Commun. Theor. Phys. (Beijing, China) 34 (2000) pp 189–192
© International Academic Publishers Vol. 34, No. 1, July 30, 2000

Infinitely Many Lax Pairs of the Korteweg-de Vries Equation

YU Jun
Department of Physics, Shaoxing College of Arts and Sciences, Shaoxing 312000, Zhejiang Province, China

(Received February 29, 2000)

Abstract *Starting from a known Lax pair, one can get some infinitely many coupled Lax pairs. In this letter, we take the well-known KdV equation as a typical example. Using infinitely many symmetries, the infinitely many inhomogeneous linear Lax pairs of KdV equation can be obtained. And considering the Darboux transformations for the KdV equation leads to the infinitely many inhomogeneous nonlinear Lax pairs.*

PACS numbers: 02.30.Jr, 11.10.Lm
Key words: Lax pairs, Darboux transformation, KdV equation

I. Introduction

Soliton systems, such as the Korteweg-de Vries (KdV), sine-Gordon, and nonlinear Schrödinger equations, have been found to enjoy many remarkable properties. For example, they may be solvable via inverse scattering transformation, have an infinite number of conservation laws (symmetries), have bi-Hamiltonian structure and Lax pairs, etc. In particular, the Lax representation has played an important role in our understanding of integrable models.[1–4] The recent study shows that infinitely many nonlocal symmetries of various integrable models are related to their Lax pairs.[5–7] It is known that for an integrable model, its Lax pairs are not unique. Therefore, how to find all the Lax pairs for a given integrable model is an interesting problem. On the other hand, the Darboux transformation (DT) provides a powerful tool to construct solutions for a partial differential equation.[8–10] Using the DT method we can obtain new solutions from old solutions. Recently, the DTs have been used to find nonlocal symmetries of integrable nonlinear models which include the KdV and Kadomtsev–Patviashvili equations.[11]

The main purpose of this letter is to show how infinitely many inhomogeneous linear and nonlinear Lax pairs can be obtained by means of the infinitely many symmetries and by using the DT respectively. To be explicit, we would like to take the most famous soliton equation — KdV equation as an example. The letter is arranged as follows. The next section includes two theorems on some inhomogeneous linear Lax pairs of the KdV equation. In Sec. III, taking into account the DT of the KdV equation we can derive some inhomogeneous nonlinear Lax pairs. The last section is a brief summary.

II. Inhomogeneous Linear Lax Pairs

The KdV equation is

$$u_t - 6uu_x + u_{xxx} = 0\,, \tag{1}$$

where the subscripts represent derivatives. The integrabilities of Eq. (1), such as the Lax pairs, Bäcklund transformation, N-soliton solutions, infinite conservation laws, Painlevé property, and so on, have been studied extensively.[1,2,12] It is well known that equation (1) possesses the following Lax pairs

$$\psi_{xx} - u\psi + \lambda\psi \equiv L_1\psi = 0\,, \tag{2}$$

$$\psi_t + 4\psi_{xxx} - 6u\psi_x - 3u_x\psi \equiv L_2\psi = 0\,. \tag{3}$$

For Eqs (2) and (3), we have

Proposition 2.1. If the functions f, g are related by

$$L_2 f = L_1 g\,, \tag{4}$$

then the KdV equation (1) possesses the following inhomogeneous Lax pairs

$$L_1 \psi_f = f\,, \tag{5}$$

$$L_2 \psi_f = g\,. \tag{6}$$

Proof. Acting L_2 on Eq. (5) and L_1 on Eq. (6), we have

$$L_2 L_1 \psi_f = L_2 f\,. \tag{7}$$

$$L_1 L_2 \psi_f = L_1 g\,. \tag{8}$$

Combining Eqs (7) and (8) will lead to

$$[L_1, L_2]\psi_f \equiv (L_1 L_2 - L_2 L_1)\psi_f \equiv L_1 g - L_2 f = 0\,. \tag{9}$$

The relation (4) has been used in the last step of Eq. (9) that is obviously true because the KdV equation (1) is equivalent to $[L_1, L_2] = 0$. Thus we have completed the proof of Proposition 2.1.

It is easy to see that there are infinitely many inhomogeneous Lax pairs which have the forms (5) and (6) because one of the functions f and g may be arbitrary while the other is given by Eq. (4). To give a detailed discussion, we now turn attention to the following proposition.

Proposition 2.2. If σ_i is a symmetry of the KdV equation (1) and the spectral function ψ satisfies the homogeneous Lax pairs (2) and (3), then

$$L_1 \psi_i = \sigma_i \psi\,, \tag{10}$$

$$L_2 \psi_i = 6\sigma_i \psi_x + 3\sigma_{ix}\psi \tag{11}$$

is an inhomogeneous Lax pair of the KdV equation (1).

Proof. A symmetry of the KdV equation is defined as a solution of the linearized equation of Eq. (1),

$$\sigma_t + \sigma_{xxx} - 6\sigma u_x - 6u\sigma_x = 0\,. \tag{12}$$

That means the KdV equation (1) is form-invariant under the transformation

$$u \to u + \varepsilon\sigma\,, \tag{13}$$

where ε is an infinitesimal parameter. Substituting $f = \sigma_i\psi$ and $g = 6\sigma_i\psi_x + 3\sigma_{ix}\psi$ into Eq. (4), we have

$$\begin{aligned}(\sigma_i\psi)_t + 4(\sigma_i\psi)_{xxx} - 6u(\sigma_i\psi)_x - 3u_x(\sigma_i\psi)\\ = (6\sigma_i\psi_x + 3\sigma_{ix}\psi)_{xx} + (\lambda - u)(6\sigma_i\psi_x + 3\sigma_{ix}\psi)\,.\end{aligned} \tag{14}$$

Using the Lax pairs (2) and (3), a straightforward calculation yields

$$(\sigma_{it} + \sigma_{ixxx} - 6\sigma_i u_x - 6u\sigma_{ix})\psi = 0\,. \tag{15}$$

Comparing Eq. (15) with the symmetry definition equation (12), we have proved the Proposition 2.2.

It must be emphasized that, by using Proposition 2.2., we will obtain a corresponding inhomogeneous linear Lax pair for every symmetry. For the KdV equation, there are infinitely many symmetries,[13,14] so its infinitely many kinds of inhomogeneous linear Lax pairs can be found.

III. Nonlinear Lax Pairs from the DT

Almost all the known integrable models possess linear Lax pairs. On the one hand, because of the linearity of the Lax pairs, one can get some DTs. On the other hand, the linear Lax pairs can also be nonlinearized by using DT.[11]

For the KdV equation (1), the DT theorem reads[8]

Proposition 3.1. Let U be a solution of the KdV equation (1), where ψ satisfies Eqs (2) and (3), then $U = u - 2(\partial^2/\partial x^2)(\ln\psi)$ is a solution of Eq. (1).

Proposition 3.2. The KdV equation (1) possesses the following coupled nonlinear Lax pairs

$$\phi_{xx} - (u + 2(\ln\phi)_{xx})\phi + \lambda\phi = 0\,, \tag{16}$$

$$\phi_t + 4\phi_{xxx} - 6(u + 2(\ln\phi)_{xx})\phi_x - 3(u + 2(\ln\phi)_{xx})_x\phi = 0\,, \tag{17}$$

$$\tilde{\phi}_{xx} - (u + 2(\ln\phi)_{xx})\tilde{\phi} + \lambda\tilde{\phi} = 0\,, \tag{18}$$

$$\tilde{\phi}_t + 4\tilde{\phi}_{xxx} - 6(u + 2(\ln\phi)_{xx})\tilde{\phi}_x - 3(u + 2(\ln\phi)_{xx})_x\tilde{\phi} = 0\,. \tag{19}$$

Proof. Let ψ_1 be a solution of spectral function in the linear Lax pairs (2) and (3). From the Proposition 3.1, we know that $U = u - 2(\ln\psi_1)_{xx}$ is also a solution of the KdV equation (1). Substituting $u = U + 2(\ln\psi_1)_{xx}$ into Eqs (2) and (3), we have

$$\psi_{xx} - (U + 2(\ln\psi_1)_{xx})\psi + \lambda\psi = 0\,, \tag{20}$$

$$\psi_t + 4\psi_{xxx} - 6(U + 2(\ln\psi_1)_{xx})\psi_x - 3(U + 2(\ln\psi_1)_{xx})_x\psi = 0\,. \tag{21}$$

Now taking $\psi_1 = \psi = \phi$ and replacing the notation U by u finishes the proof of Eqs (16) and (17). In the same way, the choices $\psi_1 = \phi$, $\psi = \tilde{\phi}$ and replacement $U \to u$ lead to the proof of Eqs (18) and (19).

Furthermore, the inhomogeneous linear Lax pairs obtained in the last section can also be nonlinearized by means of the DT. For instance, the coupled Lax pairs (10) and (11) and/or (2) and (3) can be nonlinearized as

$$L_1(u + 2(\ln\phi)_{xx})\phi_i = \sigma_i(u + 2(\ln\phi)_{xx})\phi\,, \tag{22}$$

$$L_2(u + 2(\ln\phi)_{xx})\phi_i = 6\sigma_i(u + 2(\ln\phi)_{xx})\phi_x + 3\sigma_{ix}(u + 2(\ln\phi)_{xx})\phi\,. \tag{23}$$

Besides, the DT can also be used for any times. For the KdV equation, the N-times using DT reads[8]

Proposition 3.3. If $\psi_1, \psi_2, \cdots, \psi_N$ are independent linearly solutions of Lax pairs (2) and (3) and u is a solution of the KdV equation (1), then

$$U = u - 2\frac{\partial^2}{\partial x^2}\ln W \tag{24}$$

with the usual Wronskian determinant W of N functions $\psi_1, \psi_2, \cdots, \psi_N$,

$$W \equiv W(\psi_1, \psi_2, \cdots, \psi_N) = \det A, \qquad A_{ij} = \frac{\partial^{i-1}}{\partial x^{i-1}}\psi_j \qquad (i, j = 1, 2, \cdots, N)\,, \tag{25}$$

is also a solution of the KdV equation (1).

Proposition 3.4. The KdV equation (1) possesses the following coupled nonlinear Lax pairs

$$\phi_{ixx} - (u + 2(\ln W)_{xx})\phi_i + \lambda\phi_i = 0, \qquad i = 1, 2, \cdots, N, \tag{26}$$

$$\phi_{it} + 4\phi_{ixxx} - 6(u + 2(\ln W)_{xx})\phi_{ix} - 3(u + 2(\ln W)_{xx})_x\phi_i = 0, \tag{27}$$

$$\tilde{\phi}_{ixxx} - (u + 2(\ln W)_{xx})\tilde{\phi}_i + \lambda\tilde{\phi}_i = 0, \qquad i = 1, 2, \cdots, N, \tag{28}$$

$$\tilde{\phi}_{it} + 4\tilde{\phi}_{ixxx} - 6(u + 2(\ln W)_{xx})\tilde{\phi}_{ix} - 3(u + 2(\ln W)_{xx})_x\tilde{\phi}_i = 0, \tag{29}$$

where $W = W(\phi_1, \phi_2, \cdots, \phi_N)$.

The calculation involved here is rather tedious. Since the idea employed is similar to the one used for the last proposition, we omit the proof here.

IV. Conclusion

We have shown that for an integrable model, if there is one homogeneous linear Lax pair, then starting from this Lax pair we get infinitely many inhomogeneous Lax pairs, both linear and nonlinear, by using symmetries and DT respectively. Some kinds of concrete infinitely many linear and nonlinear Lax pairs of the KdV equation are also given.

It is well known that the study of integrabilities of a nonlinear model is an important subject in soliton theory. In fact, infinitely many inhomogeneous Lax pairs also reveal some aspects of the integrability properties of the model. We would like to indicate that some types of the infinitely many nonlocal symmetries can be obtained by using the infinitely many Lax pairs (See Ref. [11]) and that similar results for other interesting models will be left to the sequel.

Acknowledgement

The author would like to thank Prof. LOU SenYue for his helpful discussions.

References

[1] M.J. Ablowitz and H. Segur, *Soliton and Inverse Scattering Transformation*, SIAM, Philadelphia (1981).

[2] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Heidelberg (1987).

[3] A.C. Newell, *Soliton in Mathematics and Physics*, SIAM, Philadelphia (1985).

[4] C.H. GU, B.L. GUO, Y.S. LI, *et al.*, *Soliton Theory and Its Applications*, Zhejiang Publishing House of Science and Technology, Hangzhou (1990).

[5] S.Y. LOU, J. Math. Phys. **35** (1994) 2390.

[6] S.Y. LOU, Phys. Lett. **A187** (1994) 239.

[7] S.Y. LOU, Commun. Theor. Phys. (Beijing, China) **25** (1996) 365.

[8] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer, New York (1990).

[9] Z.X. ZHOU, Phys. Lett. **A195** (1994) 339.

[10] Q.P. LIU, Lett. Math. Phys. **35** (1995) 115.

[11] S.Y. LOU and X.B. HU, J. Phys. A: Math. Gen. **30** (1997) L95.

[12] F. Calogero and A. Degiasperis, *Spectral Transform and Solitons*, North-Holland, Amsterdam (1982).

[13] K.M. Tamizhmami, A. Ramani and B. Grammaticos, J. Math. Phys. **32** (1991) 10.

[14] S.Y. LOU, J. Math. Phys. **35** (1994) 5.