Multi-Type Directed Scale-Free Percolation

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Abstract *In this paper, we study a long-range percolation model on the lattice $\mathbb{Z}^d$ with multi-type vertices and directed edges. Each vertex $x \in \mathbb{Z}^d$ is independently assigned a non-negative weight W_x and a type ψ_x, where $(W_x)_{x\in\mathbb{Z}^d}$ are i.i.d. random variables, and $(\psi_x)_{x\in\mathbb{Z}^d}$ are also i.i.d. Conditionally on weights and types, and given $\lambda, \alpha > 0$, the edges are independent and the probability that there is a directed edge from x to y is given by $p_{xy} = 1 - \exp(-\lambda\phi_{\psi_x\psi_y} W_x W_y/|x-y|^\alpha)$, where ϕ_{ij}'s are entries from a type matrix Φ. We show that, when the tail of the distribution of W_x is regularly varying with exponent $\tau - 1$, the tails of the out/in-degree distributions are both regularly varying with exponent $\gamma = \alpha(\tau-1)/d$. We formulate conditions under which there exist critical values $\lambda_c^{WCC} \in (0,\infty)$ and $\lambda_c^{SCC} \in (0,\infty)$ such that an infinite weak component and an infinite strong component emerge, respectively, when λ exceeds them. A phase transition is established for the shortest path lengths of directed and undirected edges in the infinite component at the point $\gamma = 2$, where the out/in-degrees switch from having finite to infinite variances. The random graph model studied here features some structures of multi-type vertices and directed edges which appear naturally in many real-world networks, such as the SNS networks and computer communication networks.*

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1 Introduction

Over the last few decades, complex networks have triggered a surge of research interest that has led to a general framework within which to analyze their topologies as well as the dynamical processes running on them.[1−3] Of special interest are the scale-free random networks in which the number of vertices with degree k falls off as an inverse power of k. Scale-free graphs are common in real-world large-scale networks and have been proposed as an approach to model the behaviors of social, technological, and biological networks.[1,4−5]

Complex networks usually incorporate a geometric component to them where the vertices have positions in space and geographic proximity plays an important role in determining which vertices get connected. Long-range percolation models (see e.g. Refs. [6–11]) provide one way to describe networks with spatial content. In the most commonly studied long-range percolation, two vertices $x, y \in \mathbb{Z}^d$ are connected by an edge with a probability that decays like $\lambda|x-y|^{-\alpha}$, for some parameters $\alpha, \lambda > 0$, as the Euclidean distance $|x-y| \to \infty$, and the presence or absence of an edge is independent on the presence or absence of other edges. We refer to $\lambda|x-y|^{-\alpha}$ as the connection function. Questions such as the appearance and uniqueness of an infinite component and the decay of connection functions can be asked for properties of the random graph $G(\lambda, \alpha)$ obtained by long-range percolation.

Another class of network models received substantial attention consists of inhomogeneous random graphs (see e.g. [12–17]), where the edge probabilities are defined in terms of weights that associated to the vertices. In contrast to classical Erdös–Rényi random graphs,[18] the edges in these models are conditionally independent, given some vertex weights. Recently, Deijfen, van der Hofstad, and Hooghiemstra[19] introduced a model $G(\lambda, \alpha, W)$ for spatial inhomogeneous random graphs on $\mathbb{Z}^d$ with long-range edges and random vertex weights. Specifically, for $\alpha, \lambda \in (0, \infty)$, the probability of having an edge between vertices $x, y \in \mathbb{Z}^d$ is defined by

$$p_{xy} = 1 - \mathrm{e}^{-\lambda W_x W_y/|x-y|^\alpha},$$

where the i.i.d. non-negative weights $(W_x)_{x\in\mathbb{Z}^d}$ follow a power law of the form

$$P(W_x > w) = w^{-(\tau-1)} L(w).$$

Here, $L(w)$ is a function that varies slowly at infinity[20] and the exponent τ satisfies $\tau > 1$. This notable model can be viewed as an interpolation between long-range percolation and inhomogeneous random graphs, and the authors in [19] showed that many behaviors of $G(\lambda, \alpha, W)$, such as the degree distribution, percolation threshold, and graph distance, share the interesting features of both these models.

To go further, in this paper we consider a sort of percolation on $\mathbb{Z}^d$, which involves not only spatial structures but also relational (i.e., non-spatial) structures. In our model, each vertex in the cubic lattice $\mathbb{Z}^d$ is associated with an attribute or type, which may affect the probability of edges coming in/out of the vertex. By doing so, we arrive at a multi-type percolation, extending the mono-type percolation studied in [19]. Another key feature of

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our model is that it accommodates directed edges, which are remarkable and pervasive in various real-life complex networks.[2,21–23] Specifically, we suppose that the edges between the same types of vertices (which means they have close relationship regardless of spatial distance) are bidirectional; while edges between different types of vertices are directed. There are quite a few such situations in real networks. For instance, in the new and fast growing SNS networks,[24–25] like Facebook, MySpace, Blogger and RenRen, people sharing common interests become "friends". A group of friends can be viewed as of the same type, and they learn each other's status via bidirectional links simultaneously. On the other hand, people from different groups of friends are also likely to pay attention to each other by an action of "subscribing". This operation, however, is unidirectional and does not occur simultaneously. Another example is the information security of computer communication.[26] In multi-layer networks, such as military communication networks, for the sake of high security and privacy, unidirectional security gateway is applied when data transport from high side to low side, and vice versa. On the other hand, communications within the same side usually go alone bidirectional routers in order to contain costs and promote efficiency. We will show that allowing multi-type vertices and directed edges modifies the picture previously drawn[19] in a nontrivial way, opening new scenarios for percolation phenomena.

1.1 *Model Definition*

We define our model over the nearest-neighbor cubic lattice $\mathbb{Z}^d$ in an integer $d \geq 1$ of dimensions. Let each vertex $x \in \mathbb{Z}^d$ be equipped with a non-negative weight W_x, where $(W_x)_{x\in\mathbb{Z}^d}$ are independent and identically distributed. Throughout the paper, we assume that the distribution F of the weights $(W_x)_{x\in\mathbb{Z}^d}$ has a regularly varying tail with exponent $\tau - 1$ for some $\tau > 1$. In other words, if we denote by W a random variable with the same distribution as W_0, we have

$$1 - F(w) = P(W > w) = w^{-(\tau-1)} L(w)\,, \tag{1}$$

where $L(w)$ is slowly varying at infinity,[16] i.e., for every $c > 0$, $\lim_{w\to\infty} L(cw)/L(w) = 1$.

Let $\Phi = (\phi_{ij})$ be a 2×2 matrix with non-negative constants as entries. Furthermore, we assume that the matrix Φ contains no row or column of all zero. For each $x \in \mathbb{Z}^d$, let $\psi_x \in \{1, 2\}$ denote the type of the vertex x. For $\beta \in (0, 1)$, vertex x belongs to type 1 with probability $P(\psi_x = 1) = \beta$ and type 2 with probability $P(\psi_x = 2) = 1-\beta$, independent to types of other vertices and all weights $(W_x)_{x\in\mathbb{Z}^d}$. Here, Φ is dubbed as the type matrix and β the type probability. We write (x, y) for a directed edge from x to y. As mentioned before, we equate (x, y) with (y, x) (i.e., bidirectional edge) when $\psi_x = \psi_y$; while (x, y) and (y, x) are different directed edges when $\psi_x \neq \psi_y$. Conditionally on the weights $(W_x)_{x\in\mathbb{Z}^d}$ and types $(\psi_x)_{x\in\mathbb{Z}^d}$, the (directed) edges in the graph are independent and the probability that there is an edge (x, y) is defined by

$$p_{xy} = 1 - \mathrm{e}^{-\lambda\phi_{\psi_x\psi_y} W_x W_y/|x-y|^\alpha}\,, \tag{2}$$

for $\alpha, \lambda \in (0, \infty)$, where $|x - y|$ denotes the Euclidean distance between x and y throughout the paper. If two vertices x and y are of the same type, the probability of presenting a non-directed edge (x, y) is $p_{xy} = p_{yx}$. We say that the edge (x, y) is occupied with probability p_{xy} and unoccupied otherwise.

We refer to the resulting random graph model as $G(\lambda, \alpha, W, \beta, \Phi)$. It is easy to see if Φ is a matrix with all entries equal to 1, we reproduce the model $G(\lambda, \alpha, W)$ with connection function $\lambda W_x W_y/|x - y|^\alpha$ proposed in [19]. There are three important parameters in our model: $\lambda > 0$ can be though of as a percolation parameter; $\alpha > 0$ describes the long-rang nature and spatial structure; and $\beta \in (0, 1)$ captures the individual attribute and relational structure. The random type-weight pairs $(\psi_x, W_x)_{x\in\mathbb{Z}^d}$ create a random environment in which we study the percolative properties of our model.

Although our definitions and all the results below are stated for two different types, the techniques generalize to the case of an arbitrarily large but bounded number k of types. We omit the details to simplify the exposition.

1.2 *Organization and Results*

The paper is organized as follows. In Sec. 2, we characterize the tail behavior of the degree distribution. Taking the weight distribution to be regularly varying with exponent $\tau - 1$ as in (1), we show that the corresponding out-degree and in-degree distributions are both regularly varying with exponent γ, where $\gamma = \alpha(\tau - 1)/d$, provided that $\alpha > d$ and $\gamma > 1$. Conditional on weights and types, the expected out-degree and in-degree are concentrated within different bounded intervals when $\alpha > d$ and $\gamma > 1$. Numerical simulations support this result, further predicting a bell-shaped distribution of the expected degrees. Note that, when $\gamma > 2$, the out/in-degrees have finite variance, while when $\gamma \in (1, 2]$, the out/in-degrees have finite mean, but infinite variance, without reference to the type matrix Φ and type probability β. Whether the out/in-degrees have infinite variance for $\gamma = 2$ depends on the precise shape of the slowly varying function involved. In the case of $\alpha \geq d$ or $\gamma < 1$, the out/in-degrees of all vertices regardless of their types are almost surely infinite. Here we stress that for $\alpha > d$ and $\gamma > 1$ the asymptotic expected out-degree and in-degree are focused on different values depending on the type matrix Φ as well as the type probability β. The inhomogeneous type matrix plays an essential role in this context which is in contrast with the mono-type percolation [19, Proposition 2.3].

Section 3 is devoted to the percolation properties of the model, where $\lambda > 0$ is our percolation parameter. The critical probability values for the infinite weakly connected component (WCC) of 0 and the infinite strongly connected component (SCC) of 0 are denoted λ_c^{WCC} and λ_c^{SCC}, respectively. When $\alpha \leq d$ or $\gamma \leq 1$, a sufficient condition for $\lambda_c^{\rm SCC}=0$ is that trace(Φ) >0, which has no counterpart in the mono-type percolation,[19] where all connected components are in the sense of WCC. In Subsec. 3.1, conditions which guarantee that $\lambda_c^{\rm WCC},\lambda_c^{\rm SCC}<\infty$ are formulated on the weights and types. If either β or $1-\beta$ is large enough, we may obtain finite $\lambda_c^{\rm SCC}$. This condition is tantamount to say that one type of vertices outnumbers the other. In Subsec. 3.2, it is shown, under some mild conditions on type matrix Φ and type probability β, that $\lambda_c^{\rm WCC},\lambda_c^{\rm SCC}>0$ if and only if the out/in-degrees have finite variance. In particular, if $\lambda_c^{\rm WCC}>0$, then at least one of the entries in matrix Φ is equal to zero.

In Sec. 4, we study the graph distance ("chemical distance") between vertices. Let $\tilde d(x,y)$ denote the undirected graph distance between x and y, that is, the minimal number of occupied edges that form an undirected path between x and y. Similarly, denote by $d(x,y)$ the length of shortest directed path, consisting of occupied edges, from x to y. We show, under the assumption that Φ is a positive matrix, that $d(0,x)$, $d(x,0)$ and $\tilde d(0,x)$ are exactly of the same order $\ln\ln|x|$ when $\gamma<2$, that is, when the out/in-degrees have infinite variance, and at least of the order $\ln|x|$ when $\gamma>2$, that is, when the out/in-degrees have finite variance. This result establishes a phase transition at the point where $\gamma=2$. Here, the condition of positivity of the type matrix Φ is naturally satisfied in the mono-type percolation where all elements of Φ are set to 1. Our results reveal that the weak limit law of graph distance can be established under the same normalization as in [19] provided every type of vertices has positive connection probability. When $\gamma>2$ and $\alpha>2d$, the aforementioned lower bound of the three distances can be raised to $|x|^\varepsilon$ for some $\varepsilon>0$, slightly improving Theorem 5.5 in [19]. Finally, we conclude the paper with some open problems in Sec. 5.

The line of the proofs mainly follows the recent beautiful paper.[19] It is often that some arguments are adapted from results for long-range percolation models in [6, 9, 11, 19, 28–29] and so the validity of these technical results under general assumptions needs to be carefully checked. We include the complete proofs of them, not only for the convenience of the reader but also to convince the reader that they do hold in our setting.

2 Vertex Degrees

In this section, we relate the tail behavior of the out/in-degree distributions in our model to that of the weight distribution. Recall that we assume the distribution function F of the weights $(W_x)_{x\in\mathbb{Z}^d}$ has a regularly varying tail satisfying (1) and the edge occupation probabilities $(p_{xy})_{x,y\in\mathbb{Z}^d}$ are as in (2). Write D_x and D'_x for the out-degree and in-degree, respectively, of vertex $x\in\mathbb{Z}^d$. By translation invariance, D_x has the same distribution as D_0; and D'_x has the same distribution as D'_0. Denote by $a\vee b$ (and $a\wedge b$, respectively) the maximum (and minimum, respectively) of two real numbers a and b.

Our first result shows that, when $\alpha\le d$ or when both $\alpha>d$ and $\gamma=\alpha(\tau-1)/d\le 1$, the model is degenerate in the sense that all vertices have infinite out/in-degrees almost surely regardless of their types.

Theorem 1 (Infinite out/in-degrees for $\alpha\le d$ or $\gamma\le 1$) Suppose that

(a) $\alpha\le d$; or

(b) $\alpha>d$ and the weight distribution satisfies

$$1-F(w)\ge cw^{-(\tau-1)},\quad w>0\,,\tag{3}$$

for some $c>0$ and $\tau>1$ such that $\gamma=\alpha(\tau-1)/d\le 1$. Then, we have $P(D_0=\infty|W_0>0)=1$ and $P(D'_0=\infty|W_0>0)=1$.

Proof We first prove the statement for out-degree D_0. Let 1_A be the indicator function of event A. By using the inequality $1-\mathrm{e}^{-x}\ge(x\wedge 1)/2$, we obtain

$$\begin{aligned}&\sum_{y\neq 0}P((0,y)\ \text{occupied}|W_0=w,\psi_0=k)\\&=\sum_{y\neq 0}E(1-\mathrm{e}^{-\lambda\phi_{k\psi_y}wW_y/|y|^\alpha})\\&\ge\frac12\sum_{y\neq 0}E\Big(\frac{\lambda\phi_{k\psi_y}wW_y}{|y|^\alpha}\wedge 1\Big)\\&\ge\frac{\lambda w}{2}\sum_{y\neq 0}\frac{E\phi_{k\psi_y}\cdot E(W_y1_{\{W_y\le|y|^\alpha/(\lambda w\|\Phi\|)\}})}{|y|^\alpha}\,,\end{aligned}\tag{4}$$

where $||\Phi||:=\max_{i,j}\phi_{ij}$. Since $E(W_y1_{\{W_y\le|y|^\alpha/(\lambda w\|\Phi\|)\}})\to EW$ as $|y|\to\infty$, and by the assumptions on the type matrix Φ, $E\phi_{k\psi_y}=\phi_{k1}\beta+\phi_{k2}(1-\beta)>0$, we have

$$\sum_{y\neq 0}P((0,y)\ \text{occupied}|W_0=w,\psi_0=k)\ge Cw\sum_{y\neq 0}\frac{1}{|y|^\alpha}\,,$$

for some constant $C>0$. If condition (a) holds, then the above summation diverges. Since the edges coming out of the origin are independent conditionally on W_0 and ψ_0, it follows from the Borel–Cantelli lemma that $P(D_0=\infty|W_0=w,\psi_0=k)=1$ for any $w>0$ and $k\in\{1,2\}$. Hence, $P(D_0=\infty|W_0>0)=1$.

If condition (b) holds, then we have $\tau\in(1,2)$. By (3) and the Fubini theorem we obtain that $EW_y=\infty$ and $E(W_y1_{\{W_y\le s\}})\ge C's^{2-\tau}$ for some constant $C'>0$. Combining this with the bound (4), we have

$$\begin{aligned}&\sum_{y\neq 0}P((0,y)\ \text{occupied}|W_0=w,\psi_0=k)\\&\ge C''w^{\tau-1}\sum_{y\neq 0}\frac{1}{|y|^{\alpha(\tau-1)}}\,,\end{aligned}$$

for some constant $C''>0$. Using a similar argument as before, we derive $P(D_0=\infty|W_0>0)=1$ when $\gamma=\alpha(\tau-1)/d\le 1$.

The statement for in-degree D'_0 can be proved similarly by considering the probability $P((y,0)$ occupied $|W_0=w,\psi_0=k)$ instead. □

Let θ_d denote the volume of the unit ball in $\mathbb{R}^d$ and $\Gamma(\cdot)$ the gamma function. The following result is a characterization of the conditional expected out/in-degrees.

Theorem 2 (Expected out/in-degrees) Suppose that the weight distribution satisfies (1) with $\alpha>d$ and $\gamma>1$.

Denote by ψ a random variable with the same distribution as ψ_0. Then

$$|E(D_0|W_0=w,\psi_0=k)-\xi w^{d/\alpha}|\le\theta_d,$$

where $\xi=\theta_d\lambda^{d/\alpha}\Gamma(1-d/\alpha)E(\phi_{k\psi}^{d/\alpha})E(W^{d/\alpha})$; and similarly,

$$|E(D_0'|W_0=w,\psi_0=k)-\xi' w^{d/\alpha}|\le\theta_d,$$

where $\xi'=\theta_d\lambda^{d/\alpha}\Gamma(1-d/\alpha)E(\phi_{\psi k}^{d/\alpha})E(W^{d/\alpha})$.

Proof First, note that $\gamma>1$ implies $\tau-1>d/\alpha$, and then $E(W^{d/\alpha})=\int_0^\infty w^{d/\alpha}\,\mathrm{d}F(w)<\infty$. We have

$$E(D_0|W_0=w,\psi_0=k)=\sum_{y\neq0}\big(1-E\big(\mathrm{e}^{-\lambda\phi_{k\psi_y}wW_y/|y|^\alpha}\big)\big)=\sum_{y\neq0}\int_0^\infty 1-\beta\,\mathrm{e}^{-\lambda\phi_{k1}wu/|y|^\alpha}-(1-\beta)\,\mathrm{e}^{-\lambda\phi_{k2}wu/|y|^\alpha}\,\mathrm{d}F(u)\,.$$

Interchanging the order of integration and summation, we first compute the sum over $y\neq0$ as follows

$$\sum_{y\neq0}1-\beta\,\mathrm{e}^{-\lambda\phi_{k1}wu/|y|^\alpha}-(1-\beta)\,\mathrm{e}^{-\lambda\phi_{k2}wu/|y|^\alpha}=\int_{|y|>1}\big(1-\beta\,\mathrm{e}^{-\lambda\phi_{k1}wu/|y|^\alpha}-(1-\beta)\,\mathrm{e}^{-\lambda\phi_{k2}wu/|y|^\alpha}\big)\,\mathrm{d}y+E_1(u)$$
$$=(\lambda uw)^{d/\alpha}\int_{|t|>(\lambda uw)^{-1/\alpha}}\big(1-\beta\,\mathrm{e}^{-\phi_{k1}/|t|^\alpha}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}/|t|^\alpha}\big)\,\mathrm{d}t+E_1(u)$$
$$=(\lambda uw)^{d/\alpha}\int_{|t|>0}\big(1-\beta\,\mathrm{e}^{-\phi_{k1}/|t|^\alpha}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}/|t|^\alpha}\big)\,\mathrm{d}t-E_2(u)+E_1(u)\,,\tag{5}$$

where $E_1(u)$ and $E_2(u)$ are error terms that will be estimated below. Converting the integral in (5) to polar coordinates and by integration by parts, we derive

$$\int_{|t|>0}\big(1-\beta\,\mathrm{e}^{-\phi_{k1}/|t|^\alpha}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}/|t|^\alpha}\big)\,\mathrm{d}t=\theta_d\int_0^\infty\big(1-\beta\,\mathrm{e}^{-\phi_{k1}/r^\alpha}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}/r^\alpha}\big)\,\mathrm{d}r^{d-1}\,\mathrm{d}r$$
$$=\theta_d\int_0^\infty r^d\,\mathrm{d}\big(\beta\,\mathrm{e}^{-\phi_{k1}/r^\alpha}+(1-\beta)\,\mathrm{e}^{-\phi_{k2}/r^\alpha}\big)=-\theta_d\int_0^\infty s^{-d/\alpha}\,\mathrm{d}\big(\beta\,\mathrm{e}^{-s\phi_{k1}}+(1-\beta)\,\mathrm{e}^{-s\phi_{k2}}\big)=\theta_d\Gamma\Big(1-\frac{\mathrm{d}}{\alpha}\Big)E(\phi_{k\psi_y}^{d/\alpha})\,,$$

for $\alpha>d$. Recall that $E(W^{d/\alpha})<\infty$, we have

$$E(D_0|W_0=0,\psi_0=k)=\theta_d\Gamma\Big(1-\frac{d}{\alpha}\Big)E(\phi_{k\psi_y}^{d/\alpha})\int_0^\infty(\lambda uw)^{d/\alpha}\,\mathrm{d}F(u)$$
$$+\int_0^\infty(E_1(u)-E_2(u))\,\mathrm{d}F(u)=\xi w^{d/\alpha}+\int_0^\infty(E_1(u)-E_2(u))\,\mathrm{d}F(u)\,,\tag{6}$$

where $\xi=\theta_d\lambda^{d/\alpha}\Gamma(1-d/\alpha)E(\phi_{k\psi}^{d/\alpha})E(W^{d/\alpha})$.

Now it remains to bound the error terms. As for $E_1(u)$, since $1-\mathrm{e}^{-c|y|^{-\alpha}}$ is monotonically decreasing as $|y|$ increases, we have

$$0\le E_1(u)=\sum_{y\neq0}\big(1-\beta\,\mathrm{e}^{-\lambda\phi_{k1}wu/|y|^\alpha}-(1-\beta)\,\mathrm{e}^{-\lambda\phi_{k2}wu/|y|^\alpha}\big)-\int_{|y|>1}\big(1-\beta\,\mathrm{e}^{-\lambda\phi_{k1}wu/|y|^\alpha}-(1-\beta)\,\mathrm{e}^{-\lambda\phi_{k2}wu/|y|^\alpha}\big)\,\mathrm{d}y\le\theta_d\,.$$

As for $E_2(u)$, a similar calculation as the one following (5) yields

$$E_2(u)=(\lambda uw)^{d/\alpha}\int_{|t|\le(\lambda uw)^{-1/\alpha}}\big(1-\beta\,\mathrm{e}^{-\phi_{k1}/|t|^\alpha}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}/|t|^\alpha}\big)\,\mathrm{d}t=\theta_d\big(1-\beta\,\mathrm{e}^{-\phi_{k1}\lambda uw}-(1-\beta)\,\mathrm{e}^{-\phi_{k2}\lambda uw}$$
$$+(\lambda uw)^{d/\alpha}\int_{\lambda uw}^\infty s^{-d/\alpha}\big(\beta\phi_{k1}\,\mathrm{e}^{-s\phi_{k1}}+(1-\beta)\phi_{k2}\,\mathrm{e}^{-s\phi_{k2}}\big)\,\mathrm{d}s\big)\,.$$

Since

$$(\lambda uw)^{d/\alpha}\int_{\lambda uw}^\infty s^{-d/\alpha}\big(\beta\phi_{k1}\,\mathrm{e}^{-s\phi_{k1}}+(1-\beta)\phi_{k2}\,\mathrm{e}^{-s\phi_{k2}}\big)\,\mathrm{d}s\le\int_{\lambda uw}^\infty\big(\beta\phi_{k1}\,\mathrm{e}^{-s\phi_{k1}}+(1-\beta)\phi_{k2}\,\mathrm{e}^{-s\phi_{k2}}\big)\,\mathrm{d}s$$
$$=\beta\,\mathrm{e}^{-\phi_{k1}\lambda uw}+(1-\beta)\,\mathrm{e}^{-\phi_{k2}\lambda uw}\,,$$

we obtain $0\le E_2(u)\le\theta_d$. Combining these bounds with (6) yields

$$|E(D_0|W_0=w,\psi_0=k)-\xi w^{d/\alpha}|\le\theta_d.$$

The statement for in-degree D_0' can be proved similarly. □

Next, we provide a simulation example to illustrate the conditional out/in-degrees of our model $G(\lambda,\alpha,W,\beta,\Phi)$. Take $d=2$, $\alpha=4$, and $\tau=2$. Hence, $\gamma=\alpha(\tau-1)/d=2$. Let the type probability $\beta=0.4$ and type matrix $\Phi=\begin{pmatrix}0.5&0.8\\0.2&0.5\end{pmatrix}$. Take the percolation parameter $\lambda=1$ and the distribution F of weights satisfying

$$F(w)=1-\Big(\frac{2}{\ln2}\Big)\frac{\ln w}{w},\quad w\ge2.$$

We can easily verify that the conditions of Theorem 2 hold. Fix $\psi_0=1$. Straightforward calculation gives $\theta_d=\pi$, $E(W^{1/2})=4(1+\ln\sqrt2)/\sqrt2\ln2$, $\Gamma(1/2)=\sqrt\pi$,

$E(\phi_{1\psi}^{1/2}) \approx 0.8$, $E(\phi_{\psi 1}^{1/2}) \approx 0.6$ and thus $\xi \approx 25.1$ and $\xi' \approx 16.9$. Given $\psi_0 = 1$, we plot the simulation results of conditional out-degree D_0 and in-degree D_0' as functions of weight w in Fig. 1. We find an excellent agreement between numerical solutions and theoretical values $\xi\sqrt{w}$ and $\xi'\sqrt{w}$. This phenomenon is further explained by Fig. 2, where we plot the histograms of 100 samples of out-degree D_0 and 100 samples of in-degree D_0' conditional on $w = 5$, $\psi_0 = 1$. We observe that the both frequency distributions are bell-like shaped with highest frequencies occurred near $\xi\sqrt{5} \approx 56.1$ and $\xi'\sqrt{5} \approx 37.7$.

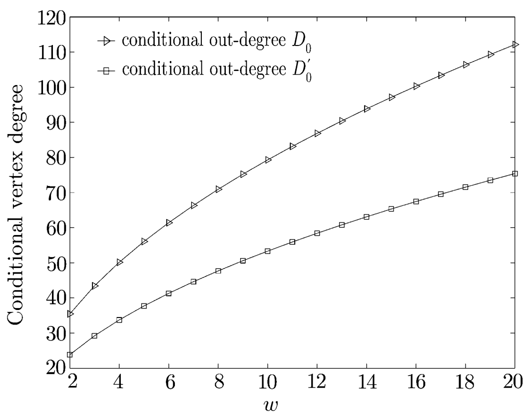

Fig. 1 Conditional vertex degrees versus weight w from numerical simulations with out-degree $(D_0|\psi_0 = 1)$ (triangles) and in-degree $(D_0'|\psi_0 = 1)$ (squares), and our exact values $\xi\sqrt{w}$ and $\xi'\sqrt{w}$ (solid curves) from Theorem 2.

Theorem 3 (Power-law out/in-degrees for power-law weights) Suppose that the weight distribution satisfies (1) with $\alpha > d$ and $\gamma > 1$. Then there exist two functions $l(s)$ and $l'(s)$ which are slowly varying at infinity such that

$$P(D_0 > s) = s^{-\gamma} l(s), \qquad P(D_0' > s) = s^{-\gamma} l'(s).$$

Under the assumption of the theorem, the out/in-degrees have finite mean, i.e., $\gamma > 1$. When $\alpha > d$, finite variance for the weights (i.e., $\tau > 3$) implies finite variance for the out/in-degrees (i.e., $\gamma > 2$). Also note that the variance of the out/in-degrees may be finite even if the weights have infinite variance, since for $\tau \in (1, 3]$ we still have $\gamma > 2$ if α is large enough. The proof of Theorem 3 relies on Theorem 2 and is adapted from [29].

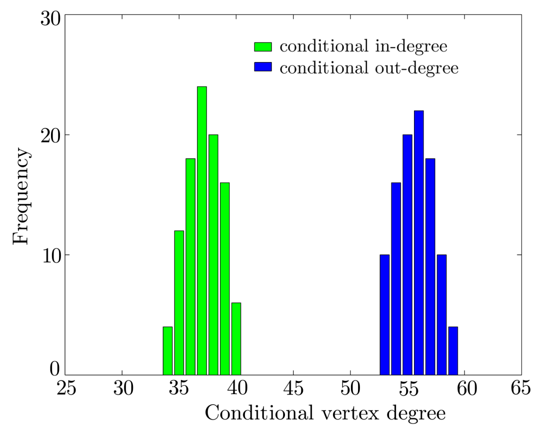

Fig. 2 Sample frequency versus conditional out-degree $(D_0|w = 5, \psi_0 = 1)$ (right part) and conditional in-degree $(D_0'|w = 5, \psi_0 = 1)$ (left part).

Proof We only prove the out-degree case, and the corresponding in-degree case can be proved likewise. For fixed s, we have

$$\begin{aligned} P(D_0 > s) &= \int_0^\infty \beta P(D_0 > s|W_0 = w, \psi_0 = 1) + (1-\beta) P(D_0 > s|W_0 = w, \psi_0 = 2)\,\mathrm{d}F(w) \\ &= \int_{I_1} \beta P(D_0 > s|W_0 = w, \psi_0 = 1) + (1-\beta) P(D_0 > s|W_0 = w, \psi_0 = 2)\,\mathrm{d}F(w) \\ &\quad + \int_{I_2} \beta P(D_0 > s|W_0 = w, \psi_0 = 1) + (1-\beta) P(D_0 > s|W_0 = w, \psi_0 = 2)\,\mathrm{d}F(w), \end{aligned}$$

where $I_1 = [0, m(s))$, $I_2 = [m(s), \infty)$, and $m(s) = [(s - s^{1/2}\ln s + O(1))/\xi]^{\alpha/d}$.

We address the above two integrals separately. For $k \in \{1, 2\}$, exploiting the Bernstein inequality, we have

$$P(D_0 > s|W_0 = w, \psi_0 = k) \le \mathrm{e}^{-(s - E(D_0|W_0 = w, \psi_0 = k))^2/(2E(D_0|W_0 = w, \psi_0 = k) + 4s/3)}.$$

From Theorem 2, we know

$$E(D_0|W_0 = w, \psi_0 = k) = \xi w^{d/\alpha} + O(1). \tag{7}$$

Therefore,

$$\inf_{w \in I_1} \{s - E(D_0|W_0 = w, \psi_0 = k)\} \ge s^{1/2}\ln s + O(1), \quad \sup_{w \in I_1} \{E(D_0|W_0 = w, \psi_0 = k)\} \le s - s^{1/2}\ln s + O(1) < s.$$

For s large enough, we then derive $P(D_0 > s|W_0 = w, \psi_0 = k) \le \mathrm{e}^{-(\ln s)^2/5} = s^{-\ln s/5}$, which implies

$$\lim_{s\to\infty} s^a \int_{I_1} \beta P(D_0 > s|W_0 = w, \psi_0 = 1) + (1-\beta) P(D_0 > s|W_0 = w, \psi_0 = 2)\,\mathrm{d}F(w) \le \lim_{s\to\infty} s^a s^{-\ln s/5} = 0, \tag{8}$$

for any $a > 0$.

As for the integral over I_2, let $X_{w,k}$ denote a random variable with the same distribution as $(D_0|W_0 = w, \psi_0 = k)$. It follows from (7) that $EX_{w,k} = \xi w^{d/\alpha} + O(1)$ and by a similar analysis as for the first moment in Theorem 2, we can show

$$\mathrm{Var} X_{w,k} = \eta w^{d/\alpha} + O(1), \tag{9}$$

for some $\eta < \xi$. Define $G(t) = \int_{m(t)}^{\infty} \beta P(X_{w,1} > t) + (1-\beta)P(X_{w,2} > t)\mathrm{d}F(w)$, and then $P(D_0 > t) = G(t) + O(t^{-a})$, for any $a > 0$, by virtue of (8).

Now, to prove the theorem, it suffices to show that (see [27] p. 275) $\lim_{t\to\infty}(P(D_0 > st)/P(D_0 > t)) = s^{-\gamma}$, for any $s \in (0,\infty)$, since $P(D_0 > s)$ is monotonic on $(0,\infty)$. From (8), this in turn follows if we can show

$$\lim_{t\to\infty} \frac{G(st)}{G(t)} = s^{-\gamma}\,, \tag{10}$$

for any $s \in (0,\infty)$. To this end, note that $G(t) \le \int_{m(t)}^{0} \mathrm{d}F(w) = 1 - F(m(t))$. On the other hand, for any $\varepsilon > 0$,

$$\begin{aligned} G(t) &\le \int_{(1+\varepsilon)^{\alpha/d}m(t)}^{\infty} \beta P(X_{w,1} > t) + (1-\beta)P(X_{w,2} > t)\mathrm{d}F(w) \\ &= 1 - F((1+\varepsilon)^{\alpha/d}m(t)) - \int_{(1+\varepsilon)^{\alpha/d}m(t)}^{\infty} \beta P(X_{w,1} \le t) + (1-\beta)P(X_{w,2} \le t)\mathrm{d}F(w) \end{aligned}$$

Note that for $w \ge (1+\varepsilon)^{\alpha/d}m(t)$ and t large enough,

$$EX_{w,k} = \xi w^{d/\alpha} + O(1) > \xi(1+\varepsilon)m(t)^{d/\alpha} + O(1) > \Bigl(1+\frac{\varepsilon}{2}\Bigr)t\,,$$

and then, by the Chebyshev inequality and (9),

$$\begin{aligned} P(X_{w,k} \le t) &= P(EX_{w,k} - X_{w,k} \ge EX_{w,k} - t) \le P(|EX_{w,k} - X_{w,k}| \ge EX_{w,k} - t) \le \frac{\mathrm{Var}X_{w,k}}{(EX_{w,k}-t)^2} \\ &\le \frac{\eta w^{d/\alpha} + O(1)}{(t\varepsilon/2)(\xi w^{d/\alpha} + O(1) - t)} \le \frac{C}{t\varepsilon}\,, \end{aligned} \tag{11}$$

for some constant $C > 0$, where the last inequality (11) follows from the fact that $\xi w^{d/\alpha} - t > (\xi - \xi/(1+\varepsilon/2))w^{d/\alpha}$. Therefore, $P(X_{w,k} \le t) \to 0$ uniformly in k and $w \ge (1+\varepsilon)^{\alpha/d}m(t)$ as $t \to \infty$. Hence, we obtain by letting $\varepsilon \to 0$,

$$\lim_{t\to\infty}\frac{G(st)}{G(t)} = \lim_{t\to\infty}\frac{1-F(m(st))}{1-F(m(t))} = \lim_{t\to\infty}\frac{1-F((1+o(1))(st/\xi)^{\alpha/d})}{1-F((1+o(1))(t/\xi)^{\alpha/d})} = \lim_{t\to\infty}\frac{(st/\xi)^{\alpha(1-\tau)/d}L((1+o(1))(st/\xi)^{\alpha/d})}{(t/\xi)^{\alpha(1-\tau)/d}L((1+o(1))(t/\xi)^{\alpha/d})} = s^{-\gamma}\,,$$

which proves (10). □

3 Percolation Properties

We investigate in this section the percolation properties of our model $G(\lambda,\alpha,W,\beta,\Phi)$, where $\lambda > 0$ is viewed as the percolation parameter.

Write $x \longleftrightarrow y$ to denote the event that there is an undirected path of occupied edges between x and y in $G(\lambda,\alpha,W,\beta,\Phi)$. Similarly, write $x \longrightarrow y$ to represent the event that there is a directed path of occupied edges from x to y in $G(\lambda,\alpha,W,\beta,\Phi)$. Denote by $\mathrm{WCC}(x) = \{y : x \longleftrightarrow y\}$ the weakly connected component of x and $\mathrm{SCC}(x) = \{y : x \longrightarrow y,\ y \longrightarrow x\}$ the strongly connected component of x. Write $|\mathrm{WCC}(x)|$ and $|\mathrm{SCC}(x)|$ the number of vertices in $\mathrm{WCC}(x)$ and $\mathrm{SCC}(x)$, respectively. The corresponding percolation probabilities are defined by $\theta^{\mathrm{WCC}}(\lambda) = P(|\mathrm{WCC}(0)| = \infty)$ and $\theta^{\mathrm{SCC}}(\lambda) = P(|\mathrm{SCC}(0)| = \infty)$, respectively. Thus, the critical percolation values are defined as

$$\lambda_c^{\mathrm{WCC}} = \inf_{\theta^{\mathrm{WCC}}(\lambda)>0} \lambda \quad \text{and} \quad \lambda_c^{\mathrm{SCC}} = \inf_{\theta^{\mathrm{SCC}}(\lambda)>0} \lambda\,,$$

respectively.

For general introductions to percolation we refer the reader to [30–31] and references therein. Clearly, $\lambda_c^{\mathrm{WCC}} \le \lambda_c^{\mathrm{SCC}}$, and by a uniqueness result in [32], the model $G(\lambda,\alpha,W,\beta,\Phi)$ contains almost surely at most one infinite weakly connected component, and hence at most one infinite strongly connected component. The following result is a consequence of Theorem 1.

Theorem 4 (Zero critical values for $\alpha \le d$ or $\gamma \le 1$) Suppose that the assumptions (a) or (b) in Theorem 1 holds. Given $W_0 > 0$, we have $\lambda_c^{\mathrm{WCC}} = 0$. In addition, if trace $(\Phi) > 0$ holds, then $\lambda_c^{\mathrm{SCC}} = 0$.

Proof It follows straightforward from Theorem 1 that $P(|\mathrm{WCC}(0)| = \infty) = 1$ for any $\lambda > 0$. Therefore, $\lambda_c^{\mathrm{WCC}} = 0$.

Now, assume that trace$(\Phi) > 0$. Without loss of generality, we may assume $\phi_{11} > 0$. For $k = 1, 2$, let V_k be the vertices in $\mathbb{Z}^d$ with type k. Then $\mathbb{Z}^d$ can be partitioned as $\mathbb{Z}^d = V_1 \cup V_2$. For $\lambda > 0$, we obtain

$$\begin{aligned} P(|\mathrm{SCC}(0)| = \infty) &\ge \beta P(|\mathrm{SCC}(0) \cap V_1| = \infty | \psi_0 = 1) \\ &= \beta P(|\mathrm{WCC}(0) \cap V_1| = \infty | \psi_0 = 1)\,, \end{aligned} \tag{12}$$

since the edges in V_1 are bidirectional. If $\phi_{12} = 0$, then by Theorem 1, (12) is lower bounded by $\beta P(D_0 = \infty | \psi_0 = 1) > 0$. If $\phi_{12} > 0$, we define a new type matrix $\hat{\Phi}$ as $\hat{\Phi} = \begin{pmatrix} \phi_{11}\wedge\phi_{12} & \phi_{11}\wedge\phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$. Then the edge configuration of $G(\lambda,\alpha,W,\beta,\Phi)$ stochastically dominates that of $\hat{G} := G(\lambda,\alpha,W,\beta,\hat{\Phi})$. Therefore, in view of Theorem 1, (12) is lower bounded by

$$\beta^2 P(|\mathrm{WCC}_{\hat{G}}(0)| = \infty | \psi_0 = 1) > 0,$$

where $\mathrm{WCC}_{\hat{G}}(0)$ represents the weakly connected component of 0 in $\hat{G}$. Hence, we have $\lambda_c^{\mathrm{SCC}} = 0$. □

In the sequel we will restrict to the case $\alpha > d$ and $\gamma > 1$.

3.1 *Finiteness of Critical Values*

A random field $(X_z)_{z\in\mathbb{Z}^d}$ is said to be r-dependent if for any two sets $A, B \subset \mathbb{Z}^d$ at L^∞-distance at least r

from each other we have that $(X_z)_{z\in A}$ is independent of $(X_z)_{z\in B}$. Here, the L^∞-distance between A and B is defined as

$$\|A-B\|_\infty = \min_{\substack{x=(x_1,\ldots,x_d)\in A\\ y=(y_1,\ldots,y_d)\in B}} \max_{1\le i\le d}\{|x_i-y_i|\}.$$

The following site percolation result regarding r-dependent random field is useful.

Lemma 1[28] For each $d\ge 2$ and $r\ge 1$ there exists a $p_c=p_c(d,r)<1$ such that the following holds. For any r-dependent random field $(X_z)_{z\in\mathbb{Z}^d}$ satisfying $P(X_z=1)=1-P(X_z=0)\ge p$, with $p>p_c$, the 1's in $(X_z)_{z\in\mathbb{Z}^d}$ percolate almost surely.

The following theorem provides sufficient conditions and necessary conditions for $\lambda_c^{\rm WCC}<\infty$ and $\lambda_c^{\rm SCC}<\infty$ so that our model percolates (in the senses of WCC and SCC) for large enough λ.

Theorem 5 (Finiteness of critical values) Suppose that $\alpha>d$ and $\gamma>1$.

(a) If $d\ge 2$, $P(W>0)=1$ and Φ is a positive matrix, then $\lambda_c^{\rm WCC}<\infty$. In addition, if $\beta\vee(1-\beta)>p_c(d,3)$, then $\lambda_c^{\rm SCC}<\infty$.

(b) If $d=1$, $\alpha\in(1,2]$, $P(W\ge w)=1$ for some $w>0$, and Φ is a positive matrix, then $\lambda_c^{\rm WCC}<\infty$. In addition, if $\beta\vee(1-\beta)$ is sufficiently close to 1, then $\lambda_c^{\rm SCC}<\infty$.

(c) If $d=1$, $\alpha>2$ and the weight distribution satisfies

$$1-F(w)\le cw^{-(\tau-1)},\quad w\ge 0\,, \tag{13}$$

for some $c>0$ and $\tau>1$ such that $\gamma=\alpha(\tau-1)/d>2$, then $\lambda_c^{\rm WCC}=\infty$, and thus $\lambda_c^{\rm SCC}=\infty$.

Note that in the mono-type percolation, all the elements in the type matrix Φ are set to 1, and we can take the value of β sufficiently close to 1 (since different value of β makes no difference in that case). Therefore, the above theorem can be seen as a generalization of [19, Theorem 3.1] to multi-type vertices case.

Proof (a) We begin with the WCC case. Denote by ϕ the minimum entry of Φ, i.e., $\phi=\min_{i,j}\phi_{ij}>0$ by assumption. We will frequently use the quantity ϕ. Say that a vertex $x\in\mathbb{Z}^d$ is ε-good, if $W_x\ge\varepsilon$. Note that if two nearest-neighbor sites x and y are both ε-good, then the probability that the edge (x,y) is occupied in $G(\lambda,\alpha,W,\beta,\Phi)$ is at least $1-{\rm e}^{-\lambda\phi\varepsilon^2}$. Therefore, it suffices to show that the edge configuration obtained by independently keeping one of the two directed edges between every pair of ε-good nearest-neighbor vertices with probability $1-{\rm e}^{-\lambda\phi\varepsilon^2}$ and removing all other edges percolates (in the sense of WCC) for some $\varepsilon>0$. To this end, we say that a vertex $z\in\mathbb{Z}^d$ is ε-open if $2d$ (directed) edges connecting its $2d$ nearest-neighbors are present in this configuration (see Fig. 3 for an illustration) and let $X_z=1$ precisely when z is ε-open. Note that this procedure defines a 3-dependent random field and that

$$\begin{aligned}P(X_z=1)&=P(z\text{ is }\varepsilon\text{-open})\\&\ge P(W\ge\varepsilon)^{2d+1}(1-{\rm e}^{-\lambda\phi\varepsilon^2})^{2d}\,.\end{aligned}$$

By virtue of the assumption $P(W>0)=1$, the first factor can be made arbitrarily close to 1 by letting ε small enough and the second factor can then be made arbitrarily close to 1 by taking λ large enough. Therefore, by Lemma 1, we can make $P(X_z=1)$ large enough to ensure that the ε-open vertices percolate. Consequently, $\lambda_c^{\rm WCC}<\infty$.

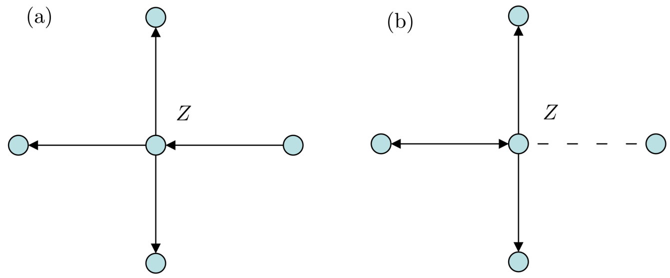

Fig. 3 (a) z is ε-open since there are 4 occupied edges connecting 4 nearest-neighbors to z. (b) z is not ε-open since there is 1 nearest-neighbor which is not connected to z. An illustration of ε-open vertex in $\mathbb{Z}^2$: directions of occupied edges are signified by arrows, and unoccupied edges are signified by dashed segments.

If $\beta\vee(1-\beta)>p_c(d,3)$, we assume $\beta\ge 1-\beta$ without loss of generality. For $z\in\mathbb{Z}^d$, define $X'_z=1$ if and only if z is ε-open and $\psi_z=1$. This then defines a 3-dependent random field and we have

$$P(X'_z=1)\ge P(W\ge\varepsilon)^{2d+1}\beta(1-{\rm e}^{-\lambda\phi\varepsilon^2})^{2d}.$$

If follows from the same reasoning as above that the 1's in $(X'_z)_{z\in\mathbb{Z}^d}$ percolate almost surely by using Lemma 1. Since the edges between vertices of type 1 are all bidirectional, the 1's percolate in the sense of SCC. Thus, we obtain $\lambda_c^{\rm SCC}<\infty$.

(b) As for WCC case, it follows from $P(W\ge w)=1$ that the edge configuration stochastically dominates a configuration with independent undirected edges between every pair $\{x,y\}$ of nearest-neighbor vertices occupied by probability $1-{\rm e}^{-\lambda\phi w^2/|x-y|^\alpha}$. Bond percolation results (see Theorem 1.1 and Theorem 1.2 in [8]) show that, for $\alpha\in(1,2]$ and $d=1$, this model percolates (in the sense of WCC) for large enough λ. Therefore, we derive $\lambda_c^{\rm WCC}<\infty$.

The SCC case is also a direct consequence of Theorem 1.1 and Theorem 1.2 in [8]. We assume $\beta\ge 1-\beta$ without loss of generality. As in Theorem 4 we have a partition $\mathbb{Z}^d=V_1\cup V_2$, since the edges in V_1 are bidirectional, we may view our model as the site-bond percolation with site "alive" probability β studied in [8]. Thus, when β is sufficiently close to 1 and λ is large enough, this model percolates, which yields that $\lambda_c^{\rm SCC}<\infty$.

(c) We modify the proofs in [9] and [19], and consider two scenarios: (i) ${\rm E}W<\infty$ and (ii) ${\rm E}W=\infty$, separately.

(i) For $x\in\mathbb{Z}$, let A_x be the event that on vertex $y\le x$ is connected to any vertex $z>x$. The sequence $(1_{A_x})_{x\in\mathbb{Z}}$ is stationary with common mean $P(A_0)$. For $n\ge 1$, write $A_0^{(n)}$ for the event that none of the $2n$ edges $(0,n),(n,0),(-1,n-1),(n-1,-1),\ldots,(-n+1,1),(1,-n+1)$ is present in the graph $G(\lambda,\alpha,W,\beta,\Phi)$

(recall that we may have e.g. $(n,0)=(0,n)$ by our convention, but we simply count as two here). By the conditional independence, we have

$$P(A_0)=E\Big(\prod_{n=1}^{\infty}P(A_0^{(n)}|(\psi_x)_{x\in\mathbb{Z}},(W_x)_{x\in\mathbb{Z}})\Big)$$
$$\geq E\Big(\mathrm{e}^{-\sum_{n=1}^{\infty}(2\lambda\|\Phi\|/n^{\alpha})(W_0W_n+\cdots+W_{-n+1}W_1)}\Big)\,,$$

where $\|\Phi\|$ is the maximum element of type matrix Φ as defined in Theorem 1. Employing the Jensen inequality, we obtain

$$P(A_0)\geq\,\mathrm{e}^{-\sum_{n=1}^{\infty}(2\lambda\|\Phi\|/n^{\alpha})E(W_0W_n+\cdots+W_{-n+1}W_1)}$$
$$=\,\mathrm{e}^{-2\lambda\|\Phi\|(\mathrm{E}W)^2\sum_{n=1}^{\infty}n^{1-\alpha}}>0\,,$$

since $\alpha>2$ and $\mathrm{E}W<\infty$. Applying the ergodic theorem to the sequence $(1_{A_x})_{x\in\mathbb{N}}$, we have almost surely

$$E\Big(\lim_{n\to\infty}\frac{\sum_{x=1}^{n}1_{A_x}}{n}\Big)=P(A_0)>0\,.$$

Thus, infinitely many of A_x's occur for $x\in\mathbb{N}$. Similarly, infinitely many of A_x's occur for $x\in\mathbb{Z}\backslash N$. Hence, all components are finite almost surely, which completes the proof of case (i).

(ii) In general, we have

$$P(A_0)\geq E\Big(\mathrm{e}^{-2\lambda\|\Phi\|\sum_{i,j\geq0:(i,j)\neq(0,0)}^{\infty}(W_{-i}W_j/(j+i)^{\alpha})}\Big)>0$$

precisely when the double sum is finite almost surely. We bound

$$\sum_{\substack{i,j\geq0\\(i,j)\neq(0,0)}}\frac{W_{-i}W_j}{(j+i)^{\alpha}}\leq Z_1Z_2\,,$$

where

$$Z_1=\sum_{j=0}^{\infty}\frac{W_j}{(j\vee1)^{\alpha/2}}\quad\text{and}\quad Z_2=\sum_{i=0}^{\infty}\frac{W_{-i}}{(i\vee1)^{\alpha/2}}.$$

These two random variables have the same distribution, and if we can check that $Z_1<\infty$ almost surely, then the remainder of the proof can be completed as in case (i).

Now, we want to prove $Z_1<\infty$ almost surely. Given $\varepsilon>0$, let $a_i=\mathrm{i}^{(1+\varepsilon)/(\tau-1)}$. Since

$$P(W_i>a_i)=1-F(a_i)\leq ca_i^{1-\tau}=c\mathrm{i}^{-(1+\varepsilon)},$$

which is summable in i, the events $\{W_i>a_i\}$ occur only finitely often by the Borel–Cantelli lemma. Hence, if we split $Z_1=Y_1+Y_2$, where

$$Y_1=\sum_{j=0}^{\infty}\frac{W_j\wedge a_j}{(j\vee1)^{\alpha/2}}\quad\text{and}\quad Y_2=\sum_{j=1}^{\infty}\frac{(W_j-a_j)1_{\{W_j>a_j\}}}{(j\vee1)^{\alpha/2}},$$

then Y_2 is finite, since there are only finite terms in the summation almost surely. To prove $Y_1<\infty$ almost surely, we note that, for any $a>0$,

$$E(W_j\wedge a)=\int_0^a1-F(y)\,\mathrm{d}y\leq1+\int_1^a1-F(y)\,\mathrm{d}y\leq1+c\int_1^ay^{1-\tau}\,\mathrm{d}y\leq\begin{cases}1+\frac{ca^{2-\tau}}{2-\tau}, & 1<\tau<2\,,\\ 1+c\ln a, & \tau=2\,,\\ 1+\frac{c}{\tau-2}, & \tau>2\,,\end{cases}$$

by the assumption (13), and hence

$$\frac{E(W_j\wedge a_j)}{(j\vee1)^{\alpha/2}}\leq\begin{cases}(j\vee1)^{-\alpha/2}+\frac{c}{2-\tau}(j\vee1)^{-\alpha/2+(1+\varepsilon)[(2-\tau)/(\tau-1)]}, & 1<\tau<2\,,\\ (j\vee1)^{-\alpha/2}+c(1+\varepsilon)(j\vee1)^{-\alpha/2}\cdot\ln(j\vee1)\,, & \tau=2\,,\\ (1+\frac{c}{\tau-2})(j\vee1)^{-\alpha/2}\,, & \tau>2\,.\end{cases}$$

Since $\gamma=\alpha(\tau-1)>2$, we have $-\alpha/2+(2-\tau)/(\tau-1)<-1$. We then have $-\alpha/2+(1+\varepsilon)(2-\tau)/(\tau-1)<-1$ for small enough ε. Combining this with the fact that $\alpha>2$, we derive $EY_1<\infty$ for all $\tau>1$. Accordingly, $Y_1<\infty$ almost surely. □

3.2 *Positivity of Critical Values*

Firstly, we want to show that when the out/in-degrees have finite variance (i.e., $\gamma>2$), there is no percolation (in the senses of WCC and SCC) for small λ, and hence $\lambda_c^{\mathrm{WCC}},\lambda_c^{\mathrm{SCC}}>0$. To this end, we first prove (see Theorem 6) that $\lambda_c^{\mathrm{WCC}},\lambda_c^{\mathrm{SCC}}>0$ in the case when the weights have finite variance (i.e., $\tau>3$); and then we extend the arguments to cover also the case $\tau\in(1,3]$ (see Theorem 7).

Secondly, we show that, under some assumptions of β and Φ, when the out/in-degrees have infinite variance (i.e., $\gamma<2$), there is percolation (in the sense of WCC and SCC) for all $\lambda>0$, and hence $\lambda_c^{\mathrm{WCC}},\lambda_c^{\mathrm{SCC}}=0$; see Theorem 8. Recall that we assume throughout that $\alpha>d$.

Theorem 6 (Positivity of critical values for finite-variance weights) Suppose that $E(W^2)<\infty$. Then, for every $\lambda<1/(\|\Phi\|E(W^2)\sum_{x\neq0}|x|^{-\alpha})$, $\theta^{\mathrm{WCC}}(\lambda)=0$. Thus,

$$\lambda_c^{\mathrm{WCC}}\geq\frac{1}{\|\Phi\|E(W^2)\sum_{x\neq0}|x|^{-\alpha}}\,,$$

and then

$$\lambda_c^{\mathrm{SCC}}\geq\frac{1}{\|\Phi\|E(W^2)\sum_{x\neq0}|x|^{-\alpha}}.$$

We remark that the lower bound of the critical value λ_c^{WCC} (and λ_c^{SCC}) relies on the maximum norm $\|\Phi\|$ of the matrix Φ. From (2), we know that the connection probability p_{xy} increases with $\phi_{\psi_x\psi_y}$. Therefore, the above result discloses the relation between the critical percolation values and the maximum connection probability between two (possibly the same) types of vertices. This important information is otherwise elusive in the scenario of monotype percolation (c.f. [19, Theorem 4.1]).

Proof First note that $\alpha>d$ implies $\sum_{x\neq0}|x|^{-\alpha}<\infty$. Since $\mathrm{E}W<\infty$, every vertex has finite out/in-degrees almost surely. We then deduce that

$$\theta^{\rm WCC}(\lambda) = P(|{\rm WCC}(0)| = \infty) \le \sum_{(x_1,\ldots,x_n)} P((0,x_1) \text{ or } (x_1,0) \text{occupied}, (x_1,x_2) \text{ or } (x_2,x_1) \text{ occupied}, \ldots,$$
$$(x_{n-1},x_n) \text{ or } (x_n,x_{n-1}) \text{ occupied})$$
$$= \sum_{(x_1,\ldots,x_n)} E\big(P((0,x_1) \text{ or } (x_1,0) \text{ occupied}, (x_1,x_2) \text{ or } (x_2,x_1) \text{ occupied}, \ldots,$$
$$(x_{n-1},x_n) \text{ or } (x_n,x_{n-1}) \text{ occupied}|(W_x)_{x\in\mathbb{Z}^d}, (\psi_x)_{x\in\mathbb{Z}^d})\big)\,,$$

where the sum is over $(x_1,\ldots,x_n) \in (\mathbb{Z}^d)^n$ such that every vertex occurs at most once in the path $(0,x_1,\ldots,x_n)$, and by "(x_i,x_j) or (x_j,x_i) occupied" we mean that at least one of them is occupied. By the conditional independence, we have

$$P((0,x_1) \text{ or } (x_1,0) \text{ occupied}, (x_1,x_2) \text{ or } (x_2,x_1) \text{ occupied}, \ldots,$$
$$(x_{n-1},x_n) \text{ or } (x_n,x_{n-1}) \text{ occupied}|(W_x)_{x\in\mathbb{Z}^d}, (\psi_x)_{x\in\mathbb{Z}^d}) \le \prod_{i=1}^n (p_{x_{i-1}x_i} + p_{x_ix_{i-1}})\,,$$

where p_{xy} is defined in (2) and $x_0 = 0$. Using the bound $1 - {\rm e}^{-x} \le x$, we obtain

$$p_{xy} \le \frac{\lambda\|\Phi\|W_xW_y}{|x-y|^\alpha}\,, \tag{14}$$

and then

$$\theta^{\rm WCC}(\lambda) \le \sum_{(x_1,\ldots,x_n)} E\Big(\prod_{i=1}^n (p_{x_{i-1}x_i} + p_{x_ix_{i-1}})\Big) \le 2\sum_{(x_1,\ldots,x_n)} E\Big(\prod_{i=1}^n \frac{\lambda\|\Phi\|W_xW_y}{|x-y|^\alpha}\Big)$$
$$= 2\lambda^n\|\Phi\|^n \sum_{(x_1,\ldots,x_n)} \Big(\prod_{i=1}^n \frac{1}{|x_{i-1}-x_i|^\alpha}\Big) \cdot E\Big(W_0W_{x_n}\prod_{i=1}^{n-1} W_{x_i}^2\Big) \le \frac{2(EW)^2}{E(W^2)}\Big(\lambda\|\Phi\|E(W^2)\sum_{x\neq 0}\frac{1}{|x|^\alpha}\Big)^n,$$

by virtue of independence of weights and translation invariance. Therefore, when $\lambda < 1/(\|\Phi\|E(W^2)\sum_{x\neq 0}|x|^{-\alpha})$, the right-hand side converges to 0 as $n \to \infty$, which implies $\theta^{\rm WCC}(\lambda) = 0$. The proof is then complete. □

We will need the following lemma, which follows from the proof of Lemma 4.3 in [19].

Lemma 2[19] Suppose that the distribution function F satisfies (13) for some $\tau > 1$ and $c > 0$. Let $g(u) = E(((W_1W_2/u) \wedge 1)^2)$, where W_1 and W_2 are independent copies of W. Then, there exists a constant $C > 0$ such that

$$g(u) \le C(1 + \ln u)^2 u^{-((\tau-1)\wedge 2)}.$$

Theorem 7 (Positivity of critical values for finite-variance out/in-degrees) Suppose that there exist $\tau > 1$ and $c > 0$ such that

$$1 - F(w) = P(W > w) \le cw^{-(\tau-1)}, \quad w \ge 0\,, \tag{15}$$

with $\gamma = \alpha(\tau-1)/d > 2$. Then, $\theta^{\rm WCC}(\lambda) = 0$ for small enough λ, that is, $\lambda_c^{\rm WCC} > 0$, and hence $\lambda_c^{\rm SCC} > 0$.

Proof The proof is adapted from that of Theorem 6. We use the more refined bound

$$p_{xy} \le \Big(\frac{\lambda\|\Phi\|W_xW_y}{|x-y|^\alpha} \wedge 1\Big)$$

instead of (14). Therefore, we have

$$\theta^{\rm WCC}(\lambda) \le 2\sum_{(x_1,\cdots,x_n)} E\Big(\prod_{i=1}^n \Big(\frac{\lambda\|\Phi\|W_{x_{i-1}}W_{x_i}}{|x_{i-1}-x_i|^\alpha} \wedge 1\Big)\Big)\,.$$

Involving the Cauchy–Schwarz inequality and the independence of weights, we obtain

$$\Big(E\Big(\prod_{i=1}^n\Big(\frac{\lambda\|\Phi\|W_{x_{i-1}}W_{x_i}}{|x_{i-1}-x_i|^\alpha} \wedge 1\Big)\Big)\Big)^2 \le E\Big(\prod_{i=1}^{\lceil n/2\rceil}\Big(\frac{\lambda\|\Phi\|W_{x_{2i-1}}W_{x_{2i}}}{|x_{2i-1}-x_{2i}|^\alpha} \wedge 1\Big)^2\Big) \cdot E\Big(\prod_{i=1}^{\lfloor n/2\rfloor}\Big(\frac{\lambda\|\Phi\|W_{x_{2i-2}}W_{x_{2i-1}}}{|x_{2i-2}-x_{2i-1}|^\alpha} \wedge 1\Big)^2\Big)$$
$$= \prod_{i=1}^n E\Big(\Big(\frac{\lambda\|\Phi\|W_{x_{i-1}}W_{x_i}}{|x_{i-1}-x_i|^\alpha} \wedge 1\Big)^2\Big)\,,$$

and then by translation invariance

$$\theta^{\rm WCC}(\lambda) \le 2\sum_{(x_1,\ldots,x_n)} \prod_{i=1}^n g\Big(\frac{|x_{i-1}-x_i|^\alpha}{\lambda\|\Phi\|}\Big)^{1/2} \le 2\Big(\sum_{x\neq 0} g\Big(\frac{|x|^\alpha}{\lambda\|\Phi\|}\Big)^{1/2}\Big)^n,$$

where the function $g(u)$ is defined as in Lemma 2.

In view of Lemma 2, we further obtain, for small enough $\lambda > 0$,

$$\theta^{\rm WCC}(\lambda) \le 2\Big(C(\lambda\|\Phi\|)^{((\tau-1)\wedge 2)/2}\sum_{x\ne 0}\Big(1+\ln\Big(\frac{|x|^\alpha}{\lambda\|\Phi\|}\Big)\Big)|x|^{-\alpha((\tau-1)\wedge 2)/2}\Big)^n \le 2\Big(C(\lambda\|\Phi\|)^{((\tau-1)\wedge 2)/4}\sum_{x\ne 0}(1+\ln(|x|^\alpha))|x|^{-\alpha((\tau-1)\wedge 2)/2}\Big)^n,$$

where the last inequality holds due to the fact that $a-\ln(\lambda\|\Phi\|)\le a(\lambda\|\Phi\|)^{-((\tau-1)\wedge 2)/4}$ for any $a>1$, when λ is small enough. Since $\alpha>d$ and $\gamma>2$, we have $\alpha((\tau-1)\wedge 2)/2>1$, and then $\sum_{x\ne 0}(1+\ln(|x|^\alpha))|x|^{-\alpha((\tau-1)\wedge 2)/2}<\infty$. Consequently, when λ is small enough, we have $\theta^{\rm WCC}(\lambda)\to 0$ as $n\to\infty$, which concludes the proof. □

Theorem 8 (Zero critical values for infinite-variance out/in-degrees) Suppose that there exist $\tau>1$ and $c>0$ such that

$$1-F(w)=P(W>w)\ge cw^{-(\tau-1)},\quad w>0\,, \tag{16}$$

with $\gamma=\alpha(\tau-1)/d<2$. Assume that Φ is a positive matrix. Then, $\theta^{\rm WCC}(\lambda)>0$ for every $\lambda>0$, that is, $\lambda_c^{\rm WCC}=0$. In addition, if $\beta\vee(1-\beta)>p_c(d,3)$, then $\lambda_c^{\rm SCC}=0$.

Proof Take a radius $r_\lambda=\lceil\lambda^{-q}\rceil$ for some $q>0$ to be determined later. Let $B(x,r)=\{y:|y-x|\le r\}$ denote the Euclidean ball of radius r around the center x, and write $B(r)=B(0,r)$. For vertex $x\in\mathbb{Z}^d$, define

$$M_x(\lambda)=r_\lambda^{-d/(\tau-1)}\max_{y\in\mathbb{Z}^d\cap B(r_\lambda x,r_\lambda)}W_y\,.$$

Thus, for small $\lambda>0$, we obtain by (16)

$$P(M_x(\lambda)\ge\varepsilon)\ge 1-F(\varepsilon r_\lambda^{d/(\tau-1)})^{r_\lambda^d} \ge 1-(1-c\varepsilon^{-(\tau-1)}r_\lambda^{-d})^{r_\lambda^d} \ge 1-{\rm e}^{-c\varepsilon^{1-\tau}},$$

which tends to 1 uniformly in λ as $\varepsilon\to 0$.

Say that $x\in\mathbb{Z}^d$ is good when $M_x(\lambda)\ge\varepsilon$. The events that two sites are good have the same probability. For two nearest-neighbor vertices $x,y\in\mathbb{Z}^d$, we say that the directed edge (x,y) is λ-occupied when there is a directed edge $(x(\lambda),y(\lambda))$ from $x(\lambda)$ to $y(\lambda)$, where $x(\lambda)$ is the vertex that maximizes W_z for $z\in B(r_\lambda x,r_\lambda)$ and $y(\lambda)$ the vertex that maximizes W_z for $z\in B(r_\lambda y,r_\lambda)$. Therefore, when x and y are both good, we have

$$\begin{aligned}&P((x,y)\ \lambda\text{-occupied}|x,y\text{ good})\\ &=E(p_{x(\lambda)y(\lambda)}|x,y\text{ good})\\ &\ge E\big(1-{\rm e}^{-\lambda\phi W_{x(\lambda)}W_{y(\lambda)}/|x(\lambda)-y(\lambda)|^\alpha}\big|x,y\text{ good}\big)\\ &\ge 1-{\rm e}^{-\lambda\phi(\varepsilon r_\lambda^{d/(\tau-1)})^2/|x(\lambda)-y(\lambda)|^\alpha}\\ &\ge 1-{\rm e}^{-\lambda\phi\varepsilon^2 r_\lambda^{2d/(\tau-1)-\alpha}3^{-\alpha}}\,,\end{aligned}$$

where $\phi>0$ is the minimum element of Φ as defined above. Since $r_\lambda=\lceil\lambda^{-q}\rceil$, we have

$$\lambda r_\lambda^{2d/(\tau-1)-\alpha}\ge\lambda^{1-q(2d/(\tau-1)-\alpha)},$$

and by the assumption $\gamma<2$, we have $\alpha(\tau-1)<2d$. By choosing $q>1/(2d/(\tau-1)-\alpha)$, we derive, for every $\varepsilon>0$,

$$\lim_{\lambda\to 0}P((x,y)\ \lambda\text{-occupied}|x,y\text{ good})=1\,.$$

Likewise, we have

$$\lim_{\lambda\to 0}P((x,y)\ \lambda\text{-occupied}|x,y\text{ good},\psi_{x(\lambda)},\psi_{y(\lambda)})=1\,. \tag{17}$$

Next, we define a nearest-neighbor directed bond percolation model on $\mathbb{Z}^d$, where the directed edge (x,y) between nearest-neighbor sites $x,y\in\mathbb{Z}^d$ is open when both x and y are good and there is a directed edge $(x(\lambda),y(\lambda))$ from $x(\lambda)$ to $y(\lambda)$, that is, when (x,y) is λ-occupied. As proved before, the probability that a vertex is good can be close enough to 1 by taking $\varepsilon>0$ small enough, and the edge probability can then be made close enough to 1 by taking λ sufficiently small. Hence, by using Lemma 1 in the same way as in the proof of Theorem 5 (a), we conclude that the model will percolate (in the sense of WCC) almost surely when ε and λ are small enough.

Let $\theta(\lambda,\varepsilon)$ be the probability that 0 percolates in the above bond percolation model. Note that 0 percolates (in the sense of WCC) in our original model $G(\lambda,\alpha,W,\beta,\Phi)$ when (i) 0 percolates in the bond model; and (ii) 0 is connected to $0(\lambda)$ by a directed edge, where $0(\lambda)$ is the vertex that maximizes W_z in $B(0,r_\lambda)$. The probability that 0 is connected to $0(\lambda)$ by a directed edge, conditionally on the event $\{W_0\ge\varepsilon\}$, is at least $1-{\rm e}^{-\lambda\phi\varepsilon^2 r_\lambda^{d/(\tau-1)-\alpha}}$. Consequently,

$$\theta^{\rm WCC}(\lambda)\ge P(W\ge\varepsilon)\big(1-{\rm e}^{-\lambda\phi\varepsilon^2 r_\lambda^{d/(\tau-1)-\alpha}}\big)\theta(\lambda,\varepsilon)>0\,.$$

As a result, we have $\lambda_c^{\rm WCC}=0$.

Now we address the SCC case. Without loss of generality, we assume that $\beta>p_c(d,3)$, where $p_c(d,3)$ is given in Lemma 1. We can derive (17) as above, and recall that for any vertex $x\in\mathbb{Z}^d$, $P(\psi_{x(\lambda)}=1)=\beta$.

Consider a nearest-neighbor directed bond percolation model on $\mathbb{Z}^d$ as in the WCC case above. Using Lemma 1 in a similar way as in the proof of Theorem 5 (a), that is, for any $z\in\mathbb{Z}^d$, say $X_z=1$ if and only if z is ε-open and $\psi_{z(\lambda)}=1$, we can derive that 1's in $(X_z)_{z\in\mathbb{Z}^d}$ percolate almost surely when ε and λ are sufficiently small.

Let $\theta(\lambda,\varepsilon)$ be the probability that 0 percolates in the above bond percolation model. Note that the edges between vertices of type 1 are all bidirectional, and that 0 percolates (in the sense of SCC) in our original model $G(\lambda,\alpha,W,\beta,\Phi)$ when (i) 0 percolates in the bond model; (ii) 0 is connected to $0(\lambda)$ by a directed edge; and (iii) $\psi_0=1$. The probability that 0 is connected to $0(\lambda)$ by a directed edge, conditionally on the event $\{W_0\ge\varepsilon\}$, is at least $1-{\rm e}^{-\lambda\phi\varepsilon^2 r_\lambda^{d/(\tau-1)-\alpha}}$. Thus,

$$\theta^{\rm SCC}(\lambda)\ge P(W\ge\varepsilon)\big(1-{\rm e}^{-\lambda\phi\varepsilon^2 r_\lambda^{d/(\tau-1)-\alpha}}\big)\beta\theta(\lambda,\varepsilon)>0.$$

Accordingly, we have $\lambda_c^{\rm SCC} = 0$, which concludes the proof. □

4 Graph Distances

For $x \in \mathbb{Z}^d$, let $\tilde{d}(0,x)$ denote the undirected graph distance between 0 and x, that is, the minimal number of occupied edges that form an undirected path between 0 and x. Similarly, denote by $d(0,x)$ (and $d(x,0)$, respectively) the length of shortest directed path, consisting of occupied edges, from 0 to x (and from x to 0, respectively). If 0 and x are not connected by undirected paths, we naturally define $\tilde{d}(0,x)=\infty$, and similar statements hold for $d(0,x)$ and $d(x,0)$.

We will first show that, conditional on different conditions, $d(0,x)$, $d(x,0)$, and $\tilde{d}(0,x)$ are of the same order $\ln\ln|x|$ as $|x|\to\infty$ when the out/in-degrees have infinite variance (see Corollary 1), by means of upper bounds (Theorem 9) and lower bounds (Theorem 10). Next, when the out/in-degrees have finite variance, we show that these distances are at least of the order $\ln|x|$ in the case of $\alpha > d$ (see Theorem 11); and at least of order $|x|^\varepsilon$ for some $\varepsilon > 0$ in the case of $\alpha > 2d$ (see Theorem 12).

Recall that $x \longleftrightarrow y$ denotes the event that there is an undirected path of occupied edges between x and y, and $x \longrightarrow y$ represents the event that there is a directed path of occupied edges from x to y in $G(\lambda,\alpha,W,\beta,\Phi)$. We mention that we assume $\alpha > d$ throughout this section.

Theorem 9 (Doubly logarithmic upper bounds for infinite-variance out/in-degrees) Suppose that there exist $\tau > 1$ and $c > 0$ such that (16) holds and such that $\gamma = \alpha(\tau-1)/d \in (1,2)$. Assume that $\lambda > 0$ and Φ is a positive matrix. Then, for any $\eta > 0$,

$$\lim_{|x|\to\infty} P\Big(\tilde{d}(0,x) \le (1+\eta)\frac{2\ln\ln|x|}{|\ln(\gamma-1)|}\Big| 0 \longleftrightarrow x\Big) = 1\,,$$

$$\lim_{|x|\to\infty} P\Big(d(0,x) \le (1+\eta)\frac{2\ln\ln|x|}{|\ln(\gamma-1)|}\Big| 0 \longrightarrow x\Big) = 1\,,$$

$$\lim_{|x|\to\infty} P\Big(d(x,0) \le (1+\eta)\frac{2\ln\ln|x|}{|\ln(\gamma-1)|}\Big| x \longrightarrow 0\Big) = 1\,.$$

Proof We first show the statement for $\tilde{d}(0,x)$. Let $(W_i)_{i=1}^n$ be i.i.d. weight variables with distribution F. For any $\delta\in(0,1)$, we have

$$P\big(\max_{1\le i\le n} W_i \le n^{(1-\delta)/(\tau-1)}\big) \le \Big(1-\frac{c}{n^{1-\delta}}\Big)^n \le \mathrm{e}^{-cn^\delta}. \quad (18)$$

Take $x\in\mathbb{Z}^d$ with $|x|$ large and let $b\in(0,1)$ be a constant, which will be determined later. For $i=0,1,2,\ldots$, write $\tilde{B}(x,b^i)$ for the ball with radius $|x|^{b^i}/4$ centered at the point at distance $|x|^{b^i}/2$ from 0 on the line segment from 0 to x. Let $z_i\in\mathbb{Z}^d$ be the random vertex in $\tilde{B}(x,b^i)$ with maximal weight. Since the number of vertices in $\mathbb{Z}^d\cap\tilde{B}(x,b^i)$ is of the order $|x|^{db^i}$, we have by using (18) that

$$P(W_{z_i} \le |x|^{db^i((1-\delta)/(\tau-1))}) \le \mathrm{e}^{-c|x|^{db^i\delta}}.$$

We then obtain

$$\begin{aligned}
P\big(\cup_{i=0}^{k-1}\{\text{neither } (z_i,z_{i+1}) \text{ nor } (z_{i+1},z_i) \text{ occupied}\}\big) &\le \sum_{i=0}^{k-1} E\big(\mathrm{e}^{-\lambda\phi W_{z_i}W_{z_{i+1}}/|z_i-z_{i+1}|^\alpha}\big)\\
&\le \sum_{i=0}^{k-1}\Big(E\big(\mathrm{e}^{-\lambda\phi W_{z_i}W_{z_{i+1}}/|z_i-z_{i+1}|^\alpha}\big|W_{z_i}\ge|x|^{db^i((1-\delta)/(\tau-1))}, W_{z_{i+1}}\ge|x|^{db^{i+1}((1-\delta)/(\tau-1))}\big)\\
&\quad + P\big(\{W_{z_i}<|x|^{db^i((1-\delta)/(\tau-1))}\}\cup\{W_{z_{i+1}}<|x|^{db^{i+1}((1-\delta)/(\tau-1))}\}\big)\Big)\\
&\le \sum_{i=0}^{k-1}\big(\mathrm{e}^{-c_1|x|^{db^i((1-\delta)/(\tau-1))}|x|^{db^{i+1}((1-\delta)/(\tau-1))}|x|^{-\alpha b^i}} + 2\,\mathrm{e}^{-c|x|^{\delta db^i}}\big)\\
&= \sum_{i=0}^{k-1}\big(\mathrm{e}^{-c_1|x|^{b^i(d(1+b)((1-\delta)/(\tau-1))-\alpha)}} + 2\,\mathrm{e}^{-c|x|^{\delta db^i}}\big)\,,
\end{aligned}$$

for some constant $c_1>0$, where, in the last inequality, we use the fact that $|z_i-z_{i+1}|\le|x|^{b^i}$. Fix $b\in(\gamma-1,1)$, we have $d(1+b)(1-\delta)/(\tau-1)-\alpha>0$ for sufficiently small δ. Employing $\sum_{i=0}^{k-1}\mathrm{e}^{-|x|^{b^i}}=\Theta(\mathrm{e}^{-|x|^{b^{k-1}}})$ as $k\to\infty$, we thus bound

$$P\big(\cup_{i=0}^{k-1}\{\text{neither } (z_i,z_{i+1}) \text{ nor } (z_{i+1},z_i) \text{ occupied}\}\big) \le c_2\,\mathrm{e}^{-c_3(|x|^{b^{k-1}})^{c_4}}\,,$$

for large k, where c_2, c_3, and c_4 are positive constants.

Fix $\varepsilon>0$. Take $A=A(\varepsilon)$ large enough so that $c_2\,\mathrm{e}^{-c_3A^{c_4}}\le\varepsilon$ and then choose k such that $|x|^{b^{k-1}}=A$, that is,

$$k = \frac{\ln\ln|x| - \ln\ln A}{|\ln b|} + 1\,.$$

Denote by B the event that either (z_i,z_{i+1}) or (z_{i+1},z_i) is occupied, for all $i=0,1,\ldots,k-1$. From the above arguments we have $P(B)\ge1-\varepsilon$, and conditional on B, we have

$$\tilde{d}(0,z_0)\le k+\tilde{d}(0,z_k)\,,$$

where $|z_k|\le|x|^{b^k}\le A$. By a similar procedure performed above (interchanging the role of 0 and x), we obtain some random vertices $z_0'=z_0,z_1',z_2',\ldots,z_k'$ such that $P(B')\ge1-\varepsilon$, with B' representing the event that either (z_i',z_{i+1}')

or (z'_{i+1}, z'_i) is occupied, for all $i = 0, 1, \ldots, k-1$. Likewise, conditional on B', we have $\tilde{d}(x, z_0) \le k + \tilde{d}(x, z'_k)$, where $|z'_k| \le A$.

Hence, for every $\eta > 0$ and $|x|$ large enough, we have

$$\begin{aligned} &P\Big(\tilde{d}(0,x) \le \frac{2(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow x\Big) \\ &\quad\ge P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}, \tilde{d}(x,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow x\Big) \\ &\quad\ge P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow z_0\Big) \cdot P\Big(\tilde{d}(x,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|x \longleftrightarrow z_0\Big) \\ &\quad\ge (1-\varepsilon)^2 \cdot P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow z_0, B\Big) \cdot P\Big(\tilde{d}(x,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|x \longleftrightarrow z_0, B'\Big)\,. \end{aligned}$$

Conditional on the events that B and $0 \longleftrightarrow z_0$ hold, we have $\tilde{d}(0, z_k) < \infty$ almost surely (since $|z_k| \le A$), and hence, for any $\kappa > 0$, $P(\tilde{d}(0, z_k) \le \kappa \ln\ln|x|) \ge 1-\varepsilon$ when k is large. Therefore, for $|x|$ large enough and κ small enough,

$$\begin{aligned} &P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow z_0, B\Big) \\ &\quad\ge P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|\tilde{d}(0,z_k) \le \kappa\ln\ln|x|, 0 \longleftrightarrow z_0, B\Big) \cdot P(\tilde{d}(0,z_k) \le \kappa\ln\ln|x|\,|0 \longleftrightarrow z_0, B) \\ &\quad\ge (1-\varepsilon) \cdot P\Big(\tilde{d}(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|\tilde{d}(0,z_k) \le \kappa\ln\ln|x|, 0 \longleftrightarrow z_0, B\Big) \\ &\quad\ge (1-\varepsilon) \cdot P(\tilde{d}(0,z_0) \le k + \tilde{d}(0,z_k)\ |\tilde{d}(0,z_k) \le \kappa\ln\ln|x|, 0 \longleftrightarrow z_0, B) \\ &\quad\ge (1-\varepsilon) \cdot P(\tilde{d}(0,z_0) \le k + \tilde{d}(0,z_k)|B) = (1-\varepsilon)\,. \end{aligned}$$

Similar, we have

$$P\Big(\tilde{d}(x,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|x \longleftrightarrow z_0, B'\Big) \ge (1-\varepsilon)\,.$$

Combining the above arguments, we deduce that

$$\lim_{|x|\to\infty} P\Big(\tilde{d}(0,x) \le \frac{2(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longleftrightarrow x\Big) \ge (1-\varepsilon)^4.$$

Taking b close enough to $\gamma - 1$ such that

$$\frac{1+\eta/2}{|\ln b|} \le \frac{1+\eta}{|\ln(\gamma-1)|}\,,$$

and then the statement for $\tilde{d}(0, x)$ follows readily.

In what follows, we show the statement for $d(0, x)$. We can similarly derive that

$$P\big(\cup_{i=0}^{k-1}\{(z_{i+1}, z_i) \text{ not occupied}\}\big) \le c_2\, \mathrm{e}^{-c_3(|x|^{b^{k-1}})^{c_4}}\,,$$

for large k, where c_2, c_3, and c_4 are positive constants.

Fix $\varepsilon > 0$. Take $A = A(\varepsilon)$ large enough so that $c_2\, \mathrm{e}^{-c_3 A^{c_4}} \le \varepsilon$ and then choose k such that $|x|^{b^{k-1}} = A$, that is,

$$k = \frac{\ln\ln|x| - \ln\ln A}{|\ln b|} + 1\,.$$

Denote by B_1 the event that each edge (z_{i+1}, z_i) is occupied, for all $i = 0, 1, \ldots, k-1$, and B_2 the event that each edge (z_i, z_{i+1}) is occupied, for all $i = 0, 1, \ldots, k-1$. From the above arguments we have $P(B_1) \ge 1-\varepsilon$, $P(B_2) \ge 1-\varepsilon$ and conditional on B_1, we have

$$d(0, z_0) \le k + d(0, z_k),$$

where $|z_k| \le |x|^{b^k} \le A$. By a similar procedure (interchanging the role of 0 and x), we obtain some random vertices $z'_0 = z_0, z'_1, z'_2, \ldots, z'_k$ such that $P(B'_1) \ge 1-\varepsilon$, with B'_1 representing the event that (z'_i, z'_{i+1}) is occupied, for all $i = 0, 1, \ldots, k-1$, and $P(B'_2) \ge 1-\varepsilon$, with B'_2 representing the event that (z'_{i+1}, z'_i) is occupied, for all $i = 0, 1, \ldots, k-1$. Likewise, conditional on B'_1, we have

$$d(z_0, x) \le k + d(z'_k, x)\,,$$

where $|z'_k| \le A$.

Hence, for every $\eta > 0$ and $|x|$ large enough, we have

$$\begin{aligned} &P\Big(d(0,x) \le \frac{2(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longrightarrow x\Big) \ge P\Big(d(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}, d(z_0,x) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longrightarrow x\Big) \\ &\quad\ge P\Big(d(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}|0 \longrightarrow z_0\Big) \cdot P\Big(d(z_0,x) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|z_0 \longrightarrow x\Big) \\ &\quad\ge (1-\varepsilon)^2 \cdot P\Big(d(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longrightarrow z_0, B_2\Big) \cdot P\Big(d(z_0,x) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|z_0 \longrightarrow x, B'_2\Big)\,. \end{aligned}$$

Conditional on the events that B_2 and $0 \longrightarrow z_0$ hold, we have $d(0, z_k) < \infty$ almost surely (since $|z_k| \le A$), and hence, for any $\kappa > 0$, $P(d(0, z_k) \le \kappa\ln\ln|x|) \ge 1-\varepsilon$ when k is large. Therefore, for $|x|$ large enough and κ small enough,

$$P\Big(d(0,z_0) \le \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0 \longrightarrow z_0, B_2\Big)$$

$$\geq P\Big(d(0,z_0)\leq \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|d(0,z_k)\leq \kappa\ln\ln|x|, 0\longrightarrow z_0, B_2\Big)\cdot P(d(0,z_k)\leq\kappa\ln\ln|x||0\longrightarrow z_0,B_2)$$
$$\geq (1-\varepsilon)\cdot P\Big((0,z_0)\leq \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|d(0,z_k)\leq \kappa\ln\ln|x|, 0\longrightarrow z_0, B_2\Big)$$
$$\geq (1-\varepsilon)^2\cdot P\Big(d(0,z_0)\leq \frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|d(0,z_k)\leq \kappa\ln\ln|x|, 0\longrightarrow z_0, B_1, B_2\Big)$$
$$\geq (1-\varepsilon)^2\cdot P(d(0,z_0)\leq k+d(0,z_k)|d(0,z_k)\leq\kappa\ln\ln|x|, 0\longrightarrow z_0, B_1,B_2)$$
$$\geq (1-\varepsilon)^2\cdot P(d(0,z_0)\leq k+d(0,z_k)|B_1)=(1-\varepsilon)^2.$$

Similar, we have
$$P\Big(d(z_0,x)\leq\frac{(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|z_0\longrightarrow x, B_2'\Big)\geq(1-\varepsilon)^2.$$
Combining the above arguments, we deduce that
$$\lim_{|x|\to\infty}P\Big(d(0,x)\leq\frac{2(1+\eta/2)\ln\ln|x|}{|\ln b|}\Big|0\longrightarrow x\Big)\geq(1-\varepsilon)^6.$$
Taking b close enough to $\gamma-1$ such that
$$\frac{1+\eta/2}{|\ln b|}\leq\frac{1+\eta}{|\ln(\gamma-1)|},$$
which ends the proof of the statement for $d(0,x)$.

The statement for $d(x,0)$ is similar with that for $d(0,x)$, and hence omitted. □

Theorem 10 (Doubly logarithmic lower bounds for infinite-variance out/in-degrees) Suppose that there exist $\tau>1$ and $c>0$ such that (15) holds and such that $\gamma=\alpha(\tau-1)/d\in(1,2)$. Assume that $\lambda>0$. Then, for any $\eta>0$,
$$\lim_{|x|\to\infty}P\Big(\tilde d(0,x)\geq(1-\eta)\frac{2\ln\ln|x|}{|\ln\kappa|}\Big|0\longleftrightarrow x\Big)=1\,,$$
$$\lim_{|x|\to\infty}P\Big(d(0,x)\geq(1-\eta)\frac{2\ln\ln|x|}{|\ln\kappa|}\Big|0\longrightarrow x\Big)=1\,,$$
$$\lim_{|x|\to\infty}P\Big(d(x,0)\geq(1-\eta)\frac{2\ln\ln|x|}{|\ln\kappa|}\Big|x\longrightarrow 0\Big)=1\,,$$
where $\kappa=\gamma-1$ when $\tau\in(1,2]$ and $\kappa=\alpha/d-1$ when $\tau>2$.

In particular, we have
$$\lim_{|x|\to\infty}P\Big(\min\{\tilde d(0,x),d(0,x),d(x,0)\}\geq(1-\eta)\frac{2\ln\ln|x|}{|\ln\kappa|}\Big)=1\,.$$

Proof We first show the statement for $\tilde d(0,x)$. Define
$$\tilde S_n(x)=\sup_{\tilde d(x,y)\leq n}\{|x-y|\}$$
to be the (undirected) distance between x and the farthest vertex $y\in\mathbb{Z}^d$ that can be reached via at most n edges. Therefore, by translation invariance, we have
$$P(\tilde d(0,x)\leq 2n|0\longleftrightarrow x)\leq P(\tilde S_n(x)\geq|x|/2|0\longleftrightarrow x)+P(\tilde S_n(0)\geq|x|/2|0\longleftrightarrow x)$$
$$=2P(\tilde S_n(0)\geq|x|/2|0\longleftrightarrow x)=2P(\tilde S_n(0)\geq|x|/2)\,. \tag{19}$$
For $s\leq t$, we derive
$$P(\tilde S_n(0)\geq t)\leq P(\tilde S_{n-1}(0)\geq s)+P(\tilde S_{n-1}(0)<s,\tilde S_n(0)\geq t)\,, \tag{20}$$
and the second term on the right-hand side of (20) can be bounded by
$$P(\tilde S_{n-1}(0)<s,\tilde S_n(0)\geq t)\leq P(\exists u,v\in\mathbb{Z}^d,\text{ such that }|u|\leq s,\ |v|\geq t,\text{ and }u\longleftrightarrow v)$$
$$\leq\sum_{u,v:\ |u|\leq s,|v|\geq t}E(p_{uv})\leq\sum_{u,v:\ |u|\leq s,|v|\geq t}E\Big(\frac{\lambda\|\Phi\|W_uW_v}{|u-v|^\alpha}\wedge 1\Big)=\sum_{u,v:\ |u|\leq s,|v|\geq t}g_1\Big(\frac{|u-v|^\alpha}{\lambda\|\Phi\|}\Big)\,,$$
where $g_1(u)=E((W_1W_2/u)\wedge 1)$ with independent copies W_1 and W_2 of weight W. From the proof of Lemma 2, we can easily derive that $g_1(u)\leq C_1(1+\ln u)^2u^{-((\tau-1)\wedge 1)}$ for some constant $C_1>0$. It follows from a similar argument as in Theorem 2 that
$$P(\tilde S_{n-1}(0)<s,\tilde S_n(0)\geq t)\leq C_1\sum_{u,v:\ |u|\leq s,|v|\geq t}|u-v|^{-\alpha((\tau-1)\wedge1)}\Big(1+\ln\Big(\frac{|u-v|^\alpha}{\lambda\|\Phi\|}\Big)\Big)^2\leq C_2|s|^d|t|^{d-\alpha((\tau-1)\wedge1)+\xi}\,,$$
where $C_2>0$ is a constant, and $\xi>0$ can be made sufficient small. Fix $A\geq1$ to be large, and take $\delta>0$ so that $\kappa-\delta\in(0,1)$. We take $t=A^{(\kappa-\delta)^{-n}}$ and $s=A^{(\kappa-\delta)^{-(n-1)}}$, so that $s=t^{\kappa-\delta}$, and
$$C_2|s|^d|t|^{d-\alpha((\tau-1)\wedge1)+\xi}=C_2|t|^{d-\alpha((\tau-1)\wedge1)+\xi+(\kappa-\delta)d}=C_2(A^{(\kappa-\delta)^{-n}})^{-\zeta},$$
where $\zeta=\alpha((\tau-1)\wedge1)-d-(\kappa-\delta)d-\xi>0$, since $\kappa=\alpha((\tau-1)\wedge1)/d-1$ by definition. Combining these with (20), we obtain by recursion
$$P(\tilde S_n(0)\geq A^{(\kappa-\delta)^{-n}})\leq P(\tilde S_{n-1}(0)\geq A^{(\kappa-\delta)^{-(n-1)}})+C_2(A^{(\kappa-\delta)^{-n}})^{-\zeta}\leq P\big(\tilde S_1(0)\geq A^{(\kappa-\delta)^{-1}}\big)+C_2\sum_{k=2}^nA^{-\zeta(\kappa-\delta)^{-k}}=o(1)\,,$$

as $A \to \infty$, due to the fact that

$$P(\tilde{S}_1(0) \geq A) \leq \sum_{u:\ |u| \geq A} g_1\Big(\frac{|u|^\alpha}{\lambda\|\Phi\|}\Big) \leq C \sum_{u:\ |u|\geq A} |u|^{-\alpha((\tau-1)\wedge 1)+\xi} = o(1)\,,$$

as $A \to \infty$, for some constant $C > 0$, and $\xi > 0$ may be taken arbitrarily small.

Now, for every $\eta > 0$, taking $n = (1-\eta)\ln\ln|x|/|\ln\kappa|$ in (19), we have $A^{(\kappa-\delta)^{-n}} \leq |x|/2$ for sufficiently small δ. Consequently, $P(\tilde{d}(0,x) \leq 2n|0 \longleftrightarrow x) = o(1)$, as $|x| \to \infty$, which ends the proof for the case of $\tilde{d}(0,x)$.

Next, we show the statement for $d(0,x)$. Define

$$S_n(x) = \sup_{d(x,y)\leq n}\{|x-y|\}$$

to be the distance between x and the farthest vertex $y \in \mathbb{Z}^d$ that can be reached via a directed path starting from x of length at most n. Similarly, we can define

$$S'_n(x) = \sup_{d(y,x)\leq n}\{|x-y|\}\,.$$

Therefore, by translation invariance, we have

$$\begin{aligned} P(d(0,x) \leq 2n|0 \longrightarrow x) &\leq P(S_n(0) \geq |x|/2|0 \longrightarrow x) + P(S'_n(x) \geq |x|/2|0 \longrightarrow x) \\ &= P(S_n(0) \geq |x|/2|0 \longrightarrow x) + P(S'_n(0) \geq |x|/2|x \longrightarrow 0) \\ &= P(S_n(0) \geq |x|/2) + P(S'_n(0) \geq |x|/2)\,. \end{aligned} \tag{21}$$

We can proceed exactly as above to deduce that $P(S_n(0) \geq A^{(\kappa-\delta)^{-n}}) = o(1)$ and $P(S'_n(0) \geq A^{(\kappa-\delta)^{-n}}) = o(1)$, as $A \to \infty$. The result then follows from (21) by taking $n = (1-\eta)\ln\ln|x|/|\ln\kappa|$ as before.

The statement for $d(x,0)$ may be proved similarly. □

Combining Theorem 9 and Theorem 10, we arrive at the following weak law of graph distances.

Corollary 1 (Doubly logarithmic weak law for infinite-variance out/in-degrees and infinite-mean weights) Suppose that there exist $\tau \in (1,2)$ and $c > 0$ such that (1) holds and such that $\gamma = \alpha(\tau-1)/d \in (1,2)$. Assume that $\lambda > 0$ and Φ is a positive matrix. Then, conditionally on $\{0 \longleftrightarrow x\}$,

$$\frac{\tilde{d}(0,x)}{\ln\ln|x|} \xrightarrow{P} \frac{2}{|\ln(\gamma-1)|}\,;$$

conditionally on $\{0 \longrightarrow x\}$,

$$\frac{d(0,x)}{\ln\ln|x|} \xrightarrow{P} \frac{2}{|\ln(\gamma-1)|}\,;$$

conditionally on $\{x \longrightarrow 0\}$,

$$\frac{d(x,0)}{\ln\ln|x|} \xrightarrow{P} \frac{2}{|\ln(\gamma-1)|}\,,$$

as $|x| \to \infty$.

Proof By the Potter theorem (see e.g. [20] p. 25), for every $\varepsilon > 0$, there exist positive constants c_ε and C_ε such that for all w large enough,

$$c_\varepsilon w^{-(\tau-1+\varepsilon)} \leq 1 - F(w) \leq C_\varepsilon w^{-(\tau-1-\varepsilon)}.$$

We choose ε sufficiently small such that the following four inequalities hold:

$$\tau+\varepsilon < 2, \quad \tau-\varepsilon > 1, \quad \frac{\alpha(\tau+\varepsilon-1)}{d} < 2 \quad \text{and} \quad \frac{\alpha(\tau-\varepsilon-1)}{d} > 1\,.$$

The results then directly follows from Theorem 9 and Theorem 10. □

Now we turn to the finite-variance out/in-degrees (i.e., $\gamma > 2$) situation.

Theorem 11 (Logarithmic lower bounds for finite-variance out/in-degrees) Suppose that there exist $\tau > 1$ and $c > 0$ such that (15) holds and such that $\gamma = \alpha(\tau-1)/d > 2$. Then, there exists some $\eta > 0$ such that

$$\lim_{|x|\to\infty} P(\tilde{d}(0,x) \geq \eta\ln|x|) = 1\,,$$

and hence

$$\lim_{|x|\to\infty} P(d(0,x) \geq \eta\ln|x|) = 1\,,$$

$$\lim_{|x|\to\infty} P(d(x,0) \geq \eta\ln|x|) = 1\,.$$

Proof In what follows, we will prove the statement for $\tilde{d}(0,x)$, and the other two statements follow immediately.

Following the proof of Theorem 7, we have

$$P(\tilde{d}(0,x) = n) \leq 2\sum_{(x_1,\ldots,x_{n-1})}\prod_{i=1}^n g\Big(\frac{|x_{i-1}-x_i|^\alpha}{\lambda\|\Phi\|}\Big)^{1/2},$$

where we utilize the convention that $x_0 = 0$ and $x_n = x$. Define a function $h(x) = (1+\ln|x|)|x|^{-\alpha(((\tau-1)/2)\wedge 1)}$ for $x \neq 0$ and $h(0) = 0$, involving Lemma 2, and then we can deduce

$$P(\tilde{d}(0,x) = n) \leq (C\lambda^{((\tau-1)/2)\wedge 1})^n h^{*n}(x)\,,$$

where h^{*n} represents the n-fold convolution of h with itself, and $C > 0$ is some constant. Notice that

$$h^{*n}(x) = \sum_{x_1+\cdots+x_n=x}\prod_{i=1}^n h(x_i),$$

and when $x_1+\cdots+x_n = x$, there must be some x_i such that $|x_i| \geq |x|/n$. Hence, we have

$$h^{*n}(x) \leq n(\sup_{|y|\geq|x|/n} h(y))\Big(\sum_{u\neq 0} h(u)\Big)^{n-1}.$$

Fix $n \leq \eta\ln|x|$, for some $\eta > 0$ which will be determined later. Since $h(y)$ is decreasing, we can define some

$\kappa > 0$ such that

$$\sup_{|y|\ge|x|/n} h(y) \le C'(\ln|x|)^\kappa |x|^{-\alpha(((\tau-1)/2)\wedge 1)},$$

for some constant $C' > 0$. Since $\alpha > d$ and $\gamma > 2$, we have $\sum_{u\neq 0} h(u) < \infty$. Therefore, for $|x|$ large enough and η small enough, we obtain

$$P(\tilde{d}(0,x)=n) \le n\Big(C\lambda^{((\tau-1)/2)\wedge 1}\Big)^n \times (\ln|x|)^\kappa |x|^{-\alpha(((\tau-1)/2)\wedge 1)} \le |x|^{-\varepsilon},$$

where $\varepsilon > 0$. Consequently, we derive

$$P(\tilde{d}(0,x) \le \eta\ln|x|) \le |x|^{-\varepsilon}\eta\ln|x| \to 0,$$

as $|x|\to\infty$, which then concludes the proof. □

The following result slightly improved the exponent ε in Theorem 5.5[19] when $\gamma\wedge(\alpha/d) < 2 + (1/d)$.

Theorem 12 (Polynomial lower bounds for finite-variance out/in-degrees when $\alpha > 2d$) Suppose that there exist $\tau > 1$ and $c > 0$ such that (15) holds and such that $\gamma = \alpha(\tau-1)/d > 2$ and $\alpha > 2d$. Then, for every

$$\varepsilon < \frac{d(\gamma\wedge(\alpha/d))-2}{d(\gamma\wedge(\alpha/d))-1},$$

we have $\lim_{|x|\to\infty} P(\tilde{d}(0,x) \ge |x|^\varepsilon) = 1$, and hence $\lim_{|x|\to\infty} P(d(0,x) \ge |x|^\varepsilon) = 1$, $\lim_{|x|\to\infty} P(d(x,0) \ge |x|^\varepsilon) = 1$.

Proof As in Theorem 11, we only need to prove the statement for $\tilde{d}(0,x)$. Following the proof of Theorem 10, we begin by investigating the probability $P(\tilde{S}_n(0)\ge t)$. For $t\to\infty$ and $n = o(t)$, we obtain

$$\begin{aligned}
P(\tilde{S}_n(0)\ge t) &\le P\Big(\tilde{S}_{n-1}(0) \ge \frac{(n-1)t}{n}\Big) + P\Big(\tilde{S}_{n-1}(0) \le \frac{(n-1)t}{n}, \tilde{S}_n(0)\ge t\Big)\\
&\le P\Big(\tilde{S}_1(0)\ge \frac{t}{n}\Big) + \sum_{k=1}^{n-1} P\Big(\tilde{S}_k(0)\le\frac{kt}{n}, \tilde{S}_{k+1}(0)\ge\frac{(k+1)t}{n}\Big)\\
&\le o(1) + \sum_{k=1}^{n-1}\sum_{\substack{u,v:\\ |u|\le kt/n,|v|\ge(k+1)t/n}} g_1\Big(\frac{|u-v|^\alpha}{\lambda\|\phi\|}\Big) \le o(1) + C\sum_{k=1}^{n-1}\Big(\frac{kt}{n}\Big)^d\Big(\frac{t}{n}\Big)^{-\alpha((\tau-1)\wedge 1)+d+\eta}\\
&\le o(1) + t^{d(2-\gamma\wedge(\alpha/d))+\eta} n^{d(\gamma\wedge(\alpha/d)-1)-\eta},
\end{aligned}$$

where $C > 0$ is some constant, $\eta > 0$ can be made sufficiently small, and we use the assumption $\gamma > 2$, $\alpha > 2d$ in the last but one inequality and use the fact $\sum_{k=1}^{n-1} k^d = \Theta(n^d)$ in the last inequality.

Accordingly, it is easy to see that $P(\tilde{S}_n(0)\ge t) = o(1)$, as $t\to\infty$, when $n \le t^\varepsilon$, where

$$\varepsilon < [d(\gamma\wedge(\alpha/d))-2]/[d(\gamma\wedge(\alpha/d))-1].$$

From (19) we have

$$P(\tilde{d}(0,x)\le 2n) \le 2P(\tilde{S}_n(0)\ge |x|/2),$$

and hence $P\Big(\tilde{d}(0,x)\le|x|^\varepsilon\Big) = o(1)$ by taking $n = |x|^\varepsilon/2$ and $t = (2n)^{1/\varepsilon}/2$. □

5 Conclusion

In this paper, we have investigated out/in-degrees, critical percolation values, and graph distances in a multi-type directed long-range percolation model $G(\lambda,\alpha,W,\beta,\Phi)$ with i.i.d. vertex weights. Our model features multiple types of vertices, directed edges, power-law degrees, small-world phenomenon, spatial structure as well as relational structure, the properties of which vary with the number of finite moments of its out/in-degree distributions in a subtle manner. Moreover, this model is based on the "scale-free percolation" model of Deijfen, van der Hofstad, and Hooghiemstra,[19] which shares a number of attractive characteristics of both inhomogeneous random graphs and long-range percolation.

In addition to the open questions raised in [19], we mention some of the problems that deserve further investigation. Firstly, in our model we have equaled the edge (x,y) to (y,x) when $\psi_x = \psi_y$ in order to serve our purpose. Although $p_{xy} = p_{yx}$ in this situation, we may still view (x,y) and (y,x) separately as two different directed edges. Some of our analyses, especially in Secs. 3 and 4 should be modified to cope with this new situation. Secondly, in view of our simulation example in Sec. 2, the conditional out/in-degrees addressed in Theorem 2 presumably have some limit laws. Thirdly, some conditions of the theorems, e.g. we have assumed that Φ is positive in some results, may be further weakened.

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