Commun. Theor. Phys. 23 (1995) pp. 175–180
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Vol. 23, No. 2, March 15, 1995

The Number of Periods in One-Dimensional Gap Map and Lorenz-Like Map

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(Received May 12, 1994)

Abstract

The problem of counting the number of periods in one-dimensional maps is extended to two discontinuous maps, the gap map and the Lorenz-like map. The numbers of periods in the gap map and the Lorenz-like map are given by $4F_2(n)$ *and* $2C_2(n)$*, respectively, where* $F_2(n)$ *and* $C_2(n)$ *are the number of periodic sequences, invariant under group* $C_n \otimes S_2$ *and* C_n*, respectively.*

I. Introduction

In a nonlinear dynamical system, which exhibits chaotic behavior, periodic orbits are in close relation with chaotic regime. When a parameter is varied, the system may undergo a series of bifurcations, leading to various periodic or aperiodic regimes. For a higher-dimensional system the problem of counting the number of the periodic regimes may be very difficult. However, topologically, higher-dimensional dissipative chaotic systems may have the same kind of attractors as low-dimensional ones. Therefore, a key problem is to count the number of periodic orbits in one-dimensional maps, then we can use them as standards to compare with that of higher-dimensional systems. Recently, we have solved this problem for one-dimensional continuous maps.[1,9,14] In this paper we mainly investigate this problem for two discontinuous maps, the gap map[2–4] and the Lorenz-like map.[5] Before tackling the problem in Secs III and IV, we summarize briefly the results for the unimodal map in the next section.

II. The Number of Periods in the Unimodal Map

The unimodal mapping function, which is shown in Fig. 1, has two monotonic branches and one critical point labelled with L, R and C, respectively. Using this symbolic description, on one hand, an arbitrary numerical orbit can be coded by a symbolic sequence, on the other hand, not every symbolic sequence, constructed with L, R and C, may correspond to a real trajectory of the dynamics at a parameter. Generally speaking, in order to be an admissible sequence, a symbolic sequence **I** must satisfy the following admissibility condition based on the nature ordering[9]

$$\mathbf{R}, \mathbf{L} \leq \mathbf{K}_C\,, \tag{1}$$

where **R** denotes the set of all subsequences that follow a letter R in **I**, the meaning of **L** is analogous, $\mathbf{K}_C$ is the ***kneading sequence***,[6] i.e., a symbolic sequence, which starts with the first iteration of the critical point C. When the kneading sequence includes the critical point C, it is called a superstable kneading sequence.

As the kneading sequence varies with the parameter, the map can also be parametrized by the kneading sequence. When varied the parameter, all kinds of admissible superstable kneading sequences appear in accordance with Eq. (1). These are the U-sequence or MSS sequence, first introduced by Metropolis, Stein and Stein.[7,8]

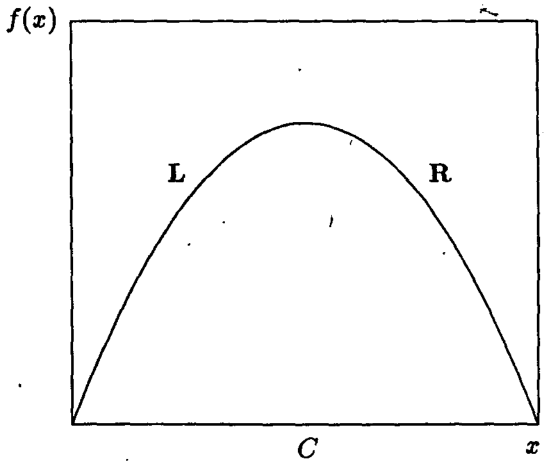

Fig. 1. The unimodal map.

In another independent way, the number of periods of the unimodal map is related to a combinatorial problem: how many different necklaces may be made from n pieces of stones, each stone having a choice from q colors. The answer naturally depends on which necklaces are considered to be different. One should allow for cyclic permutation of stones and even allow for permutation of colors. If so, the problem turns out to be a particular case of a group-theoretical problem on the number of periodic sequences, invariant under a certain discrete group, namely, the group $C_n \otimes S_q$. If a sequence is invariant only under cyclic permutation among the symbols, the group is C_n. The answer to these two cases have been known since late 1950s.[10–12]

One distinguishes the number $F_q(n)$ of primitive period n, which is not a multiple of a shorter period, from the total number $F_q^*(n)$, which is related to $F_q(n)$ by the Möbius transformation

$$F_q^*(n) = \sum_{d|n} F_q(d) , \qquad F_q(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F_q^*(d) . \tag{2}$$

For group $C_n \otimes S_q$, $F_q^*(n)$ is given explicitly by Gilbert and Riordan[11]

$$F_q^*(n) = \frac{1}{q!n} \sum_{d|n\{k_i\}} \varphi(d) N(k_1, k_2, \ldots, k_q)[m(d)]^{n/d} , \tag{3}$$

where

$$\begin{aligned} &k_1 + 2k_2 + \cdots + qk_q = q , \\ &N(k_1, k_2, \ldots, k_q) = \frac{q!}{k_1!k_2!\ldots k_q!2^{k_2}\ldots q^{k_q}} , \\ &m(d) = \sum_{c|d} ck_c , \end{aligned} \tag{4}$$

and $\varphi(d)$ is the Euler function.

We define $C_q^*(n)$ and $C_q(n)$ to be corresponding quantities for group C_n, then $C_q^*(n)$ are given by[11]

$$\begin{aligned} C_q^*(n) &= \frac{1}{n}\sum_{d|n}\varphi(d)q^{n/d}\,, \\ C_q^*(n) &= \sum_{d|n} C_q(d)\,. \end{aligned} \tag{5}$$

Metropolis, Stein and Stein realized that the number of periods in the unimodal map is the same as $F_2(n)$, the values of $F_2(n)$ for group $C_n \otimes S_2$ indeed agree with the number of periods in the MSS sequence. We list $F_2(n)$ and $C_2(n)$ in Table 1. Moreover, the relation between $F_2(n)$ and $C_2(n)$ is

$$\begin{aligned} C_2(n) &= 2F_2(n) && \text{for odd } n; \\ C_2(n) &= 2F_2(n) - F_2\left(\tfrac{n}{2}\right) && \text{for even } n. \end{aligned}$$

Table 1. Number of period n orbits for group $C_n \otimes S_2$, C_n, the gap map, and the Lorenz-like map.

n	$F_2(n)$	$C_2(n)$	Gap map	Lorenz-like map
1	1	2	2	2
2	1	1	3	2
3	1	2	4	4
4	2	3	8	6
5	3	6	12	12
6	5	9	20	18
7	9	18	36	36
8	16	30	64	60
9	28	56	112	112
10	51	99	204	198
11	93	186	372	372
12	170	335	680	670
13	315	630	1260	1260
14	585	1161	2340	2322
15	1091	2182	4364	4364
16	2048	4080	8192	8160
17	3855	7710	15420	15420
18	7280	14532	29120	29064
19	13797	27594	55188	55188
20	26214	52377	104856	104754

III. The Number of Periods in the Gap Map

The gap map is a map generated from the unimodal map by opening a gap at the critical point. A typical shape of the mapping function is shown in Fig. 2. In parlance of symbolic dynamics, there are two kneading sequences K_- and K_+, which denote the iterations of the points $f(C_-)$ and $f(C_+)$, where C_- and C_+ are the left and right limits of C, respectively. In terms of K_+ and K_- we can easily write down the admissibility condition for an arbitrary itinerary I

$$\begin{aligned} \mathbf{L} &\le \mathbf{K}_-\,, \\ \mathbf{R} &\le \mathbf{K}_+\,, \end{aligned} \tag{6}$$

where the meanings of **L** and **R** are similar to those in the foregoing section. When K_+ or K_- contains the critical point C, it is a superstable kneading sequence. The dynamical behavior of the gap map has been well investigated and the kneading plane (K_+, K_-) constructed.[4] Furthermore, one may ask how many admissible periodic kneading sequences of K_+ and K_- there are for a fixed length n in the whole kneading plane. This is the question we are going to answer. First, we point out that the number of periods of K_- and K_+ are all given by $2F_2(n)$ except for period ***one*** and ***two*** for K_-, and period ***one*** for K_+. We prove the results in the following paragraphs using the language of symbolic dynamics.

As we know, the number of superstable periodic kneading sequences for the unimodal map, which is $F_2(n)$, can be calculated directly from (1). An admissible word of (1) determines two admissible words for K_- or K_+. We prove the result for K_-, the proof for K_+ is similar.

Suppose that $I = XC$ is an admissible word of (1), i.e.,

$$\mathbf{R}, \mathbf{L}(XC) < XC\,. \tag{7}$$

According to the periodic window theorem[9] XR and XL are all admissible for (1), i.e.,

$$\begin{aligned} \mathbf{R}, \mathbf{L}(XR) &\le XR\,, \\ \mathbf{R}, \mathbf{L}(XL) &\le XL\,. \end{aligned} \tag{8}$$

Obviously, substituting the last letter of XL by C_-, XC_- is admissible in the sense of (6).

For XR, let $\mathbf{L} = \{X_1RY_1L, X_2RY_2L, \cdots, X_nRY_nL\}$, and $X_{\max}RY_{\max}L$ is the maximum element of $\mathbf{L}$, substituting the last letter L by C_-, $X_{\max}RY_{\max}C_-$ is admissible for (6), i.e.,

$$\mathbf{L}(X_{\max}RY_{\max}C_-) < X_{\max}RY_{\max}C_-\,.$$

It is obvious that XC_- is not equal to $X_{\max}RY_{\max}C_-$. Therefore, for any admissible word of (1), there must be two related admissible words of (6).

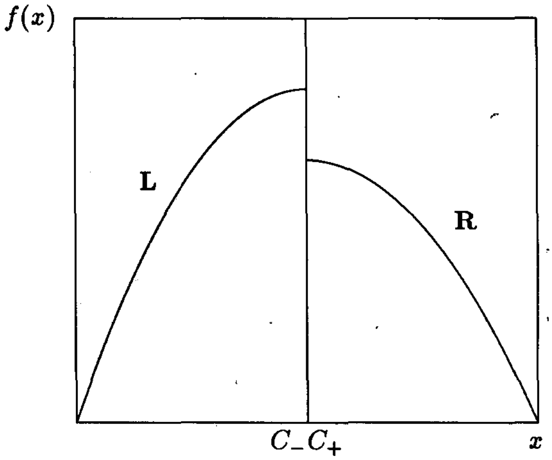

Fig. 2. The gap map.

On the contrary, one can prove that two different admissible words of (6) are related to an admissible word of (1) in a similar way. In addition, if C or RC appears as the kneading sequence in (1), the only admissible word for (6) is C_- or RC_- of (6). The reason is that the set **L** of the admissible period *one* word R^∞ of (1) is an empty set, which cannot lead to an admissible word of (6). Therefore, the number of periods in the gap map is given by $4F_2(n)$, i.e., the sum of K_+ and K_-, except for period *one* and *two*. These numbers are given in Table 1.

IV. The Number of Periods in the Lorenz-Like Map

The Lorenz-like map, being a simple geometry model, which reflects the dynamical behavior of the Lorenz system, has been well studied in terms of symbolic dynamics.[5] A typical shape of the map is shown in Fig. 3. The map has a discontinuous point C and two increasing branches, L and R. According to the kneading theory, it has two kneading sequences $K_+ = f(C_-)$ and $K_- = f(C_+)$. The admissibility condition can be written down easily

$$\begin{aligned} \mathbf{L} &\le K_+ , \\ \mathbf{R} &\le K_- . \end{aligned} \tag{9}$$

where **L** and **R** are also the same meanings as those in the foregoing section. The kneading plane (K_+, K_-) has been constructed.[5] Furthermore, the number of different periods of K_- and K_+ are all given by $C_2(n)$, except for period *one*. We give the proof as follows.

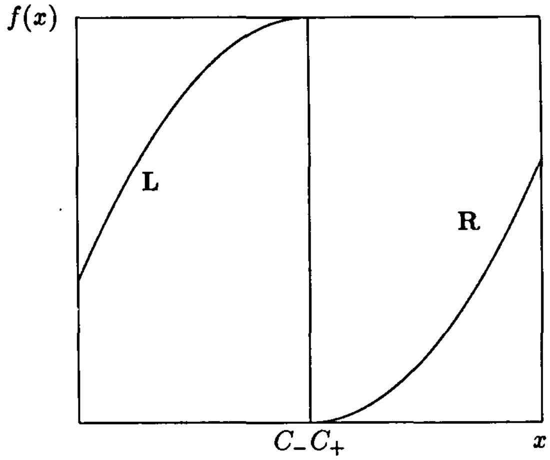

Fig. 3. The Lorenz-like map.

First, it is obvious that the parity of both R and L are even for the Lorenz-like map, therefore, there are no period-doubling bifurcation from period *one* orbit.

According to symbolic dynamics, $C_2(n)$ is the number of non-equivalent primitive period n under group C_n, which acts on two different words L and R, where L and R need not satisfy group S_2, i.e., L and R should be the same parity, so it need only satisfy the maximum shift, i.e., for a periodic symbolic sequence I of $C_2(n)$

$$\mathbf{R}, \mathbf{L}(I) \le I . \tag{10}$$

If $I = (XL)^\infty$, substituting the last letter L by C_-, XC_- is admissible in the sense of (9), because $\mathbf{L}(XC_-) < XC_-$. If $I = (XR)^\infty$, let $\mathbf{L} = \{X_1RY_1L, X_2RY_2L, \ldots, X_nRY_nL\}$, and $X_{\max}RY_{\max}L$ is the maximum word of **L**, substituting the last letter L by C_-, $X_{\max}RY_{\max}C_-$

is the only admissible word of (9), since $\mathbf{L}(X_{\mathrm{max}}RY_{\mathrm{max}}C_-) < X_{\mathrm{max}}RY_{\mathrm{max}}C_-$. On the other hand, if $I = (XC_-)^\infty$, or equivalently, $I = (XL)^\infty$, is an admissible word of (9), and suppose

$$\begin{aligned}\mathbf{R} &= \{X_1LY_1R, X_2LY_2R, \ldots, X_nLY_nR\}\,,\\ \mathbf{L} &= \{XL, X_1'LY_1'L, \ldots, X_n'LY_n'L\}\,,\end{aligned}$$

where $X_{\mathrm{max}}LY_{\mathrm{max}}R$ is the largest word of $\mathbf{R}$, XL is the largest word of $\mathbf{L}$. If XL is greater than $X_{\mathrm{max}}LY_{\mathrm{max}}R$, then XL is the only admissible word of (10), otherwise, $X_{\mathrm{max}}LY_{\mathrm{max}}R$ is the only admissible word of (10). Therefore, we have proved that there is a one–one correspondence between $C_2(n)$ and the number of period n for K_+ except for period ***one***. For $C_2(n)$ there are two period-***one*** words L and R, but they can only give one admissible word C_- of (9), because the set $\mathbf{L}$ of the word R^∞ of $C_2(n)$ is empty.

In order to calculate the number of periods of K_-, we introduce a transformation

$$\begin{aligned}L &\longleftrightarrow R\,,\\ C_- &\longleftrightarrow C_+\,.\end{aligned} \tag{11}$$

Applying the transformation (11) to an admissible word XC_- of K_+, we get only one admissible word $\bar{X}C_+$ of K_-, so the number of periods for K_- is also given by $C_2(n)$. Therefore, the number of periods in the Lorenz-like map is given by $2C_2(n)$, i.e., the sum for K_+ and K_-, except for period ***one***. The numbers are listed in the last column of Table 1.

V. Conclusion

The problem of counting the number of periods is successfully extended to two discontinuous maps, namely, the gap map and the Lorenz-like map, using the language of symbolic dynamics. Moreover, this problem can also be extended to the circle map, which is always discontinuous, but the treatment becomes more difficult due to the presence of multiple discontinuous points and several monotonic branches.[13] This will be elucidated in our further publications.

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