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Nonlocalization of Nonlocal Symmetry and Symmetry Reductions of the Burgers Equation*

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Abstract *Symmetry reduction method is one of the best ways to find exact solutions. In this paper, we study the possibility of symmetry reductions of the well known Burgers equation including the nonlocal symmetry. The related new group invariant solutions are obtained. Especially, the interactions among solitons, Airy waves, and Kummer waves are explicitly given.*

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1 Introduction

Since the Lie group theory was introduced by Sophus Lie to study differential equations,[1] the study of Lie group has been an important subject in mathematics and physics. Using both the classical, non-classical Lie group approaches[2–3] and the direct method,[4] one can reduce the dimensions of partial differential equations (PDEs) and proceed to construct analytical solutions.

Usually, one first studies the symmetry algebras of real physical problems and then the related symmetry groups via Lie's first principle. The recent studies show us that for many kinds of nonlinear partial differential equations, to directly find symmetry groups may be much more convenient than to first find related symmetry algebras.[5]

References [6–7] reveal that there are infinitely many nonlocal symmetries. It is known that utilizing Lie point symmetries, the dimensions of the related PDEs can be reduced. An important problem is how to use the nonlocal symmetries to reduce the dimensions of the original PDEs? A simple and direct way to solve this problem is the so-called localization method of nonlocal symmetries. In other words, the original system is prolonged to a larger system such that the nonlocal symmetry of the original model becomes a local one of the prolonged system.[8]

In this paper, we first write down a nonlocal symmetry for the Burgers equation. Then, we apply the localization procedure to the nonlocal symmetry to obtain new exact symmetry reductions and exact solutions. The solitons and/or solitary waves have been widely applied in almost all the physics branches.[9] Among these equations, the Burgers equation is the simplest one.[10] Burgers equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied physics such as modeling of gas dynamics and traffic flow. For a given velocity u and viscosity coefficient ν, the general form of Burgers equation can be written as

$$u_t + uu_x = \nu u_{xx}\,, \tag{1}$$

where the subscripts denote the differentiations. In Ref. [11], the infinitely many symmetries (related potential symmetries) and exact solutions of the Burgers equation are obtained by repeated symmetry reduction approach. In this paper, we will find new nonlocal symmetries related to the Schwazian Burgers equation.

2 Truncated Painlevé Expansion and Schwarzian Burgers Equation

The key point to find new nonlocal symmetries is to transform the Burgers equation to its Schwarzian form via the truncated Painlevé expansion.

The Painlevé analysis is one of the effective ways to study nonlinear systems. Using the Painlevé analysis for an integrable system, one can find not only its Painlevé property, but also the Bäcklund transformation, bilinear form, Lax pair, Schwarzian form, nonlocal symmetry, and so on.

For the Burgers equation, its truncated Painlevé expansion has the form

$$u = \frac{u_0}{f} + u_1\,, \tag{2}$$

where the function f is the singularity manifold, and the functions u_0 and u_1 are related to the derivatives of f and

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should be determined by substituting Eq. (2) into Eq. (1) and vanishing the coefficients of f^i for different i.

After substituting Eq. (2) into the Burgers equation, we can prove the following theorem.

Theorem 1 (Non-auto Bäclund transformation theorem). If f is a solution of the Schwarzian Burgers equation

$$C_t = \Big(2\nu C_x + \frac{C^2}{2} - \nu^2 S\Big)_x\,, \quad C \equiv \frac{f_t}{f_x}\,,$$
$$S \equiv \frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{f_{xx}^2}{f_x^2}\,, \tag{3}$$

then both

$$u = \nu(\ln f_x)_x - C\,, \tag{4}$$
$$u = \nu\Big(\ln\frac{f_x}{f^2}\Big)_x - C \tag{5}$$

are solutions of the Burgers equation (1).

Proof Substituting Eq. (2) into the Burgers equation, we have

$$\begin{aligned} &u_{1t} - \nu u_{1xx} + u_1 u_{1x} + \big(u_{0t} - \nu u_{0xx} + (u_0 u_1)_x\big) f^{-1} \\ &\quad + \big[2\nu u_{0x} f_x + \big(\nu f_{xx} + u_{0x} - u_1 f_x - f_t\big)\big]\, f^{-2} \\ &\quad - u_0 f_x (u_0 + 2\nu f_x) f^{-3} = 0\,. \end{aligned} \tag{6}$$

Vanishing the coefficient of f^{-3} in Eq. (6), the only nontrivial solution is

$$u_0 = -2\nu f_x\,. \tag{7}$$

Vanishing the coefficient of f^{-2} in Eq. (6) and using Eq. (7), we have

$$u_1 = \nu(\ln f_x)_x - C\,. \tag{8}$$

Using Eqs. (7) and (8), it is not difficult to find that the coefficient of f^{-1} identically becomes zero.

The coefficient of f^0 in Eq. (6) is nothing but the Burgers equation for $u = u_1$, that means u given by Eq. (4) is a solution of the Burgers equation. Equation (1) with Eqs. (7) and (8) is just the result (5). Finally, substituting Eq. (8) into the BE (i.e., the equation obtained from vanishing the coefficient of f^0 of Eq. (6) leads to the Schwarzian equation (3). The non-auto Bäclund transformation theorem 1 is proved.

3 Localization of Nonlocal Symmetry for the Burgers Equation

From the proof procedure of the Theorem 1, it is straightforward to prove the following nonlocal symmetry theorem.

Theorem 2 (Nonlocal symmetry theorem). If f is a solution of the Schwarzian Burgers equation (3), then

$$\sigma = f_x \tag{9}$$

is a nonlocal symmetry of the Burgers equation (1).

Proof A symmetry, σ, of a nonlinear system is defined by a solution of its linearized equation. For the Burgers equation (1), its symmetries are solutions of

$$\sigma_t - \nu\sigma_{xx} + (\sigma u)_x = 0\,. \tag{10}$$

That means the Burgers equation (1) is form-invariant under the transformation,

$$u \to u + \epsilon\sigma\,, \tag{11}$$

where ϵ is an infinitesimal parameter.

In the proof procedure of the Theorem 1, we know that u_1 is just a solution of the Burgers equation (1). Comparing the coefficients of f^{-1} in Eq. (6) and the symmetry definition equation (10), we know that u_0 satisfies just the symmetry definition equation. Thus, $f_x \sim u_0$ is a symmetry of the Burgers equation. Furthermore, the function f and then f_x is linked with u nonlocally via (4). Therefore, the symmetry is nonlocal. The nonlocal symmetry Theorem 2 is proved. Of course, the Theorem 2 can also be directly proved by substituting Eq. (9) into the symmetry definition equation (10) via the Schwarzian Burgers equation (3).

The symmetry transformation (11) tells us that a symmetry, $\sigma = \sigma(u)$, links two solutions approximately. To find exact relation between two exact solutions, one has to solve the initial value problem,

$$\frac{\mathrm{d}u'(\epsilon)}{\mathrm{d}\epsilon} = \sigma(u'(\epsilon))\,, \quad u'(0) = u\,, \tag{12}$$

due to the Lie's first principle.[1]

However, for the nonlocal symmetry (9), we can not solve the initial value problem (12) directly. To solve the initial problem (12) with Eq. (9), we have to prolong the original system such that the nonlocal symmetry of the original model can be changed to a local Lie point symmetry for the prolonged larger system.

To localization the nonlocal symmetry (9), we have to know what will be changed for the potential fields f and

$$g = f_x\,, \tag{13}$$

when the field u possesses the symmetry

$$\sigma^u = f_x = g\,. \tag{14}$$

That means we have to solve the symmetry definition equations,

$$f_t\sigma_x^f - f_x(\sigma_t^f - \nu\sigma_{xx}^f + f_x\sigma) - \nu f_{xx}\sigma_x^f = 0\,, \tag{15}$$
$$\sigma^g = \sigma_x^f\,, \tag{16}$$

which are the linearized equations of Eqs. (4) and (13) respectively.

It is not difficult to verify that the corresponding solution of Eqs. (15) and (16) with Eq. (14) has the form

$$\sigma^f = \frac{f^2}{2\nu} + c_1 f + c_0\,, \tag{17}$$
$$\sigma^g = \frac{g(f + c_1\nu)}{\nu}\,, \tag{18}$$

where c_1 and c_0 are arbitrary constants.

From Eqs. (14), (17), and (18), we know that the nonlocal symmetry of the single Burgers equation becomes a Lie point symmetry for the enlarged system (1), (4), and (13).

Now, the initial value problem (12) becomes

$$\frac{\mathrm{d}u'(\epsilon)}{\mathrm{d}\epsilon} = g'(\epsilon)\,, \quad u'(0) = u\,,$$

$$\frac{\mathrm{d}f'(\epsilon)}{\mathrm{d}\epsilon} = \frac{1}{2\nu}f'^2(\epsilon) + c_1 f'(\epsilon) + c_0\,, \quad f'(0) = f\,,$$

$$\frac{\mathrm{d}g'(\epsilon)}{\mathrm{d}\epsilon} = \frac{1}{\nu}[f'(\epsilon) + c_1\nu]g'(\epsilon)\,, \quad g'(0) = g\,. \tag{19}$$

Solving the initial value problem, the following auto Bäcklund transformation theorem is proved.

Theorem 3 (Auto-Bäcklund transformation theorem). If $\{u, f, g\}$ is a solution of the enlarged system $\{(1), (4), (13)\}$, so are $\{u', f', g'\}$ and $\{u'', f'', g''\}$ for

$$u' = u - 2g\nu\frac{f + c_1\nu + \alpha\tanh[\alpha a/2\nu - \operatorname{arctanh}((f + c_1\nu)/\alpha)]}{(f + c_1\nu)^2 - \alpha^2}\,,$$

$$f' = -c_1\nu - \alpha\tanh\Big[\frac{\alpha a}{2\nu} - \operatorname{arctanh}\Big(\frac{f + c_1\nu}{\alpha}\Big)\Big]\,, \quad g' = -\alpha^2 g\frac{\operatorname{sech}^2\big[\alpha a/2\nu - \operatorname{arctanh}((f + c_1\nu)/\alpha)\big]}{(f + c_1\nu)^2 - \alpha^2}\,, \tag{20}$$

with $\alpha \equiv \sqrt{c_1^2\nu^2 - 2c_0\nu} \neq 0$ and

$$u'' = u - \frac{2g\nu\epsilon}{\epsilon(f + c_1\nu) - 2\nu}\,, \quad f'' = \nu\frac{2f + c_1\epsilon(f + c_1\nu)}{2\nu - \epsilon(f + c_1\nu)}\,, \quad g'' = \frac{4\nu^2 g}{(\epsilon(f + c_1\nu) - 2\nu)^2}\,, \tag{21}$$

with $\alpha = 0$.

4 Symmetry Reductions

Though the symmetry reduction solutions of the single Burgers equation have been studied by many authors and a large number of results have been obtained, however, the symmetry reductions related to the nonlocal symmetry (14) have not yet been discussed before. To find the corresponding new symmetry reductions related to the nonlocal symmetry (14), we have to study the full Lie point symmetries of the enlarged system $\{(1), (4), (13)\}$.

The generalized Lie point symmetries of the extended system of the fields $\{u, f, g\}$ possess the form,

$$\begin{aligned} \sigma^u &= X(x, t, u, f, g)u_x + T(x, t, u, f, g)u_t - U(x, t, u, f, g)\,, \\ \sigma^f &= X(x, t, u, f, g)f_x + T(x, t, u, f, g)f_t - F(x, t, u, f, g)\,, \\ \sigma^g &= X(x, t, u, f, g)g_x + T(x, t, u, f, g)g_t - G(x, t, u, f, g)\,, \end{aligned} \tag{22}$$

where $X \equiv X(x, t, u, f, g)$, $T \equiv T(x, t, u, f, g)$, $U \equiv U(x, t, u, f, g)$, $F \equiv F(x, t, u, f, g)$, and $G \equiv G(x, t, u, f, g)$ are functions of the indicated variables.

Substituting Eq. (22) into the symmetry definition (10), (15), and (16) and using the extended system to eliminate nonindependent quantities, u_t, f_x, f_t and g_t, one can get a set of determining equations for X, T, U, F, and G. Solving the determining equation system, we get the following Lie point symmetry theorem.

Theorem 4 (Lie point symmetry theorem of the prolonged system). The prolonged Burgers system $\{(1), (4), (13)\}$ possesses the Lie point symmetries

$$\begin{aligned} \sigma^u &= (c_2tx + c_0t - c_1x + x_0)u_x + (c_2t^2 - 2c_1t + t_0)u_t + (c_2t - c_1)u - cg - c_0 - c_2x\,, \\ \sigma^f &= (c_2tx + c_0t - c_1x + x_0)f_x + (c_2t^2 - 2c_1t + t_0)f_t - \frac{cf^2}{2\nu} - a_1f - a_2\,, \\ \sigma^g &= (c_2tx + c_0t - c_1x + x_0)g_x + (c_2t^2 - 2c_1t + t_0)g_t - \frac{cgf}{\nu} + (c_2t - c_1 - a_1)g, \end{aligned} \tag{23}$$

where c_0, c_1, c_2, x_0, t_0, c, a_1, and a_2 are arbitrary constants.

It is clear that c, a_1, and a_2 related parts in the Lie point symmetry theorem is just the nonlocal symmetry of the single Burgers equation mentioned before. To find symmetry reductions of a nonlinear system means to find the group invariant solutions which is guaranteed by $\sigma^u = \sigma^f = \sigma^g = 0$,

$$\begin{aligned} 0 &= (c_2tx + c_0t - c_1x + x_0)u_x + (c_2t^2 - 2c_1t + t_0)u_t + (c_2t - c_1)u - cg - c_0 - c_2x\,, \\ 0 &= (c_2tx + c_0t - c_1x + x_0)f_x + (c_2t^2 - 2c_1t + t_0)f_t - \frac{cf^2}{2\nu} - a_1f - a_2\,, \\ 0 &= (c_2tx + c_0t - c_1x + x_0)g_x + (c_2t^2 - 2c_1t + t_0)g_t - \frac{cgf}{\nu} + (c_2t - c_1 - a_1)g\,. \end{aligned} \tag{24}$$

To solve the group invariant conditions (24) and the prolonged Burgers system, we can get four symmetry reduction theorems. Here, we just write these reduction theorems without any proofs because one can directly substitute them into the Burgers equation to verify the correctness.

Theorem 5 (Symmetry reduction theorem 1). If $G \equiv G(\xi)$ is a solution of the reduction equation

$$G_{\xi\xi} = \frac{3}{2}\frac{G_\xi^2}{G} + \frac{B^2\sqrt{t_0}G_\xi}{cA^2\nu^2 G} - f_0 G + \frac{2c^2G^3}{t_0B^2A^4} - \frac{\xi^2 G}{2\nu^2A^6} + \frac{t_0B^4}{c^2\nu^4A^4G}\,, \tag{25}$$

then

$$u = \frac{2c\nu A^2\sqrt{t_0}G\tanh[(B/2A\nu)(F + \operatorname{arctanh}((A^2-c_1^2)t + c_1t_0)/At_0)] - tB\xi(A^2-c_1^2) + U}{BA^2\sqrt{t_0}\sqrt{(c_1t-t_0)^2 - A^2t^2}} + \frac{x_0}{t_0} - \frac{x_0c_1^2}{t_0A^2} - \frac{c_0c_1}{A^2} \tag{26}$$

is a solution of the Burgers equation (1), where A, B, c, c_0, c_1, x_0, and t_0 are arbitrary constants, and the functions U and F are related to G by

$$U = Bt_0\Big(\nu A^4\frac{G_\xi}{G} + \frac{B^2A^2\sqrt{t_0}}{2c\nu G} - c_1\xi\Big)\,, \quad F = -\frac{2c\nu}{AB^2\sqrt{t_0}}\int G\,\mathrm{d}\xi\,, \tag{27}$$

with the independent variable ξ,

$$\xi = \frac{A^2(xt_0 - x_0t) + (c_1t - t_0)(x_0c_1 + c_0t_0)}{\sqrt{t_0}\sqrt{(c_1t-t_0)^2 - A^2t^2}}\,. \tag{28}$$

Theorem 6 (Symmetry reduction theorem 2). If $F \equiv F(\eta)$ is a solution of the reduction equation,

$$\begin{aligned} F_{\eta\eta\eta\eta} = {} & \frac{Bt_0\sinh(2F)}{\nu^2}\Big(F_\eta^2 - \frac{F_{\eta\eta\eta}}{4F_\eta} + \frac{F_{\eta\eta}}{2F_\eta^2}\Big) + \frac{B^2t_0^2}{64\nu^2}\sinh(4F) + 4F_\eta F_{\eta\eta} + 4\frac{F_{\eta\eta\eta}F_{\eta\eta}}{F_\eta} \\ & - 3\frac{F_{\eta\eta}^3}{F_\eta^2} - \frac{t_0}{8\nu^2}[(x_0c_1 + c_0t_0)F_\eta + 4B\cosh(2F)F_{\eta\eta}] - \frac{B^2t_0^2F_{\eta\eta}}{128\nu^4F_\eta^2}\big[\cosh(4F) - 1\big]\,, \end{aligned} \tag{29}$$

then

$$\begin{aligned} u &= \frac{4\nu F_\eta}{t_0 - c_1t}\tanh\Big[F + \frac{1}{2}\ln\frac{4t_0B\exp(Bt_0/\nu c_1(c_1t-t_0)) + \nu}{4t_0B\exp(Bt_0/(\nu c_1(c_1t) - t_0)) - \nu}\Big] - \frac{Bt_0\sinh(2F) - 8\nu^2F_{\eta\eta}}{4\nu F_\eta(c_1t-t_0)} + \frac{c_1}{2}\eta + \frac{t_0(x_0c_1 + c_0t_0)}{2c_1(c_1t-t_0)^2}\,, \\ \eta &= \frac{2x}{c_1t - t_0} + \frac{x_0c_1 + c_0(2c_1t - t_0)}{c_1^2(c_1t-t_0)^2}t_0 \end{aligned} \tag{30}$$

is a solution of the Burgers equation (1) with arbitrary constants B, c_0, c_1, t_0, and x_0.

Theorem 7 (Symmetry reduction theorem 3). If $\{U \equiv U(\tau), F \equiv F(\tau)\}$ is a solution of the reduction equation system

$$\nu U_\tau - \frac{1}{2}U^2 + x_0U - c_0\tau - C_1 = 0\,, \quad \tau = x - \frac{c_0}{2}t^2 - x_0t\,, \tag{31}$$

$$\nu F_{\tau\tau} - UF_\tau - BF_\tau^2 + x_0F_\tau - 1 = 0\,, \tag{32}$$

with arbitrary constants c_0, x_0, and B, then

$$u = U + BF_\tau\Big(1 - \tanh\frac{B(t+F)}{2\nu}\Big) + c_0t \tag{33}$$

is a solution of the Burgers equation (1).

Theorem 8 (Symmetry reduction theorem 4). If $F \equiv F(z)$ with $z = x - t/t_0$ is a solution of the reduction equation,

$$\begin{aligned} & 4\nu^2t_0F_zF_{zzz}(B - 4\nu^2F_{zz}) + F_{zz}(B - 2t_0\nu^2F_{zz}) \\ & \quad \times(B - 6\nu^2t_0F_{zz}) - 4\nu^4t_0^2(4F_z^2F_{zz} - F_{zzzz}) = 0\,, \end{aligned} \tag{34}$$

where B and t_0 are arbitrary constants, then

$$u = -2\nu F_z\tanh\Big(\frac{Bt}{2\nu t_0} + F\Big) - \frac{B}{2\nu t_0F_z} + \frac{1}{t_0} + \nu\frac{F_{zz}}{F_z} \tag{35}$$

is a solution of the Burgers equation (1).

5 Examples of Exact Solution

In this section, we solve some of reduction equations to exhibit some exact interaction solutions between different types of Burgers waves.

Example 1 (Soliton-Kummer waves). For the first type of reduction equation (25), its general solution has the form

$$\begin{aligned} G = {} & -\frac{B\sqrt{t_0}\xi}{2cA\nu} + \frac{BA^2\sqrt{t_0}}{2c} \\ & \times\Big(\ln\frac{B_1K_2(\mu_+, 3/2, y) + B_2K_1(\mu_+, 3/2, y)}{C_1K_1(\mu_-, 3/2, -y) + C_2K_2(\mu_-, 3/2, -y)}\Big)_\xi\,, \end{aligned}$$

$$\mu_\pm = \frac{3}{4} - \frac{B}{4A\nu} \pm \frac{1}{4}\nu A^3F_0\,, \quad y \equiv \frac{\xi^2}{2A^3\nu}\,, \tag{36}$$

where C_1, C_2, F_0, B_1, and B_2 are arbitrary constants, $K_1(\mu,\nu,y)$ and $K_2(\mu,\nu,y)$ are the first and second kinds of Kummer functions, which are two linearly independent solutions of the Kummer equation ($K \equiv K(y)$)

$$yK_{yy} + (\nu - y)K_y - \mu K = 0\,. \tag{37}$$

So it is clear that the solution (26) wirh Eqs. (27), (28), and (36) is a soliton-Kummer wave interaction solution.

Example 2 (Soliton-Airy waves). For the third type of reduction equation (31), its general solution can be expressed by Airy wave functions,

$$\begin{aligned} & U = x_0 - 2\nu\,[\ln(C_2\mathrm{Ai}(\tau_1) + \mathrm{C}_1\mathrm{Bi}(\tau_1))]_\tau\,, \\ & \tau_1 = \frac{1}{(4c_0\nu)^{2/3}}(x_0^2 - 2\tau c_0 - C_0)\,, \end{aligned} \tag{38}$$

$$F = \frac{\nu}{B}\Big[\ln\frac{(C_2\mathrm{Ai}(\tau_1) + \mathrm{C}_1\mathrm{Bi}(\tau_1))}{(\mathrm{C}_3\mathrm{Ai}(\tau_2) + \mathrm{C}_4\mathrm{Bi}(\tau_2))}\Big] + f_0\,, \tag{39}$$

where C_1, C_2, C_3, C_4, and f_0 are integral constants,

$\mathrm{Ai} \equiv \mathrm{Ai}(\tau_1)$ and $\mathrm{Bi} \equiv \mathrm{Bi}(\tau_1)$ are Airy Ai and Airy Bi wave functions, which are linearly independent solutions for $w \equiv w(\tau_1)$ in the Airy equation

$$w_{\tau_1\tau_1} - \tau_1 w = 0\,. \tag{40}$$

So, the solution (33) with Eqs. (38) and (39) denotes the soliton-Airy wave interaction solution of the Burgers equation (1).

Example 3 (Soliton-soliton interaction). The fourth reduction equation possesses the general solution

$$F = \frac{1}{2}\ln\frac{\cosh(\sqrt{C_1/2\nu}(z+z_0))}{\cosh(\sqrt{2t_0(C_1\nu t_0 - 2B)/2\nu t_0}(z+z_1))} + f_0\,, \tag{41}$$

with four arbitrary integral constants C_1, z_0, z_1, and f_0. Clearly, the solution (35) with Eq. (41) denotes a soliton-soliton interaction solution.

Unfortunately, for the second type of reduction equation (29), we have not yet found its general solution because of its complexity.

6 Summary and Discussions

In summary, the Burgers equation is studied by the well known effective symmetry reduction method but with nonlocal symmetry. The nonlocal symmetry of the Burgers equation (1) is firstly obtained by using the truncated Painlevé expansion and then is localized via prolongation procedure. After the localization, the Bäcklund transformation of Burgers equation (1) is obtained via the Lie's first principle. For the prolonged Burgers system $\{(1),(5),(13)\}$, the standard Lie point symmetry approach is used to find interactions between solitons and other Burgers waves including the Kummer waves and Airy waves.

The method and the results obtained here can be extended to all the integrable models. The details on the method for other nonlinear systems, other type of interaction solutions such as the solution of Eq. (29) and so on are worthy of further study.

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