# A Linearization Approach for Rational Nonlinear Models in Mathematical Physics

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***Abstract*** *In this paper, a novel method for linearization of rational second order nonlinear models is discussed. In particular, we discuss an application of the δ expansion method (created to deal with problems in Quantum Field Theory) which will enable both the linearization and perturbation expansion of such equations. Such a method allows for one to quickly obtain the order zero perturbation theory in terms of certain special functions which are governed by linear equations. Higher order perturbation theories can then be obtained in terms of such special functions. One benefit to such a method is that it may be applied even to models without small physical parameters, as the perturbation is given in terms of the degree of nonlinearity, rather than any physical parameter. As an application, we discuss a method of linearizing the six Painlevé equations by an application of the method. In addition to highlighting the benefits of the method, we discuss certain shortcomings of the method.*

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## 1 Introduction

In the late 1980's, Bender and colleagues introduced a type of perturbation technique, the $\delta$-expansion method,[1−2] in which one expands in powers of a nonlinearity present in a nonlinear differential equation. At first applied to problems in quantum field theory, the method found plenty of application to nonlinear differential equations in many areas of science (see, for instance, Ref. [3] and the references therein). Most applications of such methods to second order equations have considered nonlinear equations which depend on some power-law nonlinearity in the unknown function and or the derivatives; see, e.g., Ref. [3], for some examples in various fields. We have applied this approach to additional mathematical models recently.[4−9] Since the time when the $\delta$-expansion approach was introduced, a number of perturbation and approximation methods have been created to solve nonlinear differential equations, such as the method of homotopy analysis due to Liao.[10] In that method, one constructs a linear homotopy in operator space between the original nonlinear operator and an auxiliary linear operator. Treating the homotopy parameter as the perturbation "small parameter", one can employ the method even in the absence of small parameters native to the model.

Here we shall consider a sort of hybrid between the two. We effectively and implicitly construct a homotopy between a nonlinear rational ODE and a linear ODE in terms of the nonlinearities present in the rational functions; here the homotopy is nonlinear in nature. First we give a general outline of how one proceeds for general rational second order ODEs. As opposed to power-law nonlinearities, rational nonlinearities are notoriously difficult to deal with, for both perturbative and numerical methods, due to the appearance of poles in the solutions.

As we outline a computational method relevant to a large class of nonlinear differential equations, our approach will be more heuristic than theorem - proof, as is common when presenting with most any perturbative method. That said, we make every effort to be precise and accurate. The structure of the paper will be as follows. First, in Sec. 2, we shall outline the general method of linearizing and constructing formal perturbation solutions (the two are intimately related) for second order rational ordinary differential equations, which we take to be equations of the form $y'' = R(y, y', x)$ where $R$ is rational in each of it's arguments. We model our approach on the $\delta$-expansion method and hence the results are relevant even for such ordinary differential equations in the absence of small model or physical parameters.

In Sec. 3, we proceed to outline a method by which one applies this method in order to obtain perturbation solutions for the Painlevé transcendents. We provide the method by which one may linearize and then compute the perturbation solutions for all six Painlevé transcendents. In all cases considered, one may compute the order zero perturbation solutions by hand (as the governing equations are linear), while for the first order perturbation theory, one may construct the solutions by hand or computer algebra system. For higher order perturbation expansions, one really needs the use of a computer, and even then, symbolic computation becomes tedious if not impossible, particularly for the latter Painlevé equations. However, one may numerically solve the linearized system, obtaining the $M$-th order perturbation theory ($M > 1$)

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numerically. For each of the Painlevé equations, we give explicitly the order zero term and then give the general formulas by which one may construct an $M$-th order perturbation theory, without any assumptions of the model parameters. We note that the linearizations employed showcase the strong relation between the Painlevé transcendents and certain special functions.

We find that there is a trade off when compared to Taylor series or other methods; the approximates computed by the present method appear to converge more rapidly than, say, Taylor series. However, computing each term in the expansion is more computationally demanding than computing terms in simple Taylor series or standard perturbation schemes. As the complexity of such operations varies greatly from one specific ODE to another, the utility of this method will depend on the problem at hand. In some situations, the expansions will diverge, or will converge slowly. Additionally, over the complex plane, the solutions may have poles, depending on the form of $R(y, y', x)$ (this is certainly the case when we look at the Painlevé equations). Hence, it is natural to want a method to describe such poles and also to speed the convergence of solutions. To do so, we introduce rational Padé approximations in the auxiliary parameter and we discuss why this is useful.

## 2 Linearization and Perturbation for Second Order Rational Models

Consider the nonlinear differential equation

$$y'' = R(y, y', x)\,, \tag{1}$$

where $R$ is a rational function in each of it's arguments. Here, we take $x \in \mathbb{C}$, and note that the analogous reasoning holds over real variables and real functions. In general, one may write

$$R(y, y', x) = R_1(y, y', x)y' + R_0(y, y', x)y + r(y, y', x)\,, \tag{2}$$

and such a representation may not be unique (a fact that can be exploited in our advantage in some cases, as we shall see later).

Consider a family of functions $\varphi_\delta : \mathbb{C}^3 \to \mathbb{C}$ parameterized by $\delta \in [0, 1]$ such that the dependence on $\delta$ is continuous. We then have a homotopy between $\varphi_0$ and $\varphi_1$, where $\delta$ serves as the homotopy parameter. Along these lines, let us define families of rational functions parameterized by $\delta \in [0, 1]$,

$$S_1(y, y', x; \delta), \quad S_0(y, y', x; \delta), \quad s(y, y', x)\,, \tag{3}$$

so that

$$S_1(y, y', x; 0) = \tilde{S}_1(x)\,,$$
$$S_1(y, y', x; 1) = R_1(y, y', x)\,, \tag{4}$$
$$S_0(y, y', x; 0) = \tilde{S}_0(x)\,,$$
$$S_0(y, y', x; 1) = R_0(y, y', x)\,, \tag{5}$$
$$s(y, y', x; 0) = \tilde{s}(x)\,,$$
$$s(y, y', x; 1) = r(y, y', x)\,. \tag{6}$$

Here, when $\delta = 0$ the functions are of only one variable, the independent variable $x$, while when $\delta = 1$ we recover the rational coefficients of ($y'$, $y$, $x$, respectively) in Eq. (1). In such a way, we have developed a parameterized ODE

$$y'' = S_1(y, y', x; \delta)y' + S_0(y, y', x; \delta)y + s(y, y', x)\,. \tag{7}$$

When $\delta = 0$ we have the linear ODE

$$y'' = \tilde{S}_1(x)y' + \tilde{S}_0(x)y + \tilde{s}(x)\,, \tag{8}$$

whereas, when $\delta = 1$, we have the original nonlinear rational ODE Eq. (1). Under such a representation, one sees the similarities between Eq. (1) and special functions defined in terms of second order linear equations. Indeed, for any given linear special function defined by a second order linear ODE, there exist infinitely many such equations (1), which are linked to the special function via a parameterized ODE (7).

From here, a key observation is in order. A solution to Eq. (7) will depend on $x$ and on the parameter $\delta$. Thus, for a family of ODE (7) there in principle (assuming Eq. (7) admits solutions) corresponds a family of solutions $y = y(x; \delta)$ for $\delta \in [0, 1]$. At $\delta = 0$, $y = y(x; 0)$ is a solution to the linear equation (8) while when $\delta = 1$, $y = y(x; \delta)$ is a solution to the original nonlinear rational ODE (1). As $y$ depends on $\delta$, let us assume an expansion in $\delta$ of the form

$$y(x; \delta) = y_0(x) + \sum_{n=0}^{\infty} y_n(x)\delta^n\,. \tag{9}$$

If such an expansion converges for $\delta = 1$ then it is a solution to Eq. (1).

The benefit of such an expansion is two-fold. First, the expansion provides an iterative manner to compute the solution to Eq. (1). One can compute the terms successively to arrive at a solution, or one may truncate the expansion at $O(\delta^k)$ for some $k \geq 1$ to obtain an approximate solution. Secondly, such an expansion allows us to convert the original nonlinear ODE into successive linear ODEs, which individually are more amenable to standard solution methods.

Placing the expansion (9) into Eq. (7), we find that each of the terms in the expansion (9) are govened by

$$L[y_0] = \tilde{s}(x)\,, \tag{10}$$
$$L[y_n] = F_n(y_0, \ldots, y_{n-1}, x)\,, \tag{11}$$

where $L$ denotes the second order linear operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \tilde{S}_1(x)\frac{\mathrm{d}}{\mathrm{d}x} - \tilde{S}_0(x)\,, \tag{12}$$

$$\begin{aligned} &F_n(y_0, \ldots, y_{n-1}, x) \\ &\quad = \frac{1}{n!}\lim_{\delta\to 0}\frac{\partial^n}{\partial\delta^n}[S_1(y(x;\delta), y'(x;\delta), x;\delta)y'(x;\delta) \\ &\qquad + S_0(y(x;\delta), y'(x;\delta), x;\delta)y + s(y(x;\delta), y'(x;\delta), x) \\ &\qquad - \tilde{S}_1(x)y'(x;\delta) - \tilde{S}_0(x)y(x;\delta)]\,, \end{aligned} \tag{13}$$

are the inhomogeneities due to the lower order terms. From the form of the $F_n$'s, we see that when selecting the decomposition of $R$ as in Eq. (2) the functions $R_1$, $R_0$, and $r$ are best selected in such a manner to permit $S_1$, $S_0$, and $s$ to be analytic in $\delta$. This is not to say that we cannot apply these methods when such functions are not analytic, but, rather, that restricting to such functions will be computationally advantageous.

In order to find the $y_n$'s, we must be able to invert the linear operator $L$ given in Eq. (12). Let $v_1(x)$ and $v_2(x)$ denote the two linearly independent solutions to the homogeneous equation $L[Y] = 0$. By variation of parameters and employing the Wronskian $W(x) = v_1(x)v_2'(x) - v_1'(x)v_2(x)$, we know that the particular solution to $L[Y] = f(x)$ takes the form

$$\int^x G(x,\tau)f(\tau)\,\mathrm{d}\tau$$

where

$$G(x,\tau) = \frac{1}{W(\tau)}\{v_1(\tau)v_2(x) - v_1(x)v_2(\tau)\}\,. \tag{14}$$

In the case that $x$ is a real variable, Abel's identity for the Wronskian (see Appendix A) gives

$$G(x,\tau) = \frac{1}{W(x_0)}\{v_1(\tau)v_2(x) - v_1(x)v_2(\tau)\} \times \exp\Big(-\int_{x_0}^{\tau}\tilde{S}_1(\xi)\,\mathrm{d}\xi\Big)\,. \tag{15}$$

Here, $x_0$ is some point on the real line at which the Wronskian $W$ does not vanish.

Inverting $L$ as above, we find that the terms in the expansion (9) are given successively by

$$y_0(x) = c_1v_1(x) + c_2v_2(x) + \int^x G(x,\tau)\tilde{s}(\tau)\,\mathrm{d}\tau\,, \tag{16}$$

$$y_n(x) = \int^x G(x,\tau)F_n(y_0(\tau),\ldots,y_{n-1}(\tau),\tau)\,\mathrm{d}\tau\,, \quad n \geq 1. \tag{17}$$

To summarize, assume (i) $S_1$, $S_0$, and $s$ are analytic in $\delta$ for all $\delta \in [0,1]$, and (ii) there exists $n^*$ such that $|y_{n+1}| < |y_n|$ for all $n > n^*$. Then, the expansion $y(x;\delta)$ given by Eq. (9) is a solution to the nonlinear ODE (7) and, in particular, $y(x;1)$ is a solution to the rational ODE (1).

In the case that one of the $S_1$, $S_0$ or $s$ is not analytic in $\delta$, we may still construct formal perturbation solutions that converge, however there will be other cases which do not admit convergent solutions. However, even when the formal expansion (9) diverges, various renormalization or approximation techniques may be employed to save the expansion. In many applications, the first few terms of the divergent expansion can be retained, and actually provide quite good approximations or asymptotics depending on the problem at hand.

## 3 Application: Painlevé Equations

There are six Painlevé transcendents, corresponding to six second-order ordinary differential equations whose only movable singularities are ordinary poles (this characteristic is known as the Painlevé property) and which cannot be integrated in terms of other known functions or transcendents; see the original works Painlevé transcendents,[11−19] or any modern textbook covering the theory of nonlinear ordinary differential equations (e.g., Ref. [20]). As the solutions to the six Painlevé equations cannot be obtained exactly, one may resort to series or perturbation solutions.

Presently, we apply the method discussed in the previous section in order to arrive at some approximate solutions to the six Painlevé transcendents. The beauty of the method is that it allows us to control the manner in which the inherent nonlinearities are taken into account when constructing a linearized system. As such, the linearized equations should more accurately represent the nonlinear equations than, say, linearizations due to standard perturbation methods. The result is that, in comparison to the standard perturbation methods, the perturbation solutions obtained via the presently employed method may converge more rapidly to reasonable solutions to the Painlevé equations. This is rather important, as the equations, particularly the latter four, are relatively complicated and thus computing many iterates in the perturbation expansions, even with the resulting linearized systems, becomes tedious and computationally demanding. The linearization and series/perturbation/asymptotic expansions for the various Painlevé equations is naturally an area represented in the literature; see, for instance[21−32] and the references therein. However, to the author's knowledge, the present approach has not been applied to any of the Painlevé equations, let alone all six.

### 3.1 *Painlevé I*

Painlevé I reads

$$\ddot{y} = 6y^2 + t\,, \tag{18}$$

where the dot denotes differentiation with respect to $t$. Let us consider the related nonlinear differential equation

$$\ddot{y} = 6y^{1+\delta} + t\,, \tag{19}$$

so that when $\delta = 1$, we recover Eq. (18). We shall construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume

$$y(t;\delta) = y_0(t) + y_1(t)\delta + y_2(t)\delta^2 + \cdots \tag{20}$$

Provided this expansion converges at $\delta = 1$, we have that $y(t;1)$ is a solution to Eq. (18); i.e., $y(t;1)$ is a perturbation representation for the first Painlevé transcendent. Placing Eq. (20) back into Eq. (19), we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - 6y_0 = t\,. \tag{21}$$

Clearly,

$$y_0(t) = C_1\sinh(\sqrt{6}t) + C_2\cosh(\sqrt{6}t) - \frac{t}{6}\,, \tag{22}$$

where $C_1$ and $C_2$ are constants of integration. In general, for $k \geq 1$, we have that

$$\ddot{y}_k - 6y_k = \mathcal{F}_k(y_0, y_1, \dots, y_{k-1})\,, \tag{23}$$

where the $\mathcal{F}_k$'s satisfy

$$-6(y_0(t) + y_1(t)\delta + y_2(t)\delta^2 + \cdots)^{1+\delta} + 6\sum_{k=0}^{\infty} y_k\delta^k = \sum_{k=0}^{\infty}\mathcal{F}_k(y_0, y_1, \dots, y_{k-1})\delta^k\,. \tag{24}$$

We find that

$$y_k(t) = \frac{1}{\sqrt{6}}\int^t \sinh(\sqrt{6}(t-\tau)) \times \mathcal{F}_k(y_0(\tau), y_1(\tau), \dots, y_{k-1}(\tau))\,\mathrm{d}\tau\,. \tag{25}$$

For instance, the first order correction, $y_1(t)$, is governed by

$$\ddot{y}_1 - 6y_1 = 6y_0 \ln|y_0|\,. \tag{26}$$

The first order correction in the perturbation solution (20) is then found to be

$$y_1(t) = \sqrt{6}\int^t \sinh(\sqrt{6}(t-\tau))y_0(\tau)\ln|y_0(\tau)|\,\mathrm{d}\tau\,. \tag{27}$$

Continuing in this manner, we may construct a $\delta$-expansion solution up to $O(\delta^K)$. For the method of finding the $\mathcal{F}_k$'s, see Appendix A.

### 3.2 *Painlevé II*

Painlevé II reads

$$\ddot{y} = 2y^3 + ty + \alpha\,, \tag{28}$$

where the dot denotes differentiation with respect to $t$. Consider the related nonlinear differential equation

$$\ddot{y} = 2y^{1+2\delta} + ty + \alpha\,. \tag{29}$$

so that when $\delta = 1$, we recover Eq. (28). We shall construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume a solution of the form (20). Provided this expansion converges at $\delta = 1$, we have that $y(t;1)$ is a solution to Eq. (28); i.e., $y(t;1)$ is a perturbation representation for the second Painlevé transcendent. Placing Eq. (20) back into Eq. (29), we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - (2+t)y_0 = \alpha\,. \tag{30}$$

Solving this equation, we obtain

$$y_0(t) = C_1\mathrm{Ai}(2+t) + C_2\mathrm{Bi}(2+t) + \alpha\pi\int_0^t \{\mathrm{Ai}(2+\tau)\mathrm{Bi}(2+t) - \mathrm{Ai}(2+t)\mathrm{Bi}(2+\tau)\}\,\mathrm{d}\tau\,, \tag{31}$$

where $C_1$ and $C_2$ are constants of integration to be determined by initial or boundary conditions. Here, Ai and Bi denote the Airy functions of first and second kind, respectively. The next term in the $\delta$-expansion solution is governed by

$$\ddot{y}_1 - (2+t)y_1 = 4y_0(t)\ln|y_0(t)|\,, \tag{32}$$

and the general solution is given by

$$y_1(t) = 4\pi\int^t \{\mathrm{Ai}(2+\tau)\mathrm{Bi}(2+t) - \mathrm{Ai}(2+t)\mathrm{Bi}(2+\tau)\}y_0(\tau)\ln|y_0(\tau)|\,\mathrm{d}\tau\,. \tag{33}$$

Note that this term is held subject to homogeneous initial or boundary conditions; that is to say, the order zero term holds all relevant initial or boundary data, while the higher order terms are due only to the nonlinearities. The higher order terms $y_k(t)$ may be found in a similar manner: all such equations are of the form

$$\ddot{y}_k - (2+t)y_k = \mathcal{F}_k(t)\,, \tag{34}$$

and hence

$$y_k(t) = \pi\int^t \{\mathrm{Ai}(2+\tau)\mathrm{Bi}(2+t) - \mathrm{Ai}(2+t)\mathrm{Bi}(2+\tau)\}\mathcal{F}_k(\tau)\,\mathrm{d}\tau\,. \tag{35}$$

### 3.3 *Painlevé III*

Painlevé III reads

$$ty\ddot{y} = t(\dot{y})^2 - y\dot{y} + \mu t + \beta y + \alpha y^3 + \gamma t y^4\,, \tag{36}$$

where the dot denotes differentiation with respect to $t$. Consider the related nonlinear differential equation

$$ty^\delta\ddot{y} = t(\dot{y})^{1+\delta} - y^\delta\dot{y} + \mu t + \beta y + \alpha y^{1+2\delta} + \gamma t y^{1+3\delta}\,, \tag{37}$$

so that when $\delta = 1$, we recover Eq. (36). We shall construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume a solution of the form (20). Provided this expansion converges at $\delta = 1$, we have that $y(t;1)$ is a solution to Eq. (36); i.e., $y(t;1)$ is a perturbation representation for the third Painlevé transcendent. Placing Eq. (20) into Eq. (37), we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$t\ddot{y}_0 - (t-1)\dot{y}_0 - (\beta + \alpha + \gamma t)y_0 = \mu t\,. \tag{38}$$

Solving this equation, we find that $y_0(t)$ may be expressed in terms of Kummer functions and integrals of their products with $t$ (due to the inhomogeneity present in the governing equation for $y_0(t)$). In particular, the homogeneous part of the equation admits a solution in terms of the two linearly independent functions $v_1^{\mathrm{III}}(t)$ and $v_2^{\mathrm{III}}(t)$, where

$$v_1^{\mathrm{III}}(t) = \exp\Big[-\frac{1}{2}(\sqrt{4\gamma+1}-1)t\Big]\mathrm{M}\Big(\frac{1}{2} + \frac{2(\alpha+\beta)-1}{2\sqrt{4\gamma+1}}, 1, \sqrt{4\gamma+1}t\Big)\,, \tag{39}$$

$$v_2^{\mathrm{III}}(t) = \exp\Big[-\frac{1}{2}(\sqrt{4\gamma+1}-1)t\Big]\mathrm{U}\Big(\frac{1}{2} + \frac{2(\alpha+\beta)-1}{2\sqrt{4\gamma+1}}, 1, \sqrt{4\gamma+1}t\Big)\,, \tag{40}$$

where M and U are Kummer functions of the first and second kind. Using variation of parameters and applying Abel's identity for the Wronskian $W(v_1, v_2)(t)$ (see Ref. [33]), we find that the order zero contribution is

$$y_0(t) = C_1 v_1^{\rm III}(t) + C_2 v_2^{\rm III}(t) + \mu C_3 \int^t \big\{ v_1^{\rm III}(\tau) v_2^{\rm III}(t) - v_1^{\rm III}(t) v_2^{\rm III}(\tau) \big\} \tau^2 \,{\rm e}^{-\tau}\,{\rm d}\tau\,, \tag{41}$$

where $C_1$ and $C_2$ are constants of integration to be determined by initial or boundary conditions and $C_3$ is a fixed constant determined by the relation $C_3 \equiv t_0^{-1}\,{\rm e}^{t_0}/W(v_1^{\rm III}, v_2^{\rm III})(t_0)$ where $t_0$ is any (non-zero) fixed point in the domain of the problem. Meanwhile, the higher order terms are governed by

$$t\ddot{y}_k - (t-1)\dot{y}_k - (\beta + \alpha + \gamma t) y_k = \mathcal{F}_k(t)\,, \tag{42}$$

which admits the solution

$$y_k(t) = C_3 \int^t \big\{ v_1^{\rm III}(\tau) v_2^{\rm III}(t) - v_1^{\rm III}(t) v_2^{\rm III}(\tau) \big\} \tau \,{\rm e}^{-\tau} \mathcal{F}_k(\tau)\,{\rm d}\tau\,, \tag{43}$$

for $k = 1, 2, 3, \ldots$

### 3.4 *Painlevé IV*

Painlevé IV reads

$$y\ddot{y} = \frac{1}{2}(\dot{y})^2 + \beta + 2(t^2 - \alpha) y^2 + 4ty^3 + \frac{3}{2} y^4\,, \tag{44}$$

where the dot denotes differentiation with respect to $t$. Consider the related nonlinear differential equation

$$y^\delta \ddot{y} = \frac{1}{2}(\dot{y})^{1+\delta} + \beta + 2(t^2 - \alpha) y^{1+\delta} + 4ty^{1+2\delta} + \frac{3}{2} y^{1+3\delta}\,, \tag{45}$$

so that when $\delta = 1$, we recover Eq. (44). We shall construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume a solution of the form (20). Provided this expansion converges at $\delta = 1$, we have that $y(t;1)$ is a solution to Eq. (44); i.e., $y(t;1)$ is a perturbation representation for the fourth Painlevé transcendent. Placing Eq. (20) into Eq. (45), we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - \frac{1}{2}\dot{y}_0 - \Big(2t^2 + 4t + \frac{3}{2} - \alpha\Big) y_0 = \beta\,. \tag{46}$$

Solving this equation, we find that $y_0(t)$ may be expressed in terms of Hypergeometric functions and integrals of their products with $t$ (due to the inhomogeneity present in the governing equation for $y_0(t)$). The homogeneous part of the equation admits a solution in terms of the two linearly independent functions $v_1^{\rm IV}(t)$ and $v_2^{\rm IV}(t)$, where

$$v_1^{\rm IV}(t) = \exp\Big[ -\Big(\frac{1}{\sqrt{2}}(t+2) - \frac{1}{4}\Big)t\Big]\,{}_1F_1\Big(\frac{1}{4} - \frac{\sqrt{2}(7+16\alpha)}{128}, \frac{1}{2}, \sqrt{2}(t+1)^2\Big)\,, \tag{47}$$

$$v_2^{\rm IV}(t) = \exp\Big[ -\Big(\frac{1}{\sqrt{2}}(t+2) - \frac{1}{4}\Big)t\Big]\,{}_1F_1\Big(\frac{1}{4} - \frac{\sqrt{2}(7+16\alpha)}{128}, \frac{3}{2}, \sqrt{2}(t+1)^2\Big)\,, \tag{48}$$

where ${}_1F_1$ is a hypergeometric function. Using variation of parameters and applying Abel's identity for the Wronskian $W(v_1^{\rm IV}, v_2^{\rm IV})(t)$, we find that the order zero contribution is

$$y_0(t) = C_1 v_1^{\rm IV}(t) + C_2 v_2^{\rm IV}(t) + \beta C_3 \int^t \big\{ v_1^{\rm IV}(\tau) v_2^{\rm IV}(t) - v_1^{\rm IV}(t) v_2^{\rm IV}(\tau) \big\}\,{\rm e}^{-\tau/2}\,{\rm d}\tau\,, \tag{49}$$

where $C_1$ and $C_2$ are constants of integration to be determined by initial or boundary conditions and $C_3$ is a fixed constant determined by the relation $C_3 \equiv t_0^{-1}\,{\rm e}^{t_0}/W(v_1^{\rm IV}, v_2^{\rm IV})(t_0)$ where $t_0$ is any (non-zero) fixed point in the domain of the problem. Meanwhile, the higher order terms are governed by

$$\ddot{y}_k - \frac{1}{2}\dot{y}_k - \Big(2t^2 + 4t + \frac{3}{2} - \alpha\Big) y_k = \mathcal{F}_k(t)\,, \tag{50}$$

which admits the solution

$$y_k(t) = C_3 \int^t \big\{ v_1^{\rm IV}(\tau) v_2^{\rm IV}(t) - v_1^{\rm IV}(t) v_2^{\rm IV}(\tau) \big\}\,{\rm e}^{-\tau/2} \mathcal{F}_k(\tau)\,{\rm d}\tau\,, \tag{51}$$

for $k = 1, 2, 3, \ldots$

### 3.5 *Painlevé V*

Painlevé V reads

$$\ddot{y} = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)(\dot{y}\Big)^2 - \frac{1}{t}\dot{y} + \frac{(y-1)^2}{t}\Big(\alpha y + \frac{\beta}{y}\Big) + \gamma\frac{y}{t} + \mu\frac{y(y+1)}{y-1}\,, \tag{52}$$

where the dot denotes differentiation with respect to $t$. Consider the related nonlinear differential equation

$$\ddot{y} = \Big(\frac{1}{2y^\delta} + \frac{1}{(y-1)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{1}{t}\dot{y} + \frac{(y-1)^{1+\delta}}{t}\Big(\alpha y^\delta + \frac{\beta}{y^\delta}\Big) + \gamma\frac{y}{t} + \mu\frac{y^{\chi_1}(y+1)^{\chi_2}}{(y-1)^\delta}\,, \tag{53}$$

so that when $\delta = 1$, we recover Eq. (52). Here, either $\chi_1 = \delta$ and $\chi_2 = 1$ or $\chi_1 = 1$ and $\chi_2 = \delta$, and we shall consider each case separately. We construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume a solution of the form (20). Provided this expansion converges at $\delta=1$, we have that $y(t;1)$ is a solution to Eq. (52); i.e., $y(t;1)$ is a perturbation representation for the fifth Painlevé transcendent.

(i) ***Case 1*** $\chi_1=\delta$ and $\chi_2=1$

First we consider the case in which $\chi_1=\delta$ and $\chi_2=1$. Placing Eq. (20) into Eq. (53) and setting $\chi_1=\delta$ and $\chi_2=1$, we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0-\Big(\frac{3}{2}-\frac{1}{t}\Big)\dot{y}_0-\Big(\frac{1+\gamma}{t}+\mu\Big)y_0=\mu-\frac{\alpha+\beta}{t}\,. \tag{54}$$

Solving this equation, we find that $y_0(t)$ may be expressed in terms of Kummer functions and integrals of their products with $t$ (due to the inhomogeneity present in the governing equation for $y_0(t)$). The homogeneous part of the equation admits a solution in terms of the two linearly independent functions $v_1^{\rm V}(t)$ and $v_2^{\rm V}(t)$, where

$$v_1^{\rm V}(t)=\exp\Big[-\frac{1}{4}(\sqrt{16\mu+9}-3)t\Big]{\rm M}\Big(\frac{1}{2}+\frac{4\gamma+1}{\sqrt{16\mu+9}},1,\frac{1}{2}\sqrt{16\mu+9}t\Big)\,, \tag{55}$$

$$v_2^{\rm V}(t)=\exp\Big[-\frac{1}{4}\Big(\sqrt{16\mu+9}-3)t\Big]{\rm U}\Big(\frac{1}{2}+\frac{4\gamma+1}{\sqrt{16\mu+9}},1,\frac{1}{2}\sqrt{16\mu+9}t\Big)\,, \tag{56}$$

where M and U are Kummer functions of the first and second kind. Note that the choice of parameters $\alpha$ and $\beta$ does not effect these basis functions. Using variation of parameters and applying Abel's identity for the Wronskian $W(v_1^{\rm V},v_2^{\rm V})(t)$, we find that the order zero contribution is

$$y_0(t)=C_1v_1^{\rm V}(t)+C_2v_2^{\rm V}(t)+C_3\int^t\big\{v_1^{\rm V}(\tau)v_2^{\rm V}(t)-v_1^{\rm V}(t)v_2^{\rm V}(\tau)\big\}\big\{\mu\tau-(\alpha+\beta)\big\}\,{\rm e}^{-3\tau/2}\,{\rm d}\tau\,, \tag{57}$$

where $C_1$ and $C_2$ are constants of integration to be determined by initial or boundary conditions and $C_3$ is a fixed constant determined by the relation $C_3\equiv t_0^{-1}\,{\rm e}^{t_0}/W(v_1^{\rm V},v_2^{\rm V})(t_0)$ where $t_0$ is any (non-singular) point in the domain of the problem. The higher order terms are governed by the linear equations

$$\ddot{y}_k-\Big(\frac{3}{2}-\frac{1}{t}\Big)\dot{y}_k-\Big(\frac{1+\gamma}{t}+\mu\Big)y_k=\mathcal{F}_k(t)\,, \tag{58}$$

which admits the solution

$$y_k(t)=C_3\int^t\big\{v_1^{\rm V}(\tau)v_2^{\rm V}(t)-v_1^{\rm V}(t)v_2^{\rm V}(\tau)\big\}\tau\,{\rm e}^{-3\tau/2}\mathcal{F}_k(\tau)\,{\rm d}\tau\,, \tag{59}$$

for $k=1,2,3,\ldots$

(ii) ***Case 2*** $\chi_1=1$ and $\chi_2=\delta$

First we consider the case in which $\chi_1=1$ and $\chi_2=\delta$. Placing Eq. (20) into Eq. (53) and setting $\chi_1=1$ and $\chi_2=\delta$, we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0-\Big(\frac{3}{2}-\frac{1}{t}\Big)\dot{y}_0-\Big(\frac{1+\gamma}{t}+\mu\Big)y_0=-\frac{\alpha+\beta}{t}\,. \tag{60}$$

Solving this equation, we find that $y_0(t)$ may be expressed in terms of Kummer functions and integrals of their products with $t$ (due to the inhomogeneity present in the governing equation for $y_0(t)$). Here the choice of basis functions is the same as was considered in Case 1, the only difference here is in the inhomogeneity. The order zero term is thus given by

$$y_0(t)=C_1v_1^{\rm V}(t)+C_2v_2^{\rm V}(t)-(\alpha+\beta)C_3\int^t\big\{v_1^{\rm V}(\tau)v_2^{\rm V}(t)-v_1^{\rm V}(t)v_2^{\rm V}(\tau)\big\}\,{\rm e}^{-3\tau/2}\,{\rm d}\tau\,, \tag{61}$$

while the higher order contributions are given again by Eq. (59) (although the explicit dependence of $\mathcal{F}_k$ has changed due to the change in $y_0(t)$).

### 3.6 *Painlevé VI*

Painlevé VI reads

$$\ddot{y}=\frac{1}{2}\Big(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\Big)(\dot{y})^2-\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\Big)\dot{y}+\frac{y(y+1)(y-t)}{t^2(t-1)^2}\Big(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\mu\frac{t(t-1)}{(y-t)^2}\Big)\,, \tag{62}$$

where the dot denotes differentiation with respect to $t$. Consider the related nonlinear differential equation

$$\ddot{y}=\frac{1}{2}\Big(\frac{1}{y^\delta}+\frac{1}{(y-1)^\delta}+\frac{1}{(y-t)^\delta}\Big)(\dot{y})^{1+\delta}-\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{(y-t)^\delta}\Big)\dot{y}$$
$$+\frac{y^{\chi_1}(y+1)^{\chi_2}(y-t)^{\chi_3}}{t^2(t-1)^2}\Big(\alpha+\beta\frac{t}{y^{2\delta}}+\gamma\frac{t-1}{(y-1)^{2\delta}}+\mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,, \tag{63}$$

so that when $\delta=1$, we recover Eq. (62). Here, either (i) $\chi_1=1$ and $\chi_2=\chi_3=\delta$ or (ii) $\chi_1=\chi_3=\delta$ and $\chi_2=1$ or (iii) $\chi_1=\chi_2=\delta$ and $\chi_3=1$, and we shall consider each case separately. We construct a perturbation solution in $\delta$, about $\delta = 0$. To this end, assume a solution of the form (20). Provided this expansion converges at $\delta = 1$, we have that $y(t;1)$ is a solution to Eq. (62) i.e., $y(t;1)$ is a perturbation representation for the sixth Painlevé transcendent.

(i) ***Case 1*** $\chi_1 = 1$ and $\chi_2 = \chi_3 = \delta$

First we consider the case in which $\chi_1 = 1$ and $\chi_2 = \chi_3 = \delta$. Placing Eq. (20) into Eq. (63) and setting $\chi_1 = 1$ and $\chi_2 = \chi_3 = \delta$, we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - \Big(\frac{1}{2} - \frac{1}{t} - \frac{1}{t-1}\Big)\dot{y}_0 - \Big(\frac{\alpha + \beta t + \gamma(t-1) + \mu t(t-1)}{t^2(t-1)^2}\Big)y_0 = 0\,. \tag{64}$$

The homogeneous part of the linear equations will admit a solution in terms of the two linearly independent functions $v_1^{\rm VI}(t)$ and $v_2^{\rm VI}(t)$, where

$$v_1^{\rm VI}(t) = \,{\rm e}^{t/2}(t-1)^{\sqrt{\alpha+\beta}}t^{\sqrt{\alpha-\gamma}}\ {\rm HeunC}\Big(\frac{1}{2}, 2\sqrt{\alpha-\gamma}, 2\sqrt{\alpha+\beta}, \frac{1}{2}, 2\alpha+\beta-\gamma-\mu-\frac{1}{4}, t\Big)\,, \tag{65}$$

$$v_1^{\rm VI}(t) = \,{\rm e}^{t/2}(t-1)^{\sqrt{\alpha+\beta}}t^{-\sqrt{\alpha-\gamma}}\ {\rm HeunC}\Big(\frac{1}{2}, -2\sqrt{\alpha-\gamma}, 2\sqrt{\alpha+\beta}, \frac{1}{2}, 2\alpha+\beta-\gamma-\mu-\frac{1}{4}, t\Big)\,, \tag{66}$$

where HeunC denotes the confluent Heun function (for confluent Heun's functions, see Refs. [34–35]). Using variation of parameters and applying Abel's identity for the Wronskian $W(v_1^{\rm VI}, v_2^{\rm VI})(t)$, we find that the order zero contribution is

$$y_0(t) = C_1 v_1^{\rm VI}(t) + C_2 v_2^{\rm VI}(t)\,, \tag{67}$$

where $C_1$ and $C_2$ are constants of integration to be determined by initial or boundary conditions. The higher order terms are governed by the linear equations

$$\ddot{y}_0 - \Big(\frac{1}{2} - \frac{1}{t} - \frac{1}{t-1}\Big)\dot{y}_0 - \Big(\frac{\alpha + \beta t + \gamma(t-1) + \mu t(t-1)}{t^2(t-1)^2}\Big)y_0 = \mathcal{F}_k(t)\,, \tag{68}$$

which admits the solution

$$y_k(t) = C_3 \int^t \big\{v_1^{\rm VI}(\tau)v_2^{\rm VI}(t) - v_1^{\rm VI}(t)v_2^{\rm VI}(\tau)\big\}\tau(\tau-1)\,{\rm e}^{-\tau/2}\mathcal{F}_k(\tau)\,{\rm d}\tau\,, \tag{69}$$

for $k = 1, 2, 3, \ldots$, where $C_3$ is a fixed constant determined by the relation

$$C_3 \equiv t_0^{-1}\,{\rm e}^{t_0}/W(v_1^{\rm VI}, v_2^{\rm VI})(t_0)\,, \tag{70}$$

where $t_0$ is any (non-singular) point in the domain of the problem.

(ii) ***Case 2*** $\chi_1 = \chi_3 = \delta$ and $\chi_2 = 1$

Consider the case in which $\chi_1 = \chi_3 = \delta$ and $\chi_2 = 1$. Placing Eq. (20) into Eq. (63) and setting $\chi_1 = \chi_3 = \delta$ and $\chi_2 = 1$, we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - \Big(\frac{1}{2} - \frac{1}{t} - \frac{1}{t-1}\Big)\dot{y}_0 - \Big(\frac{\alpha + \beta t + \gamma(t-1) + \mu t(t-1)}{t^2(t-1)^2}\Big)(y_0 - 1) = 0\,. \tag{71}$$

Solving this equation, we find that $y_0(t)$ is simply

$$y_0(t) = 1 + C_1 v_1^{\rm VI}(t) + C_2 v_2^{\rm VI}(t)\,, \tag{72}$$

where $C_1$ and $C_2$ are constants of integration, while the higher order terms are still of the form (69).

(iii) ***Case 3*** $\chi_1 = \chi_2 = \delta$ and $\chi_3 = 1$

Consider the case in which $\chi_1 = \chi_2 = \delta$ and $\chi_3 = 1$. Placing Eq. (20) into Eq. (63) and setting $\chi_1 = \chi_2 = \delta$ and $\chi_3 = 1$, we find that the first approximant, $y_0(t)$, is given by the solution to the linear ordinary differential equation

$$\ddot{y}_0 - \Big(\frac{1}{2} - \frac{1}{t} - \frac{1}{t-1}\Big)\dot{y}_0 - \Big(\frac{\alpha + \beta t + \gamma(t-1) + \mu t(t-1)}{t^2(t-1)^2}\Big)(y_0 - t) = 0\,. \tag{73}$$

Solving this equation, we find that $y_0(t)$ may again be expressed in terms of confluent Heun's functions and also integrals of their products and $t$ (due to the fact that, once cleaned up a bit, the order zero equation is now inhomogenious). In particular,

$$y_0(t) = C_1 v_1^{\rm VI}(t) + C_2 v_2^{\rm VI}(t) - C_3 \int^t \big\{v_1^{\rm VI}(\tau)v_2^{\rm VI}(t) - v_1^{\rm VI}(t)v_2^{\rm VI}(\tau)\big\}\frac{\alpha + \beta\tau + \gamma(\tau-1) + \mu\tau(\tau-1)}{t(t-1)}\,{\rm e}^{-3\tau/2}{\rm d}\tau\,, \tag{74}$$

for $k = 1, 2, 3, \ldots$, where $C_3$ is a fixed constant determined by the relation

$$C_3 \equiv t_0^{-1}\,{\rm e}^{t_0}/W(v_1^{\rm VI}, v_2^{\rm VI})(t_0)\,, \tag{75}$$

where $t_0$ is any (non-singular) point in the domain of the problem. The higher order terms are still of the form (69).

(iv) A second method for computing perturbation solutions for Painlevé VI

Here we employ an alternate auxiliary nonlinear operator, which preserves a bit more of the singular structure of Painlevé VI. To this end, let us define

$$\ddot{y} = \frac{1}{2}\Big(\frac{1}{y^\delta} + \frac{1}{y^\delta - 1} + \frac{1}{y^\delta - t}\Big)(\dot{y})^{1+\delta} - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y^\delta - t}\Big)\dot{y} + \frac{y^{\chi_1}(y+1)^{\chi_2}(y-t)^{\chi_3}}{t^2(t-1)^2}\Big(\alpha + \beta\frac{t}{y^{2\delta}} + \gamma\frac{t-1}{(y-1)^{2\delta}} + \mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,, \tag{76}$$

where we have taken $(y(t)^\delta - t)$ as opposed to $(y(t)-t)^\delta$ (and so on) in the coefficients of $\dot{y}$ and $(\dot{y})^2$. Thus, when $\delta = 0$, we preserve more of the singular structure of the Painlevé VI equation. The linear equation governing the order zero term is then given by

$$\ddot{y}_0 + \Big(\frac{1}{2t} + \frac{1}{t-1}\Big)\dot{y}_0 - \Big(\frac{\alpha + \beta t + \gamma(t-1) + \mu t(t-1)}{t^2(t-1)^2}\Big)X(t) = 0\,, \tag{77}$$

where $X(t)$ is again one of $y_0(t)$, $y_0(t)-1$ or $y_0(t)-t$. The homogeneous part of the solutions will then be given in terms of the functions

$$\tilde{v}_1^{\rm VI}(t) = t^{(1/4)(1+\sqrt{1+16(\alpha-\gamma)})}(t-1)^{-\sqrt{\alpha+\beta}}\ {}_2F_1(\lambda_1^+, \lambda_2^+; \lambda_3^+; t)\,, \tag{78}$$

$$\tilde{v}_2^{\rm VI}(t) = t^{(1/4)(1-\sqrt{1+16(\alpha-\gamma)})}(t-1)^{-\sqrt{\alpha+\beta}}\ {}_2F_1(\lambda_1^-, \lambda_2^-; \lambda_3^-; t)\,, \tag{79}$$

where

$$\lambda_1^\pm = \frac{1}{2} - \sqrt{\alpha+\beta} \pm \frac{1}{4}\sqrt{1+16(\alpha-\gamma)} \mp \frac{1}{4}\sqrt{1+16\mu}\,, \tag{80}$$

$$\lambda_2^\pm = \frac{1}{2} - \sqrt{\alpha+\beta} \pm \frac{1}{4}\sqrt{1+16(\alpha-\gamma)} \pm \frac{1}{4}\sqrt{1+16\mu}\,, \tag{81}$$

$$\lambda_3^\pm = 1 \pm \frac{1}{2}\sqrt{1+16(\alpha-\gamma)}\,. \tag{82}$$

Note that the $\lambda_{1,2}$'s satisfy the simpler linear relations

$$\lambda_1^\pm + \lambda_2^\pm = -2\sqrt{\alpha+\beta} + \lambda_3^\pm\,,$$
$$\lambda_1^\pm - \lambda_2^\pm = \mp\frac{1}{2}\sqrt{1+16\mu}\,. \tag{83}$$

### 3.7 *Comments on Linearizations and Relations to Special Functions*

We have applied the $\delta$ perturbation method in order to construct perturbation solutions for the Painlevé transcendents. As we have shown in a number of explicit special cases, these expansions are capable of outperforming both Taylor series and standard perturbation solutions, even for the more strongly nonlinear Painlevé III–VI equations. In our perturbation solutions, we found that the solutions to the linearized equations will be in terms of various special functions, for Painlevé II–VI. Thus, there a strong link is present between certain special functions defined by second order linear equations and the Painlevé transcendents (which are really special functions defined by second order nonlinear equations). In particular, under our framework for linearization, the Painlevé II transcendent relates to the Airy functions of first and second kind, the Painlevé III and V transcendents relate to Kummer functions of first and second kind, the Painlevé IV transcendent relates to hypergeometric functions of type ${}_1F_1$, and the Painlevé VI transcendent relates to either confluent Heun functions or hypergeometric functions of type ${}_2F_1$, depending on the linearization method selected. Previously, the close relationship between the Airy function and Painlevé II was discussed in Ref. [36], where the Painlevé II transcendent was described as (essentially) a nonlinear Airy function. Indeed, in the limiting case $y \to 0$, Painlevé II reduces to the Airy equation as pointed out in Ref. [28], where the authors linearize Painlevé II by converting the nonlinear differential equations into a linear integral equation. While we have kept the Painlevé equations considered here in their most general form prior to linearization, we note that in some reductions and limiting cases, the equations correspond more readily with certain linear special functions. For a more thorough review of this point, see the review paper[37] and the many excellent references therein.

## 4 Improving the Usefulness of Solutions near Singularities

In certain situations, controlling error by adding more terms may be counter-productive. Indeed, when the perturbation expansion diverges, adding very many terms will make the expansion solution worse. Employing rational Padé approximations can help here. Previously, Ref. [38] applied Padé approximations to the Taylor series solutions of Painlevé I and II, and was able to recover solutions for each in regions with multiple poles. Along these lines, consider a truncated expansion in $\delta$ of the form

$$y_m(x\delta) = y_0(x) + \sum_{n=1}^{m} y_n(x)\delta^n\,. \tag{84}$$

From this expansion, we may construct the $[i,j]$ Padé approximation

$$P_{i,j}(y_m(x;\delta)) = \frac{A_0(y_0,\ldots,y_m) + A_1(y_0,\ldots,y_m)\delta + \cdots + A_i(y_0,\ldots,y_m)\delta^i}{1 + B_1(y_0,\ldots,y_m)\delta + \cdots + B_j(y_0,\ldots,y_m)\delta^j}\,. \tag{85}$$

Setting the denominator equal to zero, we can approximate the poles of the solution. Note that a Padé approximation based on Taylor series would have up to $m$ poles in the complex plane. However, due to the fact that each of the $y_k$'s is a complicated function of $x$, we may be able to capture many more poles with only an $m$ order approximation in $\delta$, depending on the structure of the solution. For instance, consider the expansion given in Sec. 3 for the second Painlevé transcendent. Any $[i, j]$ Padé approximant will necessarily involve the Airy functions. Hence, there may be many more roots to $1 + B_1(y_0, \ldots, y_m) + \cdots + B_j(y_0, \ldots, y_m) = 0$ than in the case of a comparable Padé approximant based on Taylor series.

## 5 Discussion

The beauty of the method is that it allows us to control the manner in which the inherent nonlinearities are taken into account when constructing a linearized system. As such, the linearized equations should more accurately represent the nonlinear equations than, say, linearizations due to standard perturbation methods or even low-order Taylor series solutions. We see this in the specific examples considered. Furthermore, the existence of singularities (particularly the movable poles) can greatly complicate numerical simulation of the Painlevé transcendents, especially transcendents III–VI. We then need an alternate way to compare the obtained perturbation solutions to some measure of an exact solution. As it turns out, we may construct formal Taylor series solutions to the nonlinear equations (or, at the very least, compute the first $n$ terms of such series expansions), and compare these with the perturbation solutions evaluated at $\delta = 1$, so that we may get a feel for how the perturbation solutions differ from the Taylor series, which must be a good model of the true solutions over the valid region of convergence for such series solutions. We find that, as Taylor series solutions converge over symmetric intervals, their region of convergence (as solutions to an initial value problem for one of the Painlevé transcendents) is significantly influenced by the location of any singularities. The obtained perturbation solutions do not appear to suffer so such from this, as they are constructed from non-polynomial functions, which more accurately approximate the Painlevé transcendents over the regions between singularities. Indeed, one may construct linear approximations via $\delta$-expansion which share (some of) the singular structure of the original nonlinear Painlevé equations. Note that one may also construct a Laurent series expansion around a singularity (when appropriate), however the region of convergence for such an expansion still suffers when one moves sufficiently far from the pole.

### 5.1 *Computational Considerations*

Notice the appearance of natural logarithms in these expressions. These will serve to complicate the solution process when we attempt to obtain the iterates in the perturbation expansion about $\delta$. The trade-off, we find, is that the iterates tend to converge more rapidly to the solution when we apply the $\delta$-expansion method, compared to traditional perturbation about a small parameter. This is rather important, as the equations, particularly the latter four, are relatively complicated and thus computing many iterates in the perturbation expansions, even with the resulting linearized systems, becomes tedious and computationally demanding. Very low order solutions (say, of order no greater than $\delta^2$) may be constructed by use of a computer algebra system (e.g., *Maple, Mathematica*, and such), though for anything more than this, we really do need to resort to computing the subsequent terms numerically. The numerical problem then consists of solving several linear equations in sequence, rather than solving the original nonlinear problem. Hence, expanding about $\delta$, we convert a hard numerical problem into multiple more tractable problems.

### 5.2 *Summary of the Perturbation Scheme*

To summarize, there are three parts to the method:

First, we must define and auxiliary nonlinear equation, parametrized with $\delta \in [0, 1]$ so that when $\delta = 1$ we have the original nonlinear equation while when $\delta = 0$ we have a linearized version of the original equation. This linearization is a second order linear ordinary differential equation which in some cases defines a special function.

Secondly, one defines an expansion in $\delta$ which is a formal solution to the auxiliary nonlinear equation. Computing the coefficients recursively via inversion of the linearized operator, one may construct such a solution up to a desired order in $\delta$. The solutions for these terms will be given as complicated functions of the solutions to the linearized equation. Setting $\delta = 1$ in the expansion, one recovers the formal expansion for the original equation.

Third, the choice of base function in the solution expression will depend greatly on the linearized equation. Hence, in some situations, one may wish to consider multiple linearizations to ensure the greatest rate of convergence. Furthermore, in order to control convergence of the formal solutions, additional methods such as Padé approximants may be used to construct rational solutions in $\delta$. Such methods may also be used for detecting singularities in the solutions.

## Appendix A: Forms of the Inhomogeneities $\mathcal{F}_k$ for Painlevé I–VI

Recall that each of the Painlevé equations may be represented as $L[y] + N[y] = f(t)$, where $L$ is a linear differential operator, $N[y]$ is a nonlinear operator, and $f(t)$ is some inhomogeneity (which is zero for some of the Painlevé equations). Employing the $\delta$-expansion technique, we define the nonlinear operator $N[y; \delta]$ so that $N[y; 0] = L_1[y]$ and $N[y; 1] = N[y]$. Then, as was discussed in Sec. 2, the terms in the expansion $y(t; \delta) =$

$y_0(t) + y_1(t)\delta + \cdots$ are governed by linear equations of the form $L[y_k] + L_1[y_k] = \mathcal{F}_k(y_0, y_1, \ldots, y_{k-1})$ for $k \geq 1$ and $L[y_0] + L_1[y_0] = f(t)$ for $k = 0$. We now discuss the computation of the $\mathcal{F}_k$'s.

In general, the $F_k$'s are given by the terms in the expansion

$$\sum_{k=1}^{\infty} \mathcal{F}_k \delta^k = N[y(t;\delta);0] - N[y(t;\delta);\delta] = L_1[y(t;\delta)] - N[y(t;\delta);\delta]\,. \tag{A1}$$

It is clear from this definition that $\mathcal{F}_k$ depends on $y_0, y_1, \ldots, y_{k-1}$ (and perhaps explicitly on $t$, depending on the form of $N[y]$). Defining the function $T(\delta)$ by

$$T(\delta) \equiv N[y(t;\delta);0] - N[y(t;\delta);\delta]\,, \tag{A2}$$

we may represent each $\mathcal{F}_k$ by the relation

$$\mathcal{F}_k = \frac{1}{k!}\Big(\frac{\partial^k T}{\partial \delta^k}\Big)_{\delta=0}\,. \tag{A3}$$

Thus, with a knowledge of the $\delta$-dependent nonlinear operator $N[y,\delta]$, one may in principle compute all of the inhomogeneities used in finding the $y_k$'s. For Painlevé I–IV, we have that

$$N[y;\delta] = -6y^{1+\delta}\,, \tag{A4}$$

$$N[y;\delta] = -2y^{1+2\delta}\,, \tag{A5}$$

$$N[y;\delta] = ty^\delta\ddot{y} - t(\dot{y})^{1+\delta} + y^\delta\dot{y} - \beta\delta\Big(1 - \frac{1}{y(0)}\Big)y - \alpha\delta(1 - y(0))y^{1+2\delta} - \gamma t y^{1+3\delta}\,, \tag{A6}$$

$$N[y;\delta] = y^\delta\ddot{y} - \frac{1}{2}(\dot{y})^{1+\delta} - 2(t^2 - \alpha)y^{1+\delta} - 4ty^{1+2\delta} - \frac{3}{2}y^{1+3\delta}\,, \tag{A7}$$

respectively. For each of the two cases considered for Painlevé V, we have

$$N[y;\delta] = -\Big(\frac{1}{2y^\delta} + \frac{1}{(y-1)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{(y-1)^{1+\delta}}{t}\Big(\alpha y^\delta + \frac{\beta}{y^\delta}\Big) - \mu\frac{y^\delta(y+1)}{(y-1)^\delta}\,, \tag{A8}$$

$$N[y;\delta] = -\Big(\frac{1}{2y^\delta} + \frac{1}{(y-1)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{(y-1)^{1+\delta}}{t}\Big(\alpha y^\delta + \frac{\beta}{y^\delta}\Big) - \mu\frac{y(y+1)^\delta}{(y-1)^\delta}\,, \tag{A9}$$

for case 1 and 2, respectively. Finally, for Painlevé VI, we have that

$$\begin{aligned} N[y;\delta] = &-\frac{1}{2}\Big(\frac{1}{y^\delta} + \frac{1}{(y-1)^\delta} - \frac{1}{(y-t)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{1}{(y-t)^\delta}\dot{y} \\ &- \frac{y(y+1)^\delta(y-t)^\delta}{t^2(t-1)^2}\Big(\alpha + \beta\frac{t}{y^{2\delta}} + \gamma\frac{t-1}{(y-1)^{2\delta}} + \mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,, \end{aligned} \tag{A10}$$

$$\begin{aligned} N[y;\delta] = &-\frac{1}{2}\Big(\frac{1}{y^\delta} + \frac{1}{(y-1)^\delta} - \frac{1}{(y-t)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{1}{(y-t)^\delta}\dot{y} \\ &- \frac{y^\delta(y+1)(y-t)^\delta}{t^2(t-1)^2}\Big(\alpha + \beta\frac{t}{y^{2\delta}} + \gamma\frac{t-1}{(y-1)^{2\delta}} + \mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,, \end{aligned} \tag{A11}$$

$$\begin{aligned} N[y;\delta] = &-\frac{1}{2}\Big(\frac{1}{y^\delta} + \frac{1}{(y-1)^\delta} - \frac{1}{(y-t)^\delta}\Big)(\dot{y})^{1+\delta} - \frac{1}{(y-t)^\delta}\dot{y} \\ &- \frac{y^\delta(y+1)^\delta(y-t)}{t^2(t-1)^2}\Big(\alpha + \beta\frac{t}{y^{2\delta}} + \gamma\frac{t-1}{(y-1)^{2\delta}} + \mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,, \end{aligned} \tag{A12}$$

for cases 1–3, respectively, while for the second method employed, we have

$$\begin{aligned} N[y;\delta] = &-\frac{1}{2}\Big(\frac{1}{y^\delta} + \frac{1}{(y^\delta-1)} - \frac{1}{(y^\delta-t)}\Big)(\dot{y})^{1+\delta} - \frac{1}{(y-t)^\delta}\dot{y} \\ &- \frac{y^\delta(y+1)^\delta(y-t)}{t^2(t-1)^2}\Big(\alpha + \beta\frac{t}{y^{2\delta}} + \gamma\frac{t-1}{(y-1)^{2\delta}} + \mu\frac{t(t-1)}{(y-t)^{2\delta}}\Big)\,. \end{aligned} \tag{A13}$$

From these relations, one may apply the formula (A3) to compute $\mathcal{F}_k$ in each of the cases considered. For example, consider Painlevé II. From Eqs. (A3) and (A5), we see that

$$\mathcal{F}_1 = \Big\{2\frac{\partial y(t;\delta)}{\partial\delta}((y(t;\delta))^{2\delta} - 1) + 2y(t;\delta)\frac{\partial}{\partial\delta}((y(t;\delta))^{2\delta})\Big\}_{\delta=0} = 4y_0(t)\ln|y_0(t)|\,. \tag{A14}$$

In this manner all of the inhomogeneities $\mathcal{F}_k$ may be computed for each of the Painlevé equations.

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