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Approximate Solution of D-Dimensional Klein–Gordon Equation with Hulthén-Type Potential via SUSYQM

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Abstract *Approximate analytical solutions of the D-dimensional Klein–Gordon equation are obtained for the scalar and vector general Hulthén-type potential and position-dependent mass with any l by using the concept of supersymmetric quantum mechanics (SUSYQM). The problem is numerically discussed for some cases of parameters.*

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1 Introduction

The Klein–Gordon equation has been receiving many theoretical and phenomenological attentions as it provides us with a relativistic background to study spin-zero particles. The most appealing choices for the considered potentials are perhaps the spherically symmetric ones because of their wide applications in many branches of physics including particle and nuclear physics. Many of the present works, similar to the present study, investigate the D-dimensional space for the sake of generality. Here, we consider the arbitrary dimension space and concentrate on the general Hulthén potential as it is a definitely appealing short range potential and show how the problem is solved without cumbersome algebra using the concept of SUSYQM.[1−2] For works in parallel on D-dimensional space[3−19] and references therein and relativistic wave equations one could see many papers including Refs. [19]–[30].

2 Hyperradial Part in D-Dimensions

The Klein–Gordon equation for a spherically symmetric potential in D-dimensions is

$$-\Delta_D\psi_{n,l,m}(r,\Omega_N) = \{[E_{n,l}-V(r)]^2-[m+S(r)]^2\} \times \psi_{n,l,m}(r,\Omega_D)\,, \tag{1}$$

where

$$\Delta_D \equiv \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\Big(r^{D-1}\frac{\partial}{\partial r}\Big) - \frac{\Lambda_D^2(\Omega_D)}{r^2}\,. \tag{2}$$

As the hyperspherical harmonics are eigenfunctions of operator $\Lambda_N^2(\Omega_N)$, we could write

$$\psi_{n,l,m}(r,\Omega_D) = R_{n,l}(r)\mathrm{Y}_l^m(\Omega_D)\,. \tag{3}$$

In addition, we know that $\Lambda_D^2(\Omega_D)/r^2$ is a generalization of the centrifugal barrier for the D-dimensional space and involves the angular coordinates Ω_D and the eigenvalues of the $\Lambda_D^2(\Omega_D)$ are given by

$$\Lambda_D^2(\Omega_D)\mathrm{Y}_l^m(\Omega_D) = l(l+D-2)\mathrm{Y}_l^m(\Omega_D)\,, \tag{4}$$

where $\mathrm{Y}_l^m(\Omega_D)$, $R_{n,l}(r)$, $E_{n,l}$, and l represent the hyperspherical harmonics, hyperradial part, energy eigenvalues, and orbital angular momentum, respectively. Here, we have studied potentials of the form

$$V(r) = -\frac{V_0}{\mathrm{e}^{\alpha r}-1}\,, \quad S(r) = -\frac{S_0}{\mathrm{e}^{\alpha r}-1}\,, \tag{5}$$

where r denotes the hyperradius and V_0, S_0, and α are constant coefficients. We consider a position-dependent mass of the form

$$m(r) = m_0 + \frac{m_1}{\mathrm{e}^{\alpha r}-1}\,. \tag{6}$$

On the other hand, choosing $R_{n,l}(r) = r^{-(D-1)/2}U_{n,l}(r)$ results in

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + E_{n,l}^2 + V^2(r) - 2E_{n,l}V(r) - m^2(r) - S^2(r) - 2m(r)S(r) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\Big\}U_{n,l}(r) = 0\,. \tag{7}$$

Before proceeding further, we make use of the approximation[39]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(\mathrm{e}^{\alpha r}-1)^2}\,, \tag{8}$$

which reduces our differential equation to the form

$$\Big\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_{\mathrm{eff},n,l}(r)\Big\}U_{n,l}(r) = \lambda_{n,l}U_{n,l}(r)\,, \tag{9}$$

with

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$$V_{\rm eff}(r) = \frac{-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + \frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2}{(\,{\rm e}^{\alpha r}-1)^2} - \frac{2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1}{(\,{\rm e}^{\alpha r}-1)}\,, \tag{10a}$$

$$\lambda_{n,l} = E_{n,l}^2 - m_0^2\,. \tag{10b}$$

As the latter is a Schrödinger-like equation, based on SUSY considerations, the superpotential must be the first ingredient we search for. If we choose[40]

$$\Phi(r) = \frac{-A}{{\rm e}^{\alpha r}-1} + B\,, \tag{11}$$

where

$$A = \frac{\alpha + \sqrt{\alpha^2 + 4\{-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2\}}}{2}\,,$$

$$B = \frac{(-A^2 - V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2 + 2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{2A}\,,$$

$$\lambda_{0,l} = -B^2\,,$$

and $A, B \succ 0$.[41] We find the Hamiltonian of Eq. (9) as it is related to the superpotential via[1−2]

$$V_\pm(r) = \Phi^2(r) \pm \Phi'(r)\,. \tag{12}$$

The ground state wave function $U_{0,l}(r)$ is simply calculated from

$$U_{0,l}(r) = N_{0,l}\exp\Big[-\int \Phi(r)\,{\rm d}r\Big]\,. \tag{13}$$

If the shape invariance condition exits, all desired results are directly found. The latter exists if

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1)\,, \tag{14}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F: a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ does not include the variable r. Provided that the above satisfies, everything of desire is given via the following relations[1−2]

$$H_s = -\frac{\partial^2}{\partial r^2} + V_-(a_s, r) + E_s\,, \tag{15a}$$

$$H_s\phi^-_{n-s}(a_s, r) = E_n\phi^-_{n-s}(a_s, r) \quad n \geq s\,, \tag{15b}$$

$$\phi^-_{n-s}(a_s, r) = \frac{A^\dagger}{[E_n - E_s]^{1/2}}\phi^-_{n-(s+1)}(a_{s+1}, r)\,, \tag{15c}$$

$$A_s^\dagger = -\frac{\partial}{\partial r} + \Phi(a_s, r)\,, \tag{15d}$$

$$E_n = \sum_{s=1}^{n} R(a_s)\,. \tag{15e}$$

The partner potentials are

$$\begin{aligned} V_+(r) = {} & \frac{A(A+\alpha)\,{\rm e}^{\alpha r}}{({\rm e}^{\alpha r}-1)^2} \\ & - \frac{(-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2) + (2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{{\rm e}^{\alpha r}-1} \\ & + \frac{(-A^2 - V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2 + 2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{2A}\,, \\ V_-(r) = {} & \frac{A(A-\alpha)\,{\rm e}^{\alpha r}}{({\rm e}^{\alpha r}-1)^2} - \frac{(-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + \frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2) + (2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{{\rm e}^{\alpha r}-1} \\ & + \frac{(-A^2 - V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2 + 2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{2A}\,. \end{aligned} \tag{16}$$

As it is clearly seen from the above, the partner Hamiltonians are shape invariant via a mapping of the form $A \to A+\alpha$. The energy eigenvalues of Hamiltonian $H_-(r) = -{\rm d}^2/{\rm d}r^2 + V_-(r)$ are given by $\lambda_{0,l}^{(-)} = 0$,

$$\begin{aligned} \lambda_{n,l}^{(-)} = {} & \sum_{k=1}^{n} R(a_k) \\ = {} & \left(\frac{A^2 - (-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + (1/4)(D+2l-1)(D+2l-3)\alpha^2) - (2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{2A}\right)^2 \\ & - \left(\frac{(A+n\alpha)^2 - (-V_0^2 + m_1^2 + S_0^2 - 2m_1S_0 + \frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2) - (2E_{n,l}V_0 + 2m_0S_0 - 2m_0m_1)}{2(A+n\alpha)}\right)^2. \end{aligned} \tag{17}$$

So the complete energy spectrum is given by

$$\lambda_{n,l} = \lambda_{n,l}^{(-)} + \lambda_{0,l}$$
$$= -\left(\frac{(A+n\alpha)^2 - (-V_0^2+m_1^2+S_0^2-2m_1S_0+\frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2) - (2E_{n,l}V_0+2m_0S_0-2m_0m_1)}{2(A+n\alpha)}\right)^2. \quad (18)$$

Comparing Eq. (18) with Eq. (10b), one obtains the required relativistic bound state energy spectrum

$$E_{n,l}^2 = m_0^2$$
$$-\left(\frac{(A+n\alpha)^2 - (-V_0^2+m_1^2+S_0^2-2m_1S_0+\frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2) - (2E_{n,l}V_0+2m_0S_0-2m_0m_1)}{2(A+n\alpha)}\right)^2. \quad (19)$$

In Tables 1–4 we have reported the energy values for various D, n, V_0, S_0, and ls. The results are comparable to Ref. [42]. The ground-state wave function is written as

$$U_{0,l}(r) = N_{0,l}(\,\mathrm{e}^{\alpha r}-1)^{A/\alpha}$$
$$\times \exp\left[\left\{\frac{(-A^2-V_0^2+m_1^2+S_0^2-2m_1S_0+\frac{1}{4}(D+2l-1)(D+2l-3)\alpha^2+2E_{n,l}V_0+2m_0S_0-2m_0m_1)}{2A} - A\right\}r\right]. \quad (20)$$

Table 1 Eigenvalues for $V_0 = 2$, $S_0 = 3$, $m_0 = 10$, $m_1 = 0.01$.

d	$n=0,\ l=0$	$n=1,\ l=0$	$n=1,\ l=1$	$n=2,\ l=0$	$n=2,\ l=1$	$n=2,\ l=2$
0	0.794 678	0.846 998	0.818 555	0.846 704	0.836 541	0.846 704
1	0.723 404	0.827 533	0.827 533	0.839 595	0.839 595	0.854 763
2	0.683 281	0.818 555	0.846 998	0.836 541	0.846 704	0.862 23
3	0.723 404	0.827 533	0.866 88	0.839 595	0.854 763	0.8687
4	0.794 678	0.846 998	0.883 471	0.846 704	0.862 23	0.874 192
5	0.851 318	0.866 88	0.896 565	0.854 763	0.8687	0.878 845
6	0.889 301	0.883 471	0.906 831	0.862 23	0.874 192	0.882 805
7	0.914 45	0.896 565	0.914 96	0.8687	0.878 845	0.8862
8	0.931 577	0.906 831	0.921 491	0.874 192	0.882 805	0.889 134
9	0.943 665	0.914 96	0.926 819	0.878 845	0.8862	0.891 689
10	0.952 49	0.921 491	0.931 232	0.882 805	0.889 134	0.893 932
	$n=0,\ l=0$	$n=1,\ l=0$	$n=1,\ l=1$	$n=2,\ l=0$	$n=2,\ l=1$	$n=2,\ l=2$
0	−0.709 63	−0.797 49	−0.7578	−0.809 94	−0.794 85	−0.809 94
1	−0.613 34	−0.770 29	−0.770 29	−0.799 38	−0.799 38	−0.8219
2	−0.560 53	−0.7578	−0.797 49	−0.794 85	−0.809 94	−0.832 96
3	−0.613 34	−0.770 29	−0.825 39	−0.799 38	−0.8219	−0.842 54
4	−0.709 63	−0.797 49	−0.848 74	−0.809 94	−0.832 96	−0.850 65
5	−0.788 42	−0.825 39	−0.867 18	−0.8219	−0.842 54	−0.857 51
6	−0.842 34	−0.848 74	−0.881 64	−0.832 96	−0.850 65	−0.863 34
7	−0.878 47	−0.867 18	−0.893 07	−0.842 54	−0.857 51	−0.868 32
8	−0.903 23	−0.8816 4	−0.902 23	−0.850 65	−0.863 34	−0.872 62
9	−0.920 76	−0.893 07	−0.909 69	−0.857 51	−0.868 32	−0.876 36
10	−0.933 58	−0.902 23	−0.915 85	−0.863 34	−0.872 62	−0.879 64

The higher state wavefunctions can be easily calculated from Eq. (15c). Two points are definitely worth noting about the previous sections. The first is that the position-dependent mass approach can interestingly account for the observed data in many areas of physics. The most common examples are the impurities in crystals,[43−45] the dependence of nuclear force on the relative velocity of the two nucleons,[46−47] electronic properties of quantum wells, dots and liquids,[48−49] 3He clusters,[50] and semiconductor heterostructures.[51] In some cases, the specific form of dependence may be attractive to overcome the mathematical complexity to find the eigenvalues and eigenfunctions.[52−56] In addition, just as the case used within the present case, the special forms of exponential dependence can even represent the decay of a particle.[57−58] The second one is on the consequent physical interpretation of equal and opposite vector and scalar potentials. The choice $S = +V(S = -V)$ corresponds to the negative (positive) energy states and the nonrelativistic limit of the case $S = -V$ results in a trivial non-interacting theory. For the case $S = +V$, the consequences include the singular structure of the potential, the missing negative energy spectrum, and the unexpected close affinity of the Klein–Gordon equation to the Dirac equation in the presence of interaction.[59]

Table 2 Eigenvalues for $V_0 > S_0$, $S_0 = 0.03$, $V_0 = 0.02$, $m_0 = 1$, $m_1 = 0.2$, $\alpha = 0.15$.

d	$n = 0$, $l = 0$	$n = 1$, $l = 0$	$n = 1$, $l = 1$	$n = 2$, $l = 0$	$n = 2$, $l = 1$	$n = 2$, $l = 2$
0	0.813 781	0.858 357	0.832 329	0.855 398	0.846 323	0.855 398
1	0.748 081	0.840 565	0.840 565	0.849 054	0.849 054	0.862 572
2	0.710 722	0.832 329	0.858 357	0.846 323	0.855 398	0.869 211
3	0.748 081	0.840 565	0.876 46	0.849 054	0.862 572	0.874 96
4	0.813 781	0.858 357	0.891 528	0.855 398	0.869 211	0.879 84
5	0.865 438	0.876 46	0.903 407	0.862 572	0.874 96	0.883 976
6	0.899 836	0.891 528	0.912 717	0.869 211	0.879 84	0.887 496
7	0.922 523	0.903 407	0.920 089	0.874 96	0.883 976	0.890 516
8	0.937 941	0.912 717	0.926 015	0.879 84	0.887 496	0.893 126
9	0.948 811	0.920 089	0.930 854	0.883 976	0.890 516	0.895 401
10	0.956 746	0.926 015	0.934 862	0.887 496	0.893 126	0.897 399
	$n = 0$, $l = 0$	$n = 1$, $l = 0$	$n = 1$, $l = 1$	$n = 2$, $l = 0$	$n = 2$, $l = 1$	$n = 2$, $l = 2$
0	−0.686 05	−0.783 99	−0.740 96	−0.800 19	−0.783 67	−0.800 19
1	−0.582 62	−0.754 52	−0.754 52	−0.788 64	−0.788 64	−0.813 24
2	−0.526 09	−0.740 96	−0.783 99	−0.783 67	−0.800 19	−0.8253
3	−0.582 62	−0.754 52	−0.814 18	−0.788 64	−0.813 24	−0.835 71
4	−0.686 05	−0.783 99	−0.839 41	−0.800 19	−0.8253	−0.844 53
5	−0.771 03	−0.814 18	−0.859 33	−0.813 24	−0.835 71	−0.851 98
6	−0.829 37	−0.839 41	−0.874 92	−0.8253	−0.844 53	−0.8583
7	−0.868 54	−0.859 33	−0.887 25	−0.835 71	−0.851 98	−0.8637
8	−0.895 42	−0.874 92	−0.897 12	−0.844 53	−0.8583	−0.868 36
9	−0.914 46	−0.887 25	−0.905 16	−0.851 98	−0.8637	−0.872 41
10	−0.928 38	−0.897 12	−0.911 79	−0.8583	−0.868 36	−0.875 96

Table 3 Eigenvalues for $V_0 = S_0 = 0.02$, $m_0 = 1$, $m_1 = 0.2$, $\alpha = 0.15$.

d	$n = 0$, $l = 0$	$n = 1$, $l = 0$	$n = 1$, $l = 1$	$n = 2$, $l = 0$	$n = 2$, $l = 1$	$n = 2$, $l = 2$
0	0.810 137	0.857 717	0.830 263	0.855 922	0.846 525	0.855922
1	0.740 061	0.839 061	0.839 061	0.849 385	0.849 385	0.863 192
2	0.699 248	0.830 263	0.857 717	0.846 525	0.855 922	0.869 846
3	0.740 061	0.839 061	0.876 327	0.849 385	0.863 192	0.875 569
4	0.810 137	0.857 717	0.891 616	0.855 922	0.869 846	0.880 408
5	0.863 912	0.876 327	0.903 578	0.863 192	0.875 569	0.8845
6	0.899 169	0.891 616	0.912 911	0.869 846	0.880 408	0.887 979
7	0.922 212	0.903 578	0.920 282	0.875 569	0.8845	0.890 96
8	0.937 789	0.912 911	0.926 197	0.880 408	0.887 979	0.893 536
9	0.948 735	0.920 282	0.931 021	0.8845	0.890 96	0.895 78
10	0.956 708	0.926 197	0.935 015	0.887 979	0.893 536	0.897 751
	$n = 0$, $l = 0$	$n = 1$, $l = 0$	$n = 1$, $l = 1$	$n = 2$, $l = 0$	$n = 2$, $l = 1$	$n = 2$, $l = 2$
0	−0.7254	−0.808 82	−0.769 53	−0.819 64	−0.805 14	−0.819 64
1	−0.628 37	−0.782 09	−0.782 09	−0.809 55	−0.809 55	−0.830 85
2	−0.5734	−0.769 53	−0.808 82	−0.805 14	−0.819 64	−0.841 09
3	−0.628 37	−0.782 09	−0.8356	−0.809 55	−0.830 85	−0.849 88
4	−0.7254	−0.808 82	−0.857 64	−0.819 64	−0.841 09	−0.857 29
5	−0.802 12	−0.8356	−0.874 89	−0.830 85	−0.849 88	−0.863 54
6	−0.853 43	−0.857 64	−0.888 33	−0.841 09	−0.857 29	−0.868 85
7	−0.887 34	−0.874 89	−0.898 92	−0.849 88	−0.863 54	−0.873 38
8	−0.9104	−0.888 33	−0.9074	−0.857 29	−0.868 85	−0.877 29
9	−0.926 64	−0.898 92	−0.9143	−0.863 54	−0.873 38	−0.880 69
10	−0.938 48	−0.9074	−0.919 99	−0.868 85	−0.877 29	−0.883 67

Table 4 Eigenvalues for $V_0 = -S_0$, $S_0 = 0.02$, $V_0 = -0.02$, $m_0 = 1$, $m_1 = 0.2$, $\alpha = 0.15$.

d	$n=0,\ l=0$	$n=1,\ l=0$	$n=1,\ l=1$	$n=2,\ l=0$	$n=2,\ l=1$	$n=2,\ l=2$
0	0.709 633	0.797 486	0.757 799	0.809 936	0.794 846	0.809 936
1	0.613 337	0.770 292	0.770 292	0.799 381	0.799 381	0.821 895
2	0.560 526	0.757 799	0.797 486	0.794 846	0.809 936	0.832 964
3	0.613 337	0.770 292	0.825 39	0.799 381	0.821 895	0.842 539
4	0.709 633	0.797 486	0.848 739	0.809 936	0.832 964	0.850 653
5	0.788 424	0.825 39	0.867 184	0.821 895	0.842 539	0.857 513
6	0.842 338	0.848 739	0.881 638	0.832 964	0.850 653	0.863 339
7	0.878 465	0.867 184	0.893 066	0.842 539	0.857 513	0.868 324
8	0.903 229	0.881 638	0.902 229	0.850 653	0.863 339	0.872 624
9	0.920 762	0.893 066	0.909 69	0.857 513	0.868 324	0.876 362
10	0.933 579	0.902 229	0.915 852	0.863 339	0.872 624	0.879 638
	$n=0,\ l=0$	$n=1,\ l=0$	$n=1,\ l=1$	$n=2,\ l=0$	$n=2,\ l=1$	$n=2,\ l=2$
0	−0.794 68	−0.847	−0.818 56	−0.8467	−0.836 54	−0.8467
1	−0.7234	−0.827 53	−0.827 53	−0.8396	−0.8396	−0.854 76
2	−0.683 28	−0.818 56	−0.847	−0.836 54	−0.8467	−0.862 23
3	−0.7234	−0.827 53	−0.866 88	−0.8396	−0.854 76	−0.8687
4	−0.794 68	−0.847	−0.883 47	−0.8467	−0.862 23	−0.874 19
5	−0.85132	−0.866 88	−0.896 57	−0.854 76	−0.8687	−0.878 85
6	−0.8893	−0.883 47	−0.906 83	−0.862 23	−0.874 19	−0.882 81
7	−0.914 45	−0.896 57	−0.914 96	−0.8687	−0.878 85	−0.8862
8	−0.931 58	−0.906 83	−0.921 49	−0.874 19	−0.882 81	−0.889 13
9	−0.943 67	−0.914 96	−0.926 82	−0.878 85	−0.8862	−0.891 69
10	−0.952 49	−0.921 49	−0.931 23	−0.882 81	−0.889 13	−0.893 93

3 Conclusion

We have obtained approximate analytical solutions to the Klein–Gordon equation for the Hulthén potential using the concept of SUSYQM. As illustrated in text, the solutions are simply found via the mentioned technique without much difficulty. The results are applicable to many branches of physics.

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