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General Decomposition of Spin Connection and Topological Structure in Gauss-Bonnet-Chern Density*

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Abstract *By means of devices of geometric algebra the general decomposition of spin connection on the sphere bundle of a compact n-dimensional Riemannian manifold has been studied in detail. Using this decomposition theory it is shown that the Gauss–Bonnet–Chern density of the Euler–Poincaré characteristic can be expressed in terms of a smooth vector field* $\vec{\phi}$ *and take the form of the* δ *function* $\delta(\vec{\phi})$*. The topological structure in Gauss–Bonnet–Chern theorem is reviewed.*

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I. Introduction

The Gauss–Bonnet–Chern (GBC) theorem is of great importance to both differential geometry and theoretical physics.[1,2] This theorem relates the curvature of the compact and oriented even-dimensional Riemannian manifold M with an important topological invariant, the Euler–Poincaré characteristic $\chi(M)$ of M. An elegant intrinsic proof of the theorem was given by Chern,[3,4] whose instructive idea was to work on the sphere bundle $S(M)$ rather than on M, or exactly on the individual fibre of $S(M)$. A summary and some historical comments on GBC theorem are given by Kabayashi, Nomizu,[5] and Spivak[6] respectively. Recently, a detailed review of Chern's proof of the GBC theorem is presented in Ref. [6].

Using a special decomposition of spin connections (gauge potential) of the group SO(n) in a previous work,[8] the GBC density (the Euler–Poincaré characteristic $\chi(M)$ density) can take the form of the δ function $\delta(\vec{\phi})$, which leads to that only the zeros of a smooth vector field $\vec{\phi}$ (a section of vector bundle) on manifold M contribute to $\chi(M)$. But we must point out here that the decomposition formula of spin connections in the previous paper[8] was supposed only in a special gauge condition.

In this paper we will study a general decomposition theory of spin connections for an SO(n) gauge theory in terms of the unit vector field $\vec{n}$ by means of devices of geometric algebra and give a general decomposition formula with a global property. The same result of Ref. [8] can be rigorously obtained.

This paper is arranged as follows. In Sec. II we will study a general decomposition theory of spin connections in SO(n) gauge theory on sphere bundle. In Sec. III we discuss the Chern formula on the sphere bundle and express it completely in terms of the unit vector $\vec{n}$ using the general decomposition of the spin connections. In Sec. IV we investigate the topological structure of the GBC density.

II. Sphere Bundle and General Decomposition Theory of Spin Connections

The intrinsic proof of the Gauss–Bonnet–Chern theorem of Euler–Poincaré characteristic $\chi(M)$ of the Riemannian manifold M given by Chern[3,4] made use of a unit vector field on

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M with only a finite number of isolated singular points via introducing a sphere bundle[9,10] $S(\underline{M})$ over M. On the other hand it must be kept in mind that the topological invariant $\chi(M)$ of M is identified to the sum of a smooth vector field on M at its zeros' indices, which was given by a famous Hopf theorem.[11,12] To give a unified version of the above two viewpoints, let M be the compact and oriented n-dimensional Riemannian manifold and $\vec{\phi}$ be a smooth vector field (a section of vector bundle) on M, as in Refs [13]–[16], we define a unit vector on M as

$$n^a = \frac{\phi^a}{\phi}, \qquad \phi \equiv \| \vec{\phi} \|, \qquad \phi^2 = \phi^a \phi^a, \qquad a = 1, 2, \cdots, n, \tag{1}$$

in which the superscript "a" is the local orthonormal frame index.

From Eq. (1) we see that the zeros of $\vec{\phi}$ are just the singularities of $\vec{n}$. This expression naturally gives two constraint conditions as

$$n^a n^a = 1, \qquad n^a \mathrm{d} n^a = 0. \tag{2}$$

In fact, $\vec{n}$ is identified to a section of the sphere bundle $S(M)$.[10] In the SO(n) gauge theory, i.e., the principal bundle $P(\pi, M, G)$ with a structure group $G = \mathrm{SO}(n)$, let x^μ be the local coordinates on the base manifold M, the covariant derivative 1-form of the unit vector n^a is defined as

$$D_\omega n^a = \mathrm{d} n^a - \omega^{ab} n^b, \tag{3}$$

and the curvature tensor 2-form

$$F^{ab}(\omega) = \mathrm{d}\omega^{ab} - \omega^{ac} \wedge \omega^{cb}, \tag{4}$$

where ω^{ab} is a spin connection 1-form (the SO(n) gauge potential)

$$\omega^{ab} = \omega^{ab}_\mu \mathrm{d} x^\mu, \qquad \omega^{ab} = -\omega^{ba}. \tag{5}$$

Using Eqs (2), (3) and (5) we have

$$n^a D_\omega n^a = 0. \tag{6}$$

In the following we will discuss the general decomposition formulation of the spin connections on the sphere bundle $S(M)$.

Let the Dirac matrices γ_a ($a = 1, \cdots, n$) in n-dimensional space be the bases of the Clifford algebra,[17] which satisfy $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}$, in the theory of the geometric algebra[18,19] a unit vector field $\vec{n}$ on M can be expressed in the following matrix form

$$n = n^a \gamma_a. \tag{7}$$

Using this formulation, the spin connection 1-form and curvature tensor 2-form are represented respectively

$$\omega = \frac{1}{2}\omega^{ab} I_{ab}, \qquad F(\omega) = \frac{1}{2} F^{ab}(\omega) I_{ab}, \tag{8}$$

in which I_{ab} is the generators of the group SO(n)

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \tag{9}$$

and

$$[I_{ab}, \gamma_c] = \gamma_a \delta_{bc} - \gamma_b \delta_{ac}. \tag{10}$$

Using Eq. (3) the covariant derivative 1-form can be represented in terms of n and ω

$$D_\omega n = \mathrm{d} n - [\omega, n] \tag{11}$$

and curvature tensor 2-form

$$F(\omega) = \mathrm{d}\omega - \omega \wedge \omega. \tag{12}$$

Let f and h be two vectors in geometric algebra, i.e., $f = f^a \gamma_a$, in $h = h^a \gamma_a$. It can be shown that the geometric product of f and h is

$$fh = f^a h^a + (f^a h^b - h^a f^b) I_{ab}. \tag{13}$$

Making use of Eq. (13) and from Eqs (2), (6) and (7) we can prove

$$nn = n^a n^a = 1\,, \qquad \mathrm{d}nn + n\mathrm{d}n = 0\,, \qquad D_\omega nn + nD_\omega n = 0\,. \tag{14}$$

And using Eq. (13) we have

$$\mathrm{d}nn = (\mathrm{d}n^a n^b - n^a \mathrm{d}n^b) I_{ab}\,, \tag{15}$$

$$nD_\omega n = (n^a D_\omega n^b - n^b D_\omega n^a) I_{ab}\,. \tag{16}$$

Let T_r be an r-vector in geometric algebra $T_r = (1/r!)T^{a_1 \cdots a_r}\gamma_{a_1}\cdots\gamma_{a_r}$. It can be proved that the covariant derivative 1-form of T_r

$$D_\omega T_r = \mathrm{d}T_r - [\omega, T_r\}\,. \tag{17}$$

In the gauge theory it is well-known that when connection 1-form ω takes the gauge transformation

$$\omega' = S\omega S^{-1} + \mathrm{d}SS^{-1}\,. \tag{18}$$

The curvature tensor 2-form F should be transformed as

$$F' = SFS^{-1}\,. \tag{19}$$

These are fundamental requirements of the gauge field theory. In our viewpoint,[14] the spin connection 1-form ω^{ab} can be decomposed and may has inner structure, which has been effectively used to study the magnetic monopole problems in SU(2) gauge theory,[14–16] the topological gauge theory of dislocation and disclinations in condensed matter physics,[20] and the geometrization of Planck constant in terms of space-time defect in general relativity.[21] The main feature of the theory of decomposition of connections (gauge potential) is that the connection ω in the gauge theory can be generally decomposed as follows:

$$\omega = A + b\,, \tag{20}$$

where A and b are respectively required to satisfy the gauge transformations and vector covariant transformations, i.e.,

$$A' = SAS^{-1} + \mathrm{d}SS^{-1}\,, \tag{21}$$

$$b' = SbS^{-1}\,. \tag{22}$$

From Eqs (21) and (22) it is easy to show that $\omega' = A' + b' = S(A+b)S^{-1} + \mathrm{d}SS^{-1}$. This means that the decomposition of the connection $\omega = A + b$ with Eqs (21) and (22) rigorously satisfies the fundamental requirement of the gauge theory.

By Eq. (11) and using Eq. (13) we can obtain

$$\omega = \frac{1}{2}(\mathrm{d}nn + nD_\omega n) + \frac{1}{2}J_n(\omega)\,, \tag{23}$$

where we define a useful symbol

$$J_n(\omega) = n\omega n + \omega\,. \tag{24}$$

Expression (23) with Eq. (24) is the general decomposition formula of spin connections in SO(n) gauge theory. We call the term $(1/2)(\mathrm{d}nn + nD_\omega n)$ in Eq. (24) the fundamental term of the general decomposition of connection ω respect to n and $(1/2)J_n(\omega)$, the compensative term of the general decomposition of connection ω respect to n, which makes the whole general decomposition expression of connection ω satisfy the gauge transformation rule.

Let a family $\{W, V, U, \cdots\}$ be an open cover of M and S_{VU} be the transition matrix functions which satisfy the following condition[22] $S_{UU} = 1$, $S_{VU}^{-1} = S_{UV}$, $S_{WV}S_{VU}S_{UW} = 1$ and $W \cap V \cap U \neq 0$. For any two open neighborhoods V and U, if $V \cap U \neq 0$, then

$$n_v = S_{VU} n_u S_{VU}^{-1}\,, \tag{25}$$

where n_v and n_u are two smooth vectors on V and U respectively and the spin connection ω_v and ω_u satisfy the following relation

$$\omega_v = S_{VU}\omega_u S_{VU}^{-1} + \mathrm{d}S_{VU}S_{VU}^{-1}\,, \tag{26}$$

which is the fundamental requirement of the existence of a connection on the principal bundle $P(\pi, M, G)$. In the terminology of physics S_{VU} is just the gauge transformation (18) in the gauge field theory. In the following, for abbreviation we always denote $S_{VU} = S$. From Eqs (23), (24) and making use of Eqs (25) and (26) it can be proved that[23]

$$\frac{1}{2}(\mathrm{d}n_v n_v + n_v D_{\omega_v} n_v) + \frac{1}{2} J_{n_v}(\omega_v) = S\left[\frac{1}{2}(\mathrm{d}n_u n_u + n_u D_{\omega_u} n_u) + \frac{1}{2} J_{n_u}(\omega_u)\right] S^{-1} + \mathrm{d}SS^{-1}\,. \quad (27)$$

And using Eqs (26) and (27) we find that

$$\omega_v - \left[\frac{1}{2}(\mathrm{d}n_v n_v + n_v D_{\omega_v} n_v) + \frac{1}{2} J_{n_v}(\omega_v)\right] = S\left[\omega_u - \frac{1}{2}(\mathrm{d}n_u n_u + n_u D_{\omega_u} n_u) - \frac{1}{2} J_{n_u}(\omega_u)\right] S^{-1}\,.$$

The above expression shows that if the decomposition formula on the open neighborhood U $\omega_u = (1/2)(\mathrm{d}n_u n_u + n_u D_{\omega_u} n_u) + J_{n_u}(\omega_u)/2$ holds true, then the decomposition formula on the open neighborhood V, $\omega_v = (1/2)(\mathrm{d}n_v n_v + n_v D_{\omega_v} n_v) + (1/2) J_{n_v}(\omega_v)$, must hold true also.

This means that the general decomposition formula (23) is a global property and is independent of the choice of the local coordinates.

Now let us prove that the decomposition formula (23) is independent of the choice of the unit vector $\vec{n}$. To do this let k be a unit vector in geometric algebra which is different from the unit vector n. Then nk by the terminology of geometric algebra is a 2-vector and using Eq. (17) its covariant derivative will be

$$D_\omega(nk) = \mathrm{d}(nk) - [\omega, nk]\,. \quad (28)$$

For the unit vector k which satisfies same relations in Eq. (14), so it is not difficult to get the following relation from Eq. (28)

$$\frac{1}{2}(\mathrm{d}nn + nD_\omega n) + \frac{1}{2} J_n(\omega) = \frac{1}{2}(\mathrm{d}kk + kD_\omega k) + \frac{1}{2} J_k(\omega)\,. \quad (29)$$

This means that the general decomposition formula (23) is indeed independent of the unit vector we chose.

To give a more concrete decomposition formula of the connection on the sphere bundle $S(M)$, let a be an arbitrary element of the Lie algebra so(n) which is expressed by

$$a = \frac{1}{2} a^{ab} I_{ab} \qquad \text{with } a' = SaS^{-1} \quad (30)$$

in geometric algebra, it is an antisymmetric tensor. Using the unit vector n we can always construct an argument b as

$$b = \frac{1}{2}(nan + a)\,. \quad (31)$$

Substituting Eqs (20) and (31) into Eq. (24) we have

$$J_n(\omega) = n(A + a)n + A + a\,. \quad (32)$$

As we have mentioned above that a is arbitrary, the argument $(A + a)$ in Eq. (32) should be also arbitrary and by Eqs (21) and (30) we see that it satisfies the gauge transformations. This means that

$$B = A + a \quad (33)$$

is an arbitrary connection in SO(n) gauge theory and

$$B' = SBS^{-1} + \mathrm{d}SS^{-1}\,. \quad (34)$$

Making use of Eqs (31) and (33), equation (32) can be written as

$$J_n(\omega) = J_n(B) = nBn + B\,. \quad (35)$$

In the case of the general decomposition formula (23) is rewritten as

$$\omega = \frac{1}{2}(\mathrm{d}nn + nD_\omega n) + \frac{1}{2} J_n(B)\,. \quad (36)$$

In order to study the Gauss–Bonnet–Chern density, according to the arbitrary quantity a, we can always choose it as $a = B_0 - A$, where B_0 is a flat connection

$$B_0 = \mathrm{d}UU^{-1}\,, \quad (37)$$

in which U is a local matrix in spinor representation corresponding to the group SO(n). From Eqs (35), (36) and (37) we finally get

$$\omega = \frac{1}{2}(\mathrm{d}nn + nD_\omega n) + \frac{1}{2}J_n(B_0)\,. \tag{38}$$

Substituting Eq. (38) into Eq. (12), for flat connection B_0, then $F(B_0) = 0$, one obtains

$$F(\omega) = \frac{1}{4}[-D_0 n \wedge D_0 n + D_\omega n \wedge D_\omega n + nD_0 D_\omega n - D_0 D_\omega nn]\,. \tag{39}$$

Using Eqs (8), (9), (13) and noticing $J_n(B_0) = (1/2)J_n^{ab}(B_0)I_{ab}$ (throughout this paper), from Eqs (35), (38) and (39) the component decomposition formula of $\omega, F(\omega)$ and $J_n(\omega)$ are respectively

$$\omega^{ab} = \mathrm{d}n^a n^b - \mathrm{d}n^b n^a + n^a D_\omega n^b - n^b D_\omega n^a + \frac{1}{2}J_n^{ab}(B_0)\,, \tag{40}$$

$$F^{ab}(\omega) = -D_0 n^a \wedge D_0 n^b + D_\omega n^a \wedge D_\omega n^b + n^a D_0 D_\omega n^b - n^b D_0 D_\omega n^a\,, \tag{41}$$

$$J_n^{ab}(B_0) = 2(B_0^{bc} n^c n^a - B_0^{ac} n^c n^b + B_0^{ab})\,. \tag{42}$$

It is easy to see that the special case of Eqs (40) and (41) when $B_0 = 0$ (i.e., $J_n^{ab}(B_0) = 0$) is just the result adopted in Ref. [8]. One notices that the special representation in terms of $\vec{n}$ to ω^{ab} and $F^{ab}(\omega)$ in Ref. [8] does not satisfy the requirement of the gauge transformations (18) and (19) respectively and it may be regarded as a special gauge condition.

III. Chern-Simons $(n-1)$ Form on the Sphere Bundle $S(M)$

In this section we will discuss a Chern–Simons $(n-1)$ form by means of the decomposition formula in Eq. (41). For an even dimensional compact and oriented Riemannian manifold M, there exists a unique n-form Λ over M such that

$$\Lambda = \frac{(-1)^{n/2}}{2^n \pi^{n/2}(n/2)!}\epsilon_{a_1 a_2 \cdots a_{n-1} a_n} F^{a_1 a_2}(\omega) \wedge \cdots \wedge F^{a_{n-1} a_n}(\omega)\,, \tag{43}$$

which is a closed n-form. Let π be a nature projection, thus $\pi^{-1}(p)$ is all that part of $S(M)$ lying above the point $p \in M$, this fiber is isometric to the $(n-1)$-dimensional sphere S^{n-1}.

Transformation group in fiber can be taken as SO(n) group. So, on $S(M)$, we have

$$\pi^* \Lambda = \mathrm{d}\Omega\,. \tag{44}$$

Chern[3,7] has proved that the $(n-1)$ form Ω on $S(M)$ is

$$\Omega = \frac{1}{(2\pi)^{n/2}} \sum_{k=0}^{n/2-1} (-1)^k \frac{2^{-k}}{(n-2k-1)!!k!}\Theta_k\,, \qquad n \geq 4\,, \tag{45}$$

which is called Chern formula, and Θ_k is

$$\begin{aligned}\Theta_k = {} & \epsilon_{a_1 a_2 \cdots, a_{n-2k} a_{n-2k+1} a_{n-2k+2} \cdots a_{n-1} a_n} n^{a_1} \times \\ & D_\omega n^{a_2} \wedge \cdots \wedge D_\omega n^{a_{n-2k}} \wedge \cdots \wedge F^{a_{n-2k+1} a_{n-2k+2}}(\omega) \wedge \cdots \wedge F^{a_{n-1} a_n}(\omega)\,.\end{aligned} \tag{46}$$

Λ is also independent of the connections and determines a cohomology class belonging to the cohomology group $H^n(M)$. This is equivalent to saying that the integral $\int_M$ taking over a closed manifold M is a topological invariant which is called Euler–Poincaré characteristic $\chi(M)$. Since π^* maps the cohomology class of M into that of $S(M)$, while $\vec{n}^*$ performs the inverse operation, $\vec{n}^*\pi^*$ amounts the identity. The GBC theorem can be expressed as

$$\chi(M) = \int_M \Lambda = \int_M \vec{n}^* \pi^* \Lambda = \int_M \vec{n}^* \mathrm{d}\Omega\,. \tag{47}$$

Substituting Eq. (41) into Eq. (46) and taking notice of that the last two terms in Eq. (41) contribute nothing to Θ_k due to the completely antisymmetric property of the tensor $\epsilon_{a_1 \cdots a_n}$

in expression (46) we find the expanding expression for Θ_k

$$\Theta_k = \sum_{l=0}^{k}(-1)^k C_k^l(-1)^l \times$$

$$\epsilon_{a_1 a_2 \cdots, a_{n-2k} a_{n-2k+1} a_{n-2k+2} \cdots, a_{n-2k+2l-1} a_{n-2k+2l} a_{n-2k+2l+1} a_{n-2k+2l+2} \cdots, a_{n-1} a_n} \times$$

$$n^{a_1} D_\omega n^{a_2} \wedge \cdots \wedge D_\omega n^{a_{n-2k}} \wedge D_\omega n^{a_{n-2k+1}} \wedge D_\omega n^{a_{n-2k+2}} \wedge \cdots \wedge D_\omega n^{a_{n-2k+2l-1}} \times$$

$$\wedge D_\omega n^{a_{n-2k+2l}} \wedge D_0 n^{a_{n-2k+2l+1}} \wedge D_0 n^{a_{n-2k+2l+2}} \wedge \cdots \wedge D_0 n^{a_{n-1}} \wedge D_0 n^{a_n} \,, \tag{48}$$

where $C_n^k = n!/k!(n-k)!$.

Since $D_\omega \vec{n}$ and $D_0 \vec{n}$ also are two vectors both perpendicular to $\vec{n}$. Let $D_\omega \vec{n} - D_0 \vec{n}$ be $\vec{k}$, we have

$$D_\omega n^a - D_0 n^a = \alpha k^a \,, \tag{49}$$

where α is the magnitude of the vector $D_\omega \vec{n} - D_0 \vec{n}$. Since $n^a D_\omega n^a = n^a D_0 n^a = 0$, from Eq. (49) we find that $\vec{k}$ is perpendicular to $\vec{n}$

$$n^a k^a = 0 \,, \tag{50}$$

and it follows that

$$k^a \mathrm{d} n^a + n^a \mathrm{d} k^a = 0 \,, \qquad n^a D_\omega k^a + k^a D_\omega n^a = 0 \,, \qquad n^a D_0 k^a + k^a D_0 n^a = 0 \,. \tag{51}$$

From Eqs (49) and (51) we obtain

$$D_\omega n^a = D_0 n^a - k^a n^b (D_\omega k^b - D_0 k^b) \,. \tag{52}$$

Putting this into Eq. (48), Θ_k can be reduced to[23]

$$\Theta_k = \begin{cases} \epsilon_{a_1 a_2 \cdots a_n}(n^{a_1} D_0 n^{a_2} \wedge \cdots \wedge D_0 n^{a_n}) \,, & \\ -(n-1) n^{a_1} k^{a_2} n^{b_2} (D_\omega k^{b_2} - D_0 k^{b_2}) \wedge \cdots \wedge D_0 n^{a_n} \,, & \text{if } k = 0 \,, \\ -2\epsilon_{a_1 a_2 \cdots a_n} n^{a_1} k^{a_2} n^{b_2} (D_\omega k^{b_2} - D_0 k^{b_2}) \wedge \cdots \wedge D_0 n^{a_n} \,, & \text{if } k = 1 \,, \\ 0 \,, & \text{if } k > 1 \,. \end{cases} \tag{53}$$

Using Eq. (53) we find a compact form of Ω as

$$\Omega = \frac{1}{(n-1)! A(S^{n-1})} \epsilon_{a_1 a_2 \cdots a_n} n^{a_1} D_0 n^{a_2} \wedge \cdots \wedge D_0 n^{a_n} \,, \tag{54}$$

where $A(S^{n-1})$ is the area of S^{n-1}, $A(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$.

Using the geometric algebra devices, for the flat connection $B_0 = \mathrm{d}UU^{-1}$, we take a simple matrix U as

$$U = nl \,, \qquad U^{-1} = ln \,, \tag{55}$$

this is a local spinor matrix representation of the group SO(n) (see Ref. [24]), in which n is a unit vector defined as before and l is another unit vector in geometric algebra which is perpendicular to n,

$$ln + nl = 0 \,. \tag{56}$$

It is not difficult to verify that the matrix U given by Eq. (55) transforms l into $-l$ and n into $-n$

$$UlU^{-1} = -l \,, \qquad UnU^{-1} = -n \,. \tag{57}$$

The more detailed discussion of the construction of spinor matrix representation is given in Appendix. Substituting Eq. (55) into Eq. (37) and using Eq. (56) we find that the flat connection is of the form

$$B_0 = \mathrm{d}nn + n\mathrm{d}lln \,. \tag{58}$$

By Eq. (58) we have

$$D_0 n = -\mathrm{d}n + [\mathrm{d}ll, n] \,. \tag{59}$$

From Eq. (59), using Eqs (7), (10), and $\mathrm{d}n^a l^a + n^a \mathrm{d}l^a = 0$, one obtains

$$D_0 n^a = -\mathrm{d}n^a + 2l^a l^b \mathrm{d}n^b . \tag{60}$$

Putting Eq. (60) into Eq. (54), we can reduce Ω on M to[23]

$$\Omega = \frac{1}{(n-1)!A(S^{n-1})} \epsilon_{a_1 a_2 \cdots a_n} n^{a_1} \mathrm{d}n^{a_2} \wedge \cdots \wedge \mathrm{d}n^{a_n} . \tag{61}$$

The above expression (61) is the Chern–Simons $(n-1)$-form, expressed cleanly in terms of the unit vector n^a on whole sphere bundle $S(M)$, which is a generalization of the result only on a fibre of $S(M)$ in Ref. [7].

IV. The Topological Structure of GBC Density and GBC Theorem

Using Φ-mapping method,[18] from the $(n-1)$-form Ω (61), the pull-back of the exterior derivative of Ω to M can be written as[8]

$$\vec{n}^* \mathrm{d}\Omega = \delta(\vec{\phi}) D(\phi/x) \mathrm{d}^n x . \tag{62}$$

Therefore we may define the Gauss–Bonnet–Chern density on M as

$$\rho = \frac{1}{(n-1)!A(S^{n-1})} \epsilon_{a_1 a_2 \cdots a_n} \epsilon^{\mu_1 \mu_2 \cdots \mu_n} \partial_{\mu_1} n^{a_1} \partial_{\mu_2} n^{a_2} \cdots \partial_{\mu_n} n^{a_n} = \delta(\vec{\phi}) D\Big(\frac{\phi}{x}\Big) . \tag{63}$$

We see that $\rho \neq 0$ and $\vec{n}^* \mathrm{d}\Omega \neq 0$ only when $\vec{\phi} = 0$. The above two expressions are of great importance. Suppose that $\phi^a(x)$ $(a = 1, \cdots, n)$ have N isolated zeros on M and let the ith zero be $\vec{x} = \vec{z}_i$, from the theory of δ-function $\delta(\vec{\phi})$ can be expressed as

$$\delta(\vec{\phi}) = \sum_{i=1}^{N} \frac{\beta_i \delta(\vec{x} - \vec{z})}{|D(\phi/x)|_{\vec{x}=\vec{z}_i}} , \tag{64}$$

in which β_i is a positive integer (Hopf index of the ith zero). The meaning of the Hopf index β_i is that while $\vec{x}$ covers the region neighboring of the zero $\vec{z}_i$ once, the function $\vec{\phi}$ covers the corresponding region β_i times. If let η_i be the Brouwer degree[12] $\eta_i = \operatorname{sgn} D(\phi/x)|_{\vec{x}=\vec{z}_i} = \pm 1$. From above and using Eq. (64) the GBC density on M has the following topological structure

$$\rho = \delta(\vec{\phi}) D(\phi/x) = \sum_{i=1}^{N} \beta_i \eta_i \delta(\vec{x} - \vec{z}_i) . \tag{65}$$

This shows that the local structure of ρ is labeled by the Brouwer degrees and Hopf indices, which are topological invariants.

The integration of ρ on M gives

$$\chi(M) = \int_m \rho \mathrm{d}^n x = \int_M \vec{n}^* \mathrm{d}\Omega = \sum_{i=1}^{N} \beta_i \eta_i . \tag{66}$$

The result of Eq. (66) says that the sum of indices of the zeros of $\vec{\phi}$ or the singularities of $\vec{n}$ is the Euler–Poincaré characteristic. Therefore the topological structure of GBC density shows the expected result of the Hopf theorem. In other words, we give a proof of GBC theorem. On the other hand, we must point out an important and a simple fact that from Eqs (65) and (66) we have

$$\chi(M) = \int_M \delta(\vec{\phi}) D(\phi/x) \mathrm{d}^n x = \deg \phi \int_V \delta(\vec{\phi}) \mathrm{d}^n \phi = \deg \phi \tag{67}$$

and

$$\deg \phi = \sum_{i=1}^{N} \beta_i \eta_i , \tag{68}$$

where V is the elementary volume in the vector space of $\vec{\phi}$ and the quantity $\deg \phi$ is the degree of the mapping ϕ.

Therefore in this paper we see that the decomposition theory of connection in gauge theory is an important and powerful tool in studying the topological problems not only in physics but also in mathematics.

Appendix

Theorem:[24] Let $\vec{n}$ and $\vec{l}$ be two unit vectors in an n-dimensional vector space with $\vec{n} \cdot \vec{l} \neq -1$, and $\vec{h}$ be a unit vector in the $\vec{n} - \vec{l}$ plane which bisects the angle θ between $\vec{n}$ and $\vec{l}$. Let n, l and h be vectors in the geometric algebra, then there exist two elementary transformations which bring l into n via the similarity transformations $n = L_i l L_i^{-1}$, $i = 1, 2$, where $L_1 = h$, $L_1^{-1} = L_1$, $L_2 = hl = nh$ and $L_2^{-1} = lh = hn$. In which h has been defined by $h = (n + l)/2(\cos(\theta/2))$, and θ is $\cos\theta = n^a l^a \neq -1$. L_i $(i = 1, 2)$ are regarded as the spinor matrix representation of the orthogonal transformations, L_1 represents the axial involution about h and L_2 represents the plane rotation in the $\vec{n} - \vec{l}$ plane.

Now let us consider further that the unit vector $\vec{l}$ is perpendicular to $\vec{n}$, then

$$L_2 = \frac{1}{\sqrt{2}}(1 + nl), \quad L_2^{-1} = \frac{1}{\sqrt{2}}(1 + ln).$$

Using L_2 we can construct a simple spinor matrix representation of the orthogonal transformation U as $U = L_2 L_2 = nl$ and $U^{-1} = ln$, then $UlU^{-1} = -l$, $UnU^{-1} = -n$. This shows that U brings l into $-l$, n into $-n$ respectively.

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