_0 1_{\lambda+\lambda_0} \Rightarrow 1_{\lambda+\lambda_0}$ and $\mathcal{E}_0 1_\lambda \Rightarrow \mathcal{E}_0 1_\lambda$, the composition of $\mathcal{E}_0\mathcal{F}_0 1_{\lambda+\lambda_0} \Rightarrow 1_{\lambda+\lambda_0}, \mathcal{E}_0 1_\lambda \Rightarrow \mathcal{E}_0 1_\lambda$ and $\mathcal{F}_0\mathcal{E}_0 1_\lambda \Rightarrow 1_\lambda$, the composition of $\mathcal{E}_0 1_\lambda \Rightarrow \mathcal{E}_0 1_\lambda$ and $\mathcal{F}_0\mathcal{E}_0\mathcal{F}_0\mathcal{E}_0 1_\lambda \Rightarrow 1_\lambda$.

From the above constructions, we get the following:

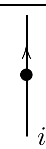
Proposition 4 $\mathcal{U}$ is a graded additive 2-category with translation consisting of

- objects: $\lambda,\ \lambda\in P$;
- 1-morphisms: formal direct sums of
  $$1_{\lambda'}\mathcal{E}_0^{\alpha_{10}}\mathcal{F}_0^{\beta_{10}}\mathcal{E}_1^{\alpha_{11}}\mathcal{F}_1^{\beta_{11}}\cdots\mathcal{E}_0^{\alpha_{m0}}\mathcal{F}_0^{\beta_{m0}}\mathcal{E}_1^{\alpha_{m1}}\mathcal{F}_1^{\beta_{m1}}1_\lambda\{s\}\,,$$
  where
  $$\lambda' = \lambda + \sum_{i=1}^m (\lambda_0(\alpha_{i0}-\beta_{i0}) + \lambda_1(\alpha_{i1}-\beta_{i1}))$$
  and $s\in\mathbb{Z}$;
- 2-morphisms: compositions of

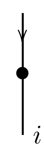
λ with $\deg = 0\,,$



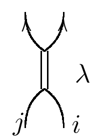
λ with $\deg = 0\,,$



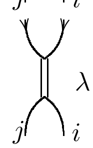
λ with $\deg = 2\,,$



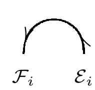
λ with $\deg = 2\,,$



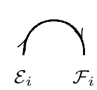
λ with $\deg = (-1)^{i+j+1}2\,,$



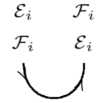
λ with $\deg = (-1)^{i+j+1}2\,,$



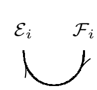
λ with $\deg = 1+a\,,$



λ with $\deg = 1-a\,,$



λ with $\deg = 1+a\,,$



λ with $\deg = 1-a\,,$

where $\lambda = (m,n,l)$, $i = 0,1$, and

$$a = \begin{cases} m, & \text{if } i = 0\,;\\ n, & \text{if } i = 1\,.\end{cases}$$

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