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Some Analytical Properties of the Model for Stochastic Evolutionary Games in Finite Populations with Non-uniform Interaction Rate*

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Abstract *Traditional evolutionary games assume uniform interaction rate, which means that the rate at which individuals meet and interact is independent of their strategies. But in some systems, especially biological systems, the players interact with each other discriminately. Taylor and Nowak (2006) were the first to establish the corresponding non-uniform interaction rate model by allowing the interaction rates to depend on strategies. Their model is based on replicator dynamics which assumes an infinite size population. But in reality, the number of individuals in the population is always finite, and there will be some random interference in the individuals' strategy selection process. Therefore, it is more practical to establish the corresponding stochastic evolutionary model in finite populations. In fact, the analysis of evolutionary games in a finite size population is more difficult. Just as Taylor and Nowak said in the outlook section of their paper, "The analysis of non-uniform interaction rates should be extended to stochastic game dynamics of finite populations." In this paper, we are exactly doing this work. We extend Taylor and Nowak's model from infinite to finite case, especially focusing on the influence of non-uniform connection characteristics on the evolutionary stable state of the system. We model the strategy evolutionary process of the population by a continuous ergodic Markov process. Based on the limit distribution of the process, we can give the evolutionary stable state of the system. We make a complete classification of the symmetric 2×2 games. For each case game, the corresponding limit distribution of the Markov-based process is given when noise intensity is small enough. In contrast with most literatures in evolutionary games using the simulation method, all our results obtained are analytical. Especially, in the dominant-case game, coexistence of the two strategies may become evolutionary stable states in our model. This result can be used to explain the emergence of cooperation in the Prisoner is Dilemma Games to some extent. Some specific examples are given to illustrate our results.*

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Key words: stochastic evolutionary games, non-uniform interaction rate, finite population, evolutionary stable state

1 Introduction

Game theory has been widely recognized as an important tool in studying the social, political and economic conflict and cooperation.[1] However, the Nash equilibrium[2] analysis, as the analysis framework for traditional game theory, has many limitations and defects. First, the Nash equilibrium is interpreted as the game player's best response to each other, but the existing mathematical methods are not mature enough to solve it in some game models.[3] Second, the arrival of Nash equilibrium is not only impeded by the rationality degree of game players, but also puzzled by multiple equilibriums.[4–5] Finally, the traditional analysis framework cannot explain the process of choosing and arriving of Nash equilibrium. In contrast, evolutionary game theory, which is a theory frame suitable for bounded rationality, can solve these problems to some extent.

Evolutionary game theory, which is based on the population of individuals, was pioneered by Maynard and Price.[6] Unlike the static analysis methods of rational inference in the traditional game theory, in the evolutionary game theory, a learning and strategy adjustment process of individuals is introduced. Theoretical research of the evolutionary game theory mainly focuses on this learning process or the strategy adjustment dynamics of the population, and then the corresponding stable states of the system. Many mathematical approaches describing this dynamics have been proposed, such as models based on ordinary differential equations,[7–10] partial differential equations,[11–12] stochastic differential equations,[13–15] cellular automata,[16–19] and stochastic processes.[20–25] The applied research of the evolutionary games mainly focuses on exploring the evolution of cooperation.[22,26–32]

Traditional evolutionary games assume uniform interaction rate which means that the rate at which individuals

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meet and interact is independent of their strategies. But in some systems, especially biological systems, the players interact with each other discriminately. As we know, Taylor and Nowak[33] were the first to study evolutionary game dynamics with non-uniform interaction rate. Especially, Tang *et al.*[34] found the relationship between the interaction rate of strategies and the structure coefficient of the population very recently. Structured population is a hot topic in evolutionary games these years and many interesting results have been obtained in the area.[35–40] This makes the research of non-uniform interaction rate model more meaningful. However, Taylor and Nowak's model is based on replicator dynamics which assumes an infinite population. In reality, the number of individuals in the population is always finite, and there will be some random interference in the individuals' strategy selection process. Therefore, it is more practical to establish the stochastic evolutionary game model with non-uniform interaction rate in finite populations. In fact, the analysis of evolutionary games in a finite size population is more difficult. Just as Taylor and Nowak said in the outlook section of their paper, "The analysis of non-uniform interaction rates should be extended to stochastic game dynamics of finite populations.[33]" This is our motivation for this paper.

In this paper, we use the stochastic model of Amir and Berninghaus[21] in 2×2 symmetric games with finite populations. Instead of well-mixed interaction, we use the non-uniform interaction which allows the interaction rates to depend on the strategies. This extension leads to non-linear payoff functions which result in more subtle results and their conclusions are the special case of ours. We make a complete classification of the symmetric 2×2 games and give the corresponding evolutionary stable state in each case. All of the results given are analytical. Especially, in the dominant-case game, coexistence of the two strategies may become evolutionary stable states in our model. To illustrate our results, some examples are given and we plot the corresponding area of each case.

The rest of the paper is organized as follows. In Sec. 2, we describe our model in details. In Sec. 3, we give some analytical results prepared for the model. In Sec. 4, we apply our theory to all cases of symmetric 2×2 games and give the main results of this paper. Some numerical examples to illustrate our results are given in Sec. 5. And the paper is concluded in the last section. The appendix in the end contains all the proof of the theorems in this paper.

2 Model

Consider a finite population with size N playing symmetric 2×2 games. The strategy set of the game is $\{A, B\}$ and the payoff matrix is

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array},$$

which means an A player will obtain a when playing against another A or b when playing against B. Choosing strategy B results in either obtaining c (against A) or d (against B).

Suppose that the probability of interaction between two players is not independent of their strategies. Let r_1, r_2, r_3 denote the reaction rate of two A players, an A player and a B player, two B players respectively ($r_1, r_2, r_3 > 0$). The payoff of each individual is determined by the average payoff over a large number of interactions. In order to simplify the situation to obtain the analytical results, we consider an outside player playing against the population.[21] When the population size is small, there will be some difference between the outsider and insider methods owing to the slight difference of the payoff functions. But when the population size is large enough, the results of the two methods are the same. For such a player, when the number of individuals in the population choosing A is i, the expected payoff of choosing strategy A and B are respectively:

$$\pi_A^i = \frac{ar_1 i + br_2(N-i)}{r_1 i + r_2(N-i)}, \qquad \pi_B^i = \frac{cr_2 i + dr_3(N-i)}{r_2 i + r_3(N-i)}. \tag{1}$$

Introduce stochastic process $z(t)$ that denotes the number of the individuals choosing strategy A at time t. $S = \{0, 1, \ldots, N\}$ is the state space of the process. Players update strategy by their payoff. The three hypotheses: inertia, myopia and mutation to describe the bounded rational behavior in Ref. [21] are used in this paper. Because of inertia, we can suppose that there is no chance that two or more players change their strategies at the same time. Thus we can model this evolutionary learning system as a birth-death process. Let

$$\lambda_i = \varepsilon + \kappa \cdot (\pi_A^i - \pi_B^i)^+, \quad \text{for} \quad i \in S - \{N\}, \qquad \lambda_N = 0, \tag{2}$$

$$\mu_i = \varepsilon + \kappa \cdot (\pi_B^i - \pi_A^i)^+, \quad \text{for} \quad i \in S - \{0\}, \qquad \mu_0 = 0, \tag{3}$$

where ε is a small positive number, $\kappa > 0$ is constant,

$$f^+ = \begin{cases} f, & \text{if } f \geq 0, \\ 0, & \text{if } f < 0. \end{cases}$$

When $z(t) = i$ ($i \in S$), a player switches from strategy B to strategy A with rate λ_i and from strategy A to strategy B with rate μ_i. Thus, given that $\pi_A^i > \pi_B^i$ (in this case $\mu_i = \varepsilon$) and a transfer has occurred, the player has more incentives to shift from strategy B to strategy A; that is, $z(t)$ jumps upward to $i + 1$ with probability

$\lambda_i/(\lambda_i+\varepsilon)$ and also downward to $i-1$ with probability $\varepsilon/(\lambda_i+\varepsilon)$ because of noise or some uncertain factors in the decision making process. Here, ε denotes noise intensity, and κ denotes the speed at which the players react to the environment.

The evolutionary process $z(t)$ is a homogenous Markov chain in continuous time with state space S. For any $t>0$, we have an $(N+1)\times(N+1)$ transition matrix $P(t)=\{p_{ij}(t)\}_{i,j\in S}$. $p_{ij}(t)$ is the probability that $z(t)$ will be in state j after time t given that presently it is in state i.

$$p_{ij}(t)=p\{z(s+t)=j|z(s)=i\},\quad \forall s\geq 0\,. \tag{4}$$

In our model,

$$\begin{aligned}&p_{i,i+1}(t)=\lambda_i t+o(t),\ \text{for } i\in S-\{N\}\,,\\&p_{i,i-1}(t)=\mu_i t+o(t),\ \text{for } i\in S-\{0\}\,,\\&p_{i,i}(t)=1-(\lambda_i+\mu_i)t+o(t),\ \text{for } i\in S\,,\\&p_{i,j}(t)=o(t),\ \text{otherwise}\,.\end{aligned} \tag{5}$$

As $\varepsilon>0$, the Markov chain is ergodic. Based on the results in the birth-death process, for any $i\in S$, $\lim\limits_{t\to+\infty}p_{ij}(t)$ exists and all of them are equal. Let

$$\lim_{t\to+\infty}p_{ij}(t)=v_j^{\varepsilon}=\frac{\xi_j}{\sum_{k=0}^{N}\xi_k},\quad \forall i\in S\,,$$

$$\text{with}\quad \xi_0=1,\quad \xi_k=\frac{\lambda_0\lambda_1\cdots\lambda_{k-1}}{\mu_1\mu_2\cdots\mu_k}\quad (1\leq k\leq N)\,. \tag{6}$$

So $v^{\varepsilon}=(v_0^{\varepsilon},v_1^{\varepsilon},\ldots,v_N^{\varepsilon})$ is the limit distribution of the process when noise intensity is ε.

3 Analytical Properties of the Limit Distribution

3.1 *Classification of Cases*

Without loss of generality, let $r_2=1$ (otherwise, divide the denominator and numerator of π_A^i and π_B^i by r_2, then replace r_1/r_2 by r_1, r_3/r_2 by r_3). Let $x=i/N$, $y=1-x$. Then

$$\begin{aligned}\pi_A(i)&=\frac{ar_1i+b(N-i)}{r_1i+(N-i)}=\frac{ar_1x+by}{r_1x+y}\,,\\ \pi_B(i)&=\frac{ci+dr_3(N-i)}{i+r_3(N-i)}=\frac{cx+dr_3y}{x+r_3y}\,.\end{aligned} \tag{7}$$

Denote

$$\begin{aligned}&\alpha=r_1r_3(a-d)+(b-c),\ \beta=r_1(a-c)\,,\\&\gamma=r_3(b-d)\,,\end{aligned} \tag{8}$$

$$h(x)=(\beta+\gamma-\alpha)x^2+(\alpha-2\gamma)x+\gamma\,. \tag{9}$$

Thus

$$\begin{aligned}\pi_A(i)-\pi_B(i)&=\frac{ar_1x+by}{r_1x+y}-\frac{cx+dr_3y}{x+r_3y}\\&=\frac{h(x)}{(r_1x+y)(x+r_3y)}\,.\end{aligned} \tag{10}$$

As $(r_1x+y)(x+r_3y)>0$ for any values of x,y,r_1,r_3, so

$$\operatorname{sign}(\pi_A^i-\pi_B^i)=\operatorname{sign}\,h(x)\,. \tag{11}$$

The following cases should be distinguished.

Case 1 $\beta+\gamma-\alpha=0$.

In this case, $h(x)$ is a linear function of x. $h(x)=(\alpha-2\gamma)x+\gamma=0\Rightarrow x^*=\gamma/(2\gamma-\alpha)$.

(1-i) $x^*\in[0,1]$, let $i^*=\max\{i|i\in S,i\leq N\cdot x^*\}$.

(1-i-a) $\gamma\leq 0$,

(1-i-b) $\gamma>0$.

(1-ii) $x^*\notin[0,1]$, so $\forall x\in[0,1]$:

(1-ii-a) $h(x)>0$,

(1-ii-b) $h(x)<0$.

(1-iii) $\forall\ x\in\{1/N,2/N,\ldots,1\}$, $h(x)=0$, in this case $\alpha=\beta=\gamma=0$.

Case 2 $\beta+\gamma-\alpha>0$.

In this case, $h(x)$ is a parabola opening upward.

Let $\Delta=(\alpha-2\gamma)^2-4(\beta+\gamma-\alpha)\gamma=\alpha^2-4\beta\gamma$.

(2-i) $\alpha^2-4\beta\gamma\leq 0$. In this case, for any x, $h(x)\geq 0$.

(2-ii) $\alpha^2-4\beta\gamma>0$, the two roots of $h(x)=0$ are

$$\begin{aligned}x_1^*&=\frac{-(\alpha-2\gamma)-\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}\,,\\ x_2^*&=\frac{-(\alpha-2\gamma)+\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}\quad (x_1^*<x_2^*)\,.\end{aligned}$$

(2-ii-a) $x_1^*\leq 0$, $x_2^*\geq 1$. In this case, for any $x\in(0,1)$, $h(x)<0$.

(2-ii-b) $x_1^*\leq 0$, $0<x_2^*<1$. Let $i_2^*=\max\{i|i\in S,i\leq N\cdot x_2^*\}$.

(2-ii-c) $0<x_1^*<1$, $x_2^*\geq 1$. Let $i_1^*=\max\{i|i\in S,i\leq N\cdot x_1^*\}$.

(2-ii-d) $0<x_1^*<1$, $0<x_2^*<1$.

Let $i_1^*=\max\{i|i\in S,i\leq N\cdot x_1^*\}$, $i_2^*=\max\{i|i\in S,i\leq N\cdot x_2^*\}$, $i_1^*<i_2^*$.

Case 3 $\beta+\gamma-\alpha<0$.

In this case, $h(x)$ is a parabola opening downward.

(3-i) $\alpha^2-4\beta\gamma\leq 0$. In this case, for any x, $h(x)\leq 0$.

(3-ii) $\alpha^2-4\beta\gamma>0$, the two roots of $h(x)=0$ are

$$\begin{aligned}x_1^*&=\frac{-(\alpha-2\gamma)+\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}\,,\\ x_2^*&=\frac{-(\alpha-2\gamma)-\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}(x_1^*<x_2^*)\,.\end{aligned}$$

(3-ii-a) $x_1^*\leq 0$, $x_2^*\geq 1$. In this case, for any $x\in(0,1)$, $h(x)>0$.

(3-ii-b) $0<x_1^*<1$, $x_2^*\geq 1$. Let $i_1^*=\max\{i|i\in S,i\leq N\cdot x_1^*\}$.

(3-ii-c) $x_1^*\leq 0$, $0<x_2^*<1$. Let $i_2^*=\max\{i|i\in S,i\leq N\cdot x_2^*\}$.

(3-ii-d) $0<x_1^*<1$, $0<x_2^*<1$.

Let $i_1^*=\max\{i|i\in S,i\leq N\cdot x_1^*\}$, $i_2^*=\max\{i|i\in S,i\leq N\cdot x_2^*\}$, $i_1^*<i_2^*$.

3.2 *Analysis Results of* v^{ε} *When* $\varepsilon\to 0^+$

Let

$$\lim_{\varepsilon\to 0^+}v^{\varepsilon}=(v_0,v_1,\ldots,v_N)=v^*. \tag{12}$$

Lemma 1, 2, 3 give the distribution of v^* for each case listed above. They can be regarded as the preparation before giving the main results. All the proofs are in the appendix.

Lemma 1 In Case 1, that is, $\beta+\gamma-\alpha=0$.

In case (1-i-a), when $i^*<(N-1)/2$, v^* puts probability 1 on state N; when $i^*>(N-1)/2$, v^* puts probability 1 on state 0; when $i^*=(N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with

$$B_N(0)=\frac{a_{(N-1)/2+1}\cdots a_{N-1}}{a_1\cdots a_{(N-1)/2}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le (N-1)/2\,,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i>(N-1)/2\,.\end{cases}$$

In case (1-i-b), v^* puts probability p on state i^* and $1-p$ on state i^*+1, with

$$p=\frac{a_{i^*+1}}{a_{i^*+1}+a_{i^*}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le i^*\,,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i>i^*\,,\end{cases}$$

In case (1-ii-a), v^* puts probability 1 on state N.

In case (1-ii-b), v^* puts probability 1 on state 0.

In case (1-iii), v^* puts the same probability on all the states in the state space $\{0,1,2,\ldots,N\}$.

Lemma 2 In Case 2, that is, $\beta+\gamma-\alpha>0$.

In case (2-i), v^* puts probability 1 on state N.

In case (2-ii-a), v^* puts probability 1 on state 0.

In case (2-ii-b), when $i_2^*<(N-1)/2$, v^* puts probability 1 on state N; when $i_2^*>(N-1)/2$, v^* puts probability 1 on state 0; when $i_2^*=(N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with

$$B_N(0)=\frac{a_{(N-1)/2+1}\cdots a_{N-1}}{a_1\cdots a_{(N-1)/2}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le (N-1)/2\,,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i>(N-1)/2\,.\end{cases}$$

In case (2-ii-c), v^* puts probability p on state i_1^* and $1-p$ on state i_1^*+1, with

$$p=\frac{a_{i_1^*+1}}{a_{i_1^*+1}+a_{i_1^*}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_A(i)-\pi_B(i)), & i\le i_1^* \text{ or } i>i_2^*\,,\\ \kappa\cdot(\pi_B(i)-\pi_A(i)), & i_1^*<i\le i_2^*\,.\end{cases}$$

In case (2-ii-d), when $2i_2^*-i_1^*>N$, v^* puts probability p on state i_1^* and $1-p$ on state i_1^*+1, with

$$p=\frac{a_{i_1^*+1}}{a_{i_1^*+1}+a_{i_1^*}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_A(i)-\pi_B(i)), & i\le i_1^* \text{ or } i>i_2^*\,,\\ \kappa\cdot(\pi_B(i)-\pi_A(i)), & i_1^*<i\le i_2^*\,,\end{cases}$$

when $2i_2^*-i_1^*<N$, v^* puts probability 1 on state N; when $2i_2^*-i_1^*=N$, v^* puts probability p_1 on state i_1^*, p_2 on state i_1^*+1 and $1-p_1-p_2$ on state N, with

$$p_1=\frac{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*}}{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*}+a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*}+a_{i_1^*}\cdot a_{i_2^*+1}\cdots a_{N-1}}\,,$$

$$p_2=\frac{a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*}}{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*}+a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*}+a_{i_1^*}\cdot a_{i_2^*+1}\cdots a_{N-1}}\,.$$

□

Lemma 3 In Case 3, that is, $\beta+\gamma-\alpha<0$.

In case (3-i), v^* puts probability 1 on state 0.

In case (3-ii-a), v^* puts probability 1 on state N.

In case (3-ii-b), when $i_1^*<(N-1)/2$, v^* puts probability 1 on state N; when $i_1^*>(N-1)/2$, v^* puts probability 1 on state 0; when $i_1^*=(N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with

$$B_N(0)=\frac{a_{(N-1)/2+1}\cdots a_{N-1}}{a_1\cdots a_{(N-1)/2}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le (N-1)/2\,,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i>(N-1)/2\,.\end{cases}$$

In case (3-ii-c), v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1, with

$$p=\frac{a_{i_2^*+1}}{a_{i_2^*+1}+a_{i_2^*}}\,,$$

$$a_i=\begin{cases}\kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le i_1^* \text{ or } i>i_2^*\,,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i_1^*<i\le i_2^*\,.\end{cases}$$

In case (3-ii-d), when $2i_1^*+1<i_2^*$, v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1, with $p=a_{i_2^*+1}/(a_{i_2^*+1}+a_{i_2^*})$; when $2i_1^*+1>i_2^*$, v^* puts probability 1 on state 0; when $2i_1^*+1=i_2^*$, v^* puts probability p_1 on state 0, p_2 on state i_2^* and $1-p_1-p_2$ on state i_2^*+1, with

$$p_1=\frac{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1}}{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1}+a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1}+a_{i_1^*+1}\cdots a_{i_2^*}}\,,$$

$$p_2=\frac{a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1}}{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1}+a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1}+a_{i_1^*+1}\cdots a_{i_2^*}}\,.$$

□

4 Main Results for all Cases of Symmetric 2×2 Games

2×2 symmetric games can be classified as dominant-case game, coordinate-case game, and coexistent-case game. The following Theorem gives all the possible distribution of v^* for each case game. The proof of the Theorem is in the appendix.

Theorem

(i) In the dominant-case games,

(a) If strategy A is strictly dominant than strategy B, that is $a > c$, $b > d$, only the following four cases can occur.

Case (1-ii-a): v^* puts probability 1 on state N.

Case (2-i): v^* puts probability 1 on state N.

Case (2-ii-d): when $2i_2^* - i_1^* > N$, v^* puts probability p on state i_1^* and $1-p$ on state i_1^*+1, with

$$p = \frac{a_{i_1^*+1}}{a_{i_1^*+1} + a_{i_1^*}}\,,$$

$$a_i = \begin{cases} \kappa \cdot (\pi_A(i) - \pi_B(i)), & i \le i_1^* \text{ or } i > i_2^*\,, \\ \kappa \cdot (\pi_B(i) - \pi_A(i)), & i_1^* < i \le i_2^*\,, \end{cases}$$

when $2i_2^* - i_1^* < N$, v^* puts probability 1 on state N; when $2i_2^* - i_1^* = N$, v^* puts probability p_1 on state i_1^*, p_2 on state i_1^*+1 and $1-p_1-p_2$ on state N, with

$$p_1 = \frac{a_{i_1^*+1} \cdot a_{i_1^*+2} \cdots a_{i_2^*}}{a_{i_1^*+1} \cdot a_{i_1^*+2} \cdots a_{i_2^*} + a_{i_1^*} \cdot a_{i_1^*+2} \cdots a_{i_2^*} + a_{i_1^*} \cdot a_{i_2^*+1} \cdots a_{N-1}}\,,$$

$$p_2 = \frac{a_{i_1^*} \cdot a_{i_1^*+2} \cdots a_{i_2^*}}{a_{i_1^*+1} \cdot a_{i_1^*+2} \cdots a_{i_2^*} + a_{i_1^*} \cdot a_{i_1^*+2} \cdots a_{i_2^*} + a_{i_1^*} \cdot a_{i_2^*+1} \cdots a_{N-1}}\,.$$

Case (3-ii-a), v^* puts probability 1 on state N.

(b) If strategy B is strictly dominant than strategy A, that is $a < c, b < d$, only the following four cases can occur.

Case (1-ii-b): v^* puts probability 1 on state 0.

Case (2-ii-a): v^* puts probability 1 on state 0.

Case (3-i): v^* puts probability 1 on state 0.

Case (3-ii-d): when $2i_1^* + 1 < i_2^*$, v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1, with $p = a_{i_2^*+1}/(a_{i_2^*+1} + a_{i_2^*})$; when $2i_1^*+1 > i_2^*$, v^* puts probability 1 on state 0; when $2i_1^*+1 = i_2^*$, v^* puts probability p_1 on state 0, p_2 on state i_2^* and $1-p_1-p_2$ on state i_2^*+1, with

$$p_1 = \frac{a_1 \cdots a_{i_1^*} \cdot a_{i_2^*+1}}{a_1 \cdots a_{i_1^*} \cdot a_{i_2^*+1} + a_{i_1^*+1} \cdots a_{i_2^*-1} \cdot a_{i_2^*+1} + a_{i_1^*+1} \cdots a_{i_2^*}}\,,$$

$$p_2 = \frac{a_{i_1^*+1} \cdots a_{i_2^*-1} \cdot a_{i_2^*+1}}{a_1 \cdots a_{i_1^*} \cdot a_{i_2^*+1} + a_{i_1^*+1} \cdots a_{i_2^*-1} \cdot a_{i_2^*+1} + a_{i_1^*+1} \cdots a_{i_2^*}}\,.$$

(ii) In the coordinate-case games, that is $a > c$, $b < d$, only the following three cases can occur.

Case (1-i-a): when $i^* < (N-1)/2$, v^* puts probability 1 on state N; when $i^* > (N-1)/2$, v^* puts probability 1 on state 0; when $i^* = (N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with $B_N(0) = a_{(N-1)/2+1} \cdots a_{N-1}/a_1 \cdots a_{(N-1)/2}$,

$$a_i = \begin{cases} \kappa \cdot (\pi_B(i) - \pi_A(i)), & i \le (N-1)/2, \\ \kappa \cdot (\pi_B(i) - \pi_B(i)), & i > (N-1)/2\,. \end{cases}$$

Case (2-ii-b): when $i_2^* < (N-1)/2$, v^* puts probability 1 on state N; when $i_2^* > (N-1)/2$, v^* puts probability 1 on state 0; when $i_2^* = (N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with $B_N(0) = a_{(N-1)/2+1} \cdots a_{N-1}/a_1 \cdots a_{(N-1)/2}$,

$$a_i = \begin{cases} K \cdot (\pi_B(i) - \pi_A(i)), & i \le (N-1)/2, \\ K \cdot (\pi_B(i) - \pi_B(i)), & i > (N-1)/2\,. \end{cases}$$

Case (3-ii-b): when $i_2^* < (N-1)/2$, v^* puts probability 1 on state N; when $i_1^* > (N-1)/2$, v^* puts probability 1 on state 0; when $i_1^* = (N-1)/2$, v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N, with $B_N(0) = a_{(N-1)/2+1} \cdots a_{N-1}/a_1 \cdots a_{(N-1)/2}$,

$$a_i = \begin{cases} \kappa \cdot (\pi_B(i) - \pi_A(i)), & i \le (N-1)/2\,, \\ \kappa \cdot (\pi_A(i) - \pi_B(i)), & i > (N-1)/2\,. \end{cases}$$

(iii) In the coexistent-case games, that is $a < c$, $b > d$, only the following three cases can occur.

Case (1-i-b): v^* puts probability p on state i^* and $1-p$ on state i^*+1, with

$$p = \frac{a_{i^*+1}}{a_{i^*+1} + a_{i^*}}\,,$$

$$a_i = \begin{cases} \kappa \cdot (\pi_B(i) - \pi_A(i), & i \le i^*\,, \\ \kappa \cdot (\pi_A(i) - \pi_B(i), & i > i^*\,. \end{cases}$$

Case (2-ii-c): v^* puts probability p on state i_1^* and $1-p$ on state i_1^*+1, with

$$p = \frac{a_{i_1^*+1}}{a_{i^*+1} + a_{i_1^*}}\,,$$

$$a_i = \begin{cases} \kappa \cdot (\pi_A(i) - \pi_B(i)), & i \le i_1^* \text{ or } i > i_2^*\,, \\ \kappa \cdot (\pi_B(i) - \pi_A(i)), & i_1^* < i \le i_2^*\,. \end{cases}$$

Case (3-ii-c): v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1, with

$$p = \frac{a_{i_2^*+1}}{a_{i_2^*+1} + a_{i_2^*}}\,,$$

$$a_i = \begin{cases} \kappa \cdot (\pi_B(i) - \pi_A(i)), & i \le i_1^* \text{ or } i > i_2^*\,, \\ \kappa \cdot (\pi_A(i) - \pi_B(i), & i_1^* < i \le i_2^*\,. \end{cases}$$

□

Given the limit distribution of the process, we can get the evolutionary stable state of the system. It is necessary to emphasize that the evolutionary stable state we discussed in this paper is not the same, but more stronger than the evolutionary stable strategy (ESS) proposed by Maynard Smith and Price in 1973.[6] According to Maynard Smith and Price, a strategy is ESS, if it cannot be invaded by a small number of individuals playing a different strategy. In this paper, what we discussed is a state on which the probability does not go to zero when the noise is vanishing. This is equivalent with the "stochastic stable equilibrium (SSE)" proposed by Dean Foster and Peyton Young.[13,41] According to their theory, a state P is an SSE, if in the long run, it is nearly certain that the system lies within every small neighborhood of P as the noise tends slowly to zero. And it can be seen as a refinement of the ESS.

5 Numerical Examples and Discussion

Example 1 Payoff matrix is $\left(\begin{smallmatrix}3 & 0\\ 5 & 1\end{smallmatrix}\right)$, population size $N = 10$.

It is a dominant-case game and strategy B is strictly dominant than strategy A. According to the Theorem, the following three situations can occur: v^* puts probability 1 on state 0; v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1; v^* puts probability p_1 on state 0, p_2 on state i_2^* and $1-p_1-p_2$ on state i_2^*+1. Figure 1 gives the corresponding areas about each situation when $0 < r_1$, $r_3 < 5$.

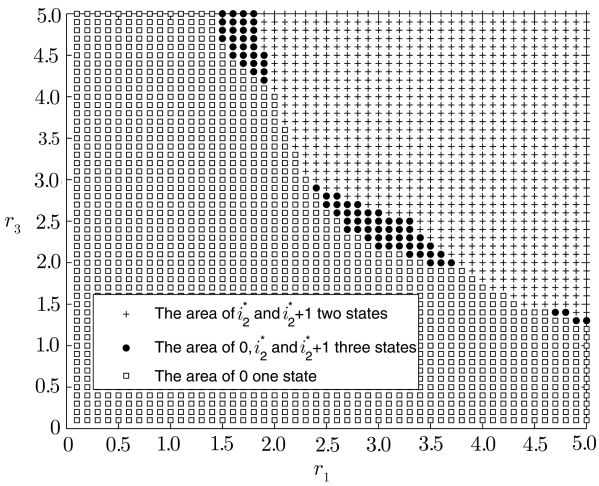

Fig. 1 The corresponding areas of the three situations when $0 < r_1$, $r_3 < 5$ in the B-dominant game.

Example 2 Payoff matrix is $\left(\begin{smallmatrix}5 & 4\\ 0 & 6\end{smallmatrix}\right)$, population size $N = 11$.

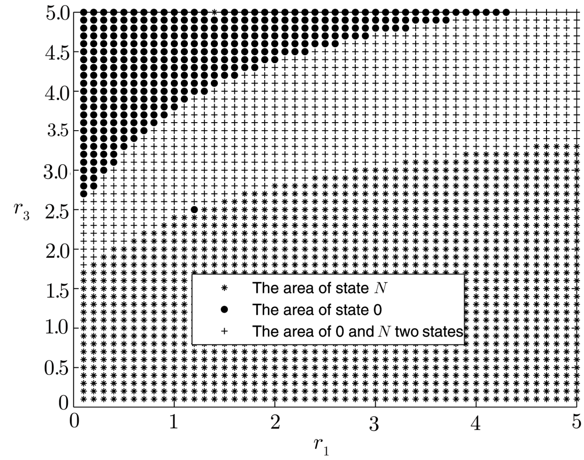

Fig. 2 The corresponding areas of the three situations when $0 < r_1$, $r_3 < 5$ in the coordinate-case game.

It is a coordinate-case game. According to the Theorem, the following three situations can occur: v^* puts probability 1 on state N; v^* puts probability 1 on state 0; v^* puts probability $1/(1 + B_N(0))$ on 0 and $B_N(0)/(1 + B_N(0))$ on N. Figure 2 gives the corresponding areas about each situation when $0 < r_1$, $r_3 < 5$.

Example 3 Payoff matrix is $\left(\begin{smallmatrix}-5 & 3\\ 0 & 1\end{smallmatrix}\right)$, population size $N = 10$.

It is a coexistent-case game. According to the Theorem, the following three situations can occur: v^* puts probability p on state i^* and $1-p$ on state i^*+1; v^* puts probability p on state i_1^* and $1-p$ on state i_1^*+1; v^* puts probability p on state i_2^* and $1-p$ on state i_2^*+1. Figure 3 gives the corresponding areas about each situation when $0 < r_1$, $r_3 < 5$.

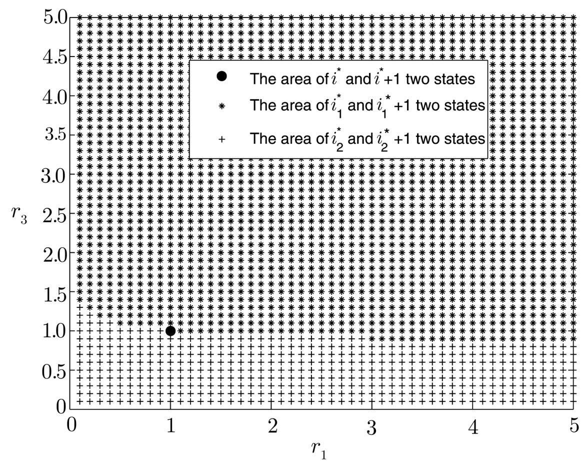

Fig. 3 The corresponding areas of the three situations when $0 < r_1$, $r_3 < 5$ in the coexistent-case game.

6 Conclusions

In summary, the motivation of this paper is based on the idea of non-uniform interaction rate model proposed by Taylor and Nowak in 2006. We have completed the work listed in the outlook part of their paper to extend the analysis of non-uniform interaction rates from deterministic dynamics of infinite populations to stochastic game dynamics of finite populations.

Especially, in the stochastic framework, we also get the following results for symmetric 2×2 games: (i) In the dominant-case games, i.e., strategy A is strictly dominant than strategy B, the non-uniform interactive assumption may change the results of the evolutionary dynamics, and coexistence of the two strategies may become the evolutionary stable state of the system. However, the non-uniform interactive assumption cannot change the results of the invasion dynamics. That is, strategy A can invade strategy B while strategy B cannot invade strategy A. (ii) In the coordinate-case games, the non-uniform interactive assumption cannot change the results of the evolution dynamics. "all A" or "all B" are still the two evolutionary stable states of the system. (iii) In the coexistent-case games, the non-uniform interactive assumption cannot change the results of the evolution dynamics, but only change the specific location of equilibrium point. The coexistence of the two strategies is still the only evolutionary stable state of the system.

These results are the same as that of Taylor and Nowak's in the deterministic case. The main discovery which is different from that of their deterministic framework can be summarized as follows: (i) The equilibrium discussed in our paper is stochastically stable which means that the equilibrium reached is independent of the initial state.[13,41] But in the deterministic framework, this is not the case. More specifically, in the dominant-case games, i.e., strategy A is strictly dominant than strategy B, when in the deterministic framework, if most of the individuals choose strategy A in the initial state, "all individuals choose strategy A" will be the evolutionary stable state of the system; However, if most of the individuals choose strategy B in the initial state, "coexistence of strategy A and B" will be the evolutionary stable state of the system.[33] But in the stochastic framework, the initial state of choosing the two strategies will not affect the final equilibrium at all. (ii) In the stochastic framework, the introduction of a vanishingly small noise term can cause the system to select among the ESS: some of the ESS may be stochastically stable, while others are not. But in our paper, we prove all types of ESS in the deterministic framework can be preserved to be stochastically stable with non-uniform interaction rate. The results we obtained will be much more stronger than those of Taylor and Nowak's. Specifically, in the coordinate-case games, both strategies are the ESS, but only the risk dominant strategy is stochastically stable in the uniform interaction rate case.[13] So the non-uniform interaction rate condition changes the stochastic stable equilibrium in the stochastic framework, but not changes the ESS in the deterministic framework in this coordinate-case games.

Appendix

Proof of Lemma 1

In case (1-i-a), when $i \le i^*$, $\pi_B(i) \ge \pi_A(i)$; when $i > i^*$, $\pi_B(i) < \pi_A(i)$.

Let

$$a_i = \begin{cases} \kappa\cdot(\pi_B(i)-\pi_A(i)), & i\le i^*,\\ \kappa\cdot(\pi_A(i)-\pi_B(i)), & i> i^*,\end{cases}$$

thus

$$\lambda_i = \begin{cases} \varepsilon, & i\le i^*\\ a_i+\varepsilon, & i>i^*\end{cases} \quad \text{and} \quad \mu_i = \begin{cases} a_i+\varepsilon, & i\le i^*\,,\\ \varepsilon, & i>i^*\,.\end{cases}$$

Denote

$$A_j = A_j(\varepsilon) = \frac{1}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} \quad (1\le j\le i^*)\,,$$

$$B_j = B_j(\varepsilon) = \frac{(a_{i^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i^*}+\varepsilon)} \quad (i^*+1<j\le N)\,,$$

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \begin{cases} \frac{\varepsilon^j}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} = \varepsilon^j A_j, & 1\le j\le i^*\,,\\ \frac{\varepsilon^{i^*+1}}{(a_1+\varepsilon)\cdots(a_{i^*}+\varepsilon)\cdot\varepsilon} = \varepsilon^{i^*}A_{i^*}, & j=i^*+1\,,\\ \frac{\varepsilon^{i^*+1}(a_{i^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i^*}+\varepsilon)\varepsilon^{j-i^*}} = \varepsilon^{2i^*-j+1}B_j, & j>i^*+1\,,\end{cases}$$

$$\sum_{k=0}^N \xi_k = [1+\varepsilon A_1+\cdots+\varepsilon^{i^*}A_{i^*}] + [\varepsilon^{i^*}A_{i^*}] + [\varepsilon^{i^*-1}B_{i^*+2}+\cdots+\varepsilon^{2i^*-N+1}B_N]\,,$$

when $i^*<(N-1)/2$, $2i^*-N+1<0$,

$$v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N\xi_k} = \begin{cases} 1, & j=N\,,\\ 0, & \text{otherwise}\,,\end{cases}$$

v^* puts probability 1 on state N;

when $i^*>(N-1)/2$, $2i^*-N+1>0$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} 1, & j=0\,,\\ 0, & \text{otherwise}\,,\end{cases}$

v^* puts probability 1 on state 0;

when $i^* = (N-1/2)$, $2i^*-N+1=0$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} p, & j=0\,, \\ 1-p, & j=N\,, \\ 0, & \text{otherwise}\,, \end{cases}$ with $p = 1/(1+B_N(0))$,

v^* puts probability $1/(1+B_N(0))$ on 0 and $B_N(0)/(1+B_N(0))$ on N.

In case (1-i-b), when $i \le i^*$, $\pi_A(i) \ge \pi_B(i)$; when $i > i^*$, $\pi_A(i) < \pi_B(i)$.

Let $a_i = \begin{cases} \kappa\cdot(\pi_A(i)-\pi_B(i)), & i\le i^*\,, \\ \kappa\cdot(\pi_B(i)-\pi_A(i)), & i > i^*\,, \end{cases}$ thus $\lambda_i = \begin{cases} a_i+\varepsilon, & i\le i^*\,, \\ \varepsilon, & i>i^*\,, \end{cases}$ and $\mu_i = \begin{cases} \varepsilon, & i \le i^*\,, \\ a_i+\varepsilon, & i>i^*\,. \end{cases}$

Denote

$$A_j = A_j(\varepsilon) = (a_0+\varepsilon)\cdots(a_{j-1}+\varepsilon) \quad (1\le j\le i^*)\,,$$

$$B_j = B_j(\varepsilon) = \frac{(a_0+\varepsilon)\cdots(a_{i^*}+\varepsilon)}{(a_{i^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} \quad (i^*+1\le j\le N)\,,$$

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \begin{cases} \frac{(a_0+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{\varepsilon^j} = \frac{A_j}{\varepsilon^j}, & 1\le j\le i^*\,, \\ \frac{(a_0+\varepsilon)\cdots(a_{i^*}+\varepsilon)\varepsilon^{j-(i^*+1)}}{\varepsilon^{i^*}(a_{i^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} = \frac{B_j}{\varepsilon^{2i^*-j+1}}, & j\ge i^*+1\,, \end{cases}$$

$$\sum\nolimits_{k=0}^N \xi_k = \Big[1+\frac{A_1}{\varepsilon}+\cdots+\frac{A_{i^*}}{\varepsilon^{i^*}}\Big] + \Big[\frac{B_{i^*+1}}{\varepsilon^{i^*}}+\cdots+\frac{B_N}{\varepsilon^{2i^*-N+1}}\Big]\,,$$

$$v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N \xi_k} = \begin{cases} p, & j=i^*\,, \\ 1-p, & j=i^*+1\,, \\ 0, & \text{otherwise}\,, \end{cases} \quad \text{with} \quad p = \frac{a_{i^*+1}}{a_{i^*+1}+a_{i^*}}\,.$$

v^* puts probability p on state i^* and $1-p$ on state i^*+1.

In case (1-ii-a), $\pi_A(i) > \pi_B(i)$. Let $a_i = \kappa\cdot(\pi_A(i)-\pi_B(i)) > 0$ for any i, so $\lambda_i = a_i+\varepsilon$ and $\mu_i = \varepsilon$ $(0\le i\le N)$.

Denote $A_j = A_j(\varepsilon) = (a_0+\varepsilon)\cdots(a_{j-1}+\varepsilon)$ $(1\le j\le N)$.

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \frac{(a_0+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{\varepsilon^j} = \frac{A_j}{\varepsilon^j} \quad (1\le j\le N), \quad \xi_0 = 1\,,$$

$$\sum\nolimits_{k=0}^N \xi_k = 1+\frac{A_1}{\varepsilon}+\cdots+\frac{A_N}{\varepsilon^N}, \quad v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N\xi_k} = \begin{cases} 1, & j=N\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

v^* puts probability 1 on state N.

In case (1-ii-b), $\pi_A(i) < \pi_B(i)$. Let $a_i = \kappa\cdot(\pi_B(i)-\pi_A(i)) > 0$ for any i, so $\lambda_i = \varepsilon$ and $\mu_i = a_i+\varepsilon$ $(0\le i\le N)$.

Denote

$$A_j = A_j(\varepsilon) = \frac{1}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} \quad (1\le j\le N)\,,$$

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \frac{\varepsilon^j}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} = \varepsilon^j A_j \quad (1\le j\le N), \quad \xi_0 = 1\,,$$

$$\sum\nolimits_{k=0}^N \xi_k = 1+\varepsilon A_1+\cdots+\varepsilon^N A_N, \quad v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N\xi_k} = \begin{cases} 1, & j=0\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

v^* puts probability 1 on state 0.

In case (1-iii), $\pi_A(i) \equiv \pi_B(i)$ for any i. So $\lambda_i = \mu_i = \varepsilon$, $\xi_j = 1$, $v_j = 1/(N+1)$ $(0\le j\le N)$, v^* puts the same probability on all the states in the state space $\{0,1,2,\ldots,N\}$. □

Proof of Lemma 2

We only give the proof of case (2-ii-d), proof of other cases are similar but simpler.

When $i\le i_1^*$, $\pi_A(i)\ge\pi_B(i)$; when $i_1^* < i \le i_2^*$, $\pi_A(i)\le\pi_B(i)$; when $i > i_2^*$, $\pi_A(i) > \pi_B(i)$.

Let $a_i = \begin{cases} \kappa\cdot(\pi_A(i)-\pi_B(i)), & i\le i_1^* \text{ or } i>i_2^*\,, \\ \kappa\cdot(\pi_B(i)-\pi_A(i)), & i_1^*<i\le i_2^*\,, \end{cases}$ thus $\lambda_i = \begin{cases} a_i+\varepsilon, & i\le i_1^* \text{ or } i>i_2^*\,, \\ \varepsilon, & i_1^*<i\le i_2^*\,, \end{cases}$

and $\mu_i = \begin{cases} \varepsilon, & i\le i_1^* \text{ or } i>i_2^*\,, \\ a_i+\varepsilon, & i_1^*<i\le i_2^*\,. \end{cases}$

Denote $A_j = A_j(\varepsilon) = (a_0+\varepsilon)(a_1+\varepsilon)\cdots(a_{j-1}+\varepsilon)$ $(1\le j\le i_1^*+1)$,

$$B_j = B_j(\varepsilon) = \frac{(a_0+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)}{(a_{i_1^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} \quad (i_1^*+1<j\le i_2^*)\,,$$

$$C_j = C_j(\varepsilon) = \frac{(a_0+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\cdot(a_{i_2^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)} \quad (i_2^*+1 < j \le N)\,.$$

So

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \begin{cases} \frac{(a_0+\varepsilon)(a_1+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{\varepsilon^j} = \frac{A_j}{\varepsilon^j}, & 1 \le j \le i_1^*\,, \\ \frac{(a_0+\varepsilon)(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)} = \frac{A_{i_1^*+1}}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)}, & j = i_1^*+1\,, \\ \frac{(a_0+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{j-(i_1^*+1)}}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} = \frac{B_j}{\varepsilon^{2i_1^*-j+1}}, & i_1^*+1 < j \le i_2^*\,, \\ \frac{(a_0+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{i_2^*-i_1^*}}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)\varepsilon} = \frac{B_{i_2^*}}{\varepsilon^{2i_1^*-i_2^*+1}}, & j = i_2^*+1\,, \\ \frac{(a_0+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{i_2^*-i_1^*}(a_{i_2^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)\varepsilon^{j-i_2^*}} = \frac{C_j}{\varepsilon^{2i_1^*-2i_2^*+j}}, & j > i_2^*+1\,, \end{cases}$$

$$\sum\nolimits_{k=0}^N \xi_k = \Big[1 + \frac{A_1}{\varepsilon} + \cdots + \frac{A_{i_1^*}}{\varepsilon^{i_1^*}}\Big] + \Big[\frac{A_{i_1^*+1}}{\varepsilon^{i_1^*}(a_{i_1^*+1}+\varepsilon)}\Big] + \Big[\frac{B_{i_1^*+2}}{\varepsilon^{i_1^*-1}} + \cdots + \frac{B_{i_2^*}}{\varepsilon^{2i_1^*-i_2^*+1}}\Big] + \Big[\frac{B_{i_2^*}}{\varepsilon^{2i_1^*-i_2^*+1}}\Big] + \Big[\frac{C_{i_2^*+2}}{\varepsilon^{2i_1^*-2i_2^*+2}} + \cdots + \frac{C_N}{\varepsilon^{2i_1^*-2i_2^*+N}}\Big]\,.$$

When $2i_2^* - i_1^* > N$, $i_1^* > 2i_1^* - 2i_2^* + N$,

$$v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N \xi_k} = \begin{cases} p, & j = i_1^*\,, \\ 1-p, & j = i_1^*+1\,, \\ 0, & \text{otherwise}\,. \end{cases} \quad \text{with} \quad p = \frac{a_{i_1^*+1}}{a_{i_1^*+1} + a_{i_1^*}}\,.$$

So $v^* = (v_0, \ldots, v_N)$ puts probability p on state i_1^* and $1-p$ on state i_1^*+1.

When $2i_2^* - i_1^* < N$, $i_1^* < 2i_1^* - 2i_2^* + N$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} 1, & j = N\,, \\ 0, & \text{otherwise}\,. \end{cases}$

So v^* puts probability 1 on state N.

When $2i_2^* - i_1^* = N$, $i_1^* = 2i_1^* - 2i_2^* + N$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} p_1, & j = i_1^*\,, \\ p_2, & j = i_1^*+1\,, \\ 1-p_1-p_2, & j = N\,, \\ 0, & \text{otherwise}\,, \end{cases}$

with

$$p_1 = \frac{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*}}{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*} + a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*} + a_{i_1^*}\cdot a_{i_2^*+1}\cdots a_{N-1}}\,,$$

$$p_2 = \frac{a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*}}{a_{i_1^*+1}\cdot a_{i_1^*+2}\cdots a_{i_2^*} + a_{i_1^*}\cdot a_{i_1^*+2}\cdots a_{i_2^*} + a_{i_1^*}\cdot a_{i_2^*+1}\cdots a_{N-1}}\,.$$

So v^* puts probability p_1 on state i_1^*, p_2 on state i_1^*+1 and $1-p_1-p_2$ on state N. □

Proof of Lemma 3

We only give the proof of case (3-ii-4), proof of other cases are similar but simpler. When $i \le i_1^*$, $\pi_A(i) \le \pi_B(i)$; when $i_1^* < i \le i_2^*$, $\pi_A(i) \ge \pi_B(i)$; when $i > i_2^*$, $\pi_A(i) < \pi_B(i)$. Let

$$a_i = \begin{cases} \kappa\cdot(\pi_B(i) - \pi_A(i)), & i \le i_1^* \text{ or } i > i_2^*\,, \\ \kappa\cdot(\pi_A(i) - \pi_B(i)), & i_1^* < i \le i_2^*\,, \end{cases}$$

thus $\lambda_i = \begin{cases} \varepsilon, & i \le i_1^* \quad \text{or} \quad i > i_2^*\,, \\ a_i+\varepsilon, & i_1^* < i \le i_2^*\,, \end{cases}$ and $\mu_i = \begin{cases} a_i+\varepsilon, & i \le i_1^* \quad \text{or} \quad i > i_2^*\,, \\ \varepsilon, & i_1^* < i \le i_2^*\,. \end{cases}$

Denote

$$A_j = A_j(\varepsilon) = \frac{1}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} \quad (1 \le j \le i_1^*)\,,$$

$$B_j = B_j(\varepsilon) = \frac{(a_{i_1^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)} \quad (i_1^*+1 < j \le i_2^*+1)\,,$$

$$C_j = C_j(\varepsilon) = \frac{(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\cdot(a_{i_2^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} \quad (j > i_2^*+1)\,.$$

So

$$\xi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} = \begin{cases} \frac{\varepsilon^j}{(a_1+\varepsilon)\cdots(a_j+\varepsilon)} = \varepsilon^j A_j, & 1 \le j \le i_1^*\,, \\ \frac{\varepsilon^{i_1^*+1}}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\cdot\varepsilon} = \varepsilon^{i_1^*} A_{i_1^*}, & j = i_1^*+1\,, \\ \frac{\varepsilon^{i_1^*+1}(a_{i_1^*+1}+\varepsilon)\cdots(a_{j-1}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{j-i_1^*}} = \varepsilon^{2i_1^*-j+1} B_j, & i_1^*+1 < j \le i_2^*\,, \\ \frac{\varepsilon^{i_1^*+1}(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{i_2^*-i_1^*}(a_{i_2^*+1}+\varepsilon)} = \varepsilon^{2i_1^*-i_2^*+1}\frac{B_{i_2^*+1}}{a_{i_2^*+1}+\varepsilon}, & j = i_2^*+1\,, \\ \frac{\varepsilon^{i_1^*+1}(a_{i_1^*+1}+\varepsilon)\cdots(a_{i_2^*}+\varepsilon)\varepsilon^{j-i_2^*-1}}{(a_1+\varepsilon)\cdots(a_{i_1^*}+\varepsilon)\varepsilon^{i_2^*-i_1^*}(a_{i_2^*+1}+\varepsilon)\cdots(a_j+\varepsilon)} = \varepsilon^{2i_1^*-2i_2^*+j} C_j, & j > i_2^*+1\,, \end{cases}$$

$$\begin{aligned}\sum\nolimits_{k=0}^{N}\xi_k &= [1+\varepsilon A_1+\cdots+\varepsilon^{i_1^*}A_{i_1^*}] + [\varepsilon^{i_1^*}A_{i_1^*}] + [\varepsilon^{i_1^*-1}B_{i_1^*+2}+\cdots+\varepsilon^{2i_1^*-i_2^*+1}B_{i_2^*}] \\ &\quad + \Big[\varepsilon^{2i_1^*-i_2^*+1}\frac{B_{i_2^*+1}}{a_{i_2^*+1}+\varepsilon}\Big] + [\varepsilon^{2i_1^*-i_2^*+2}C_{i_2^*+2}+\cdots+\varepsilon^{2i_1^*-2i_2^*+N}C_N]\,.\end{aligned}$$

When $2i_1^*+1 < i_2^*$, $2i_1^*-i_2^*+1 < 0$,

$$v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \lim_{\varepsilon\to 0}\frac{\xi_j}{\sum_{k=0}^N \xi_k} = \begin{cases} p, & j = i_2^*\,, \\ 1-p, & j = i_2^*+1\,, \\ 0, & \text{otherwise}\,, \end{cases} \quad \text{with} \quad p = \frac{a_{i_2^*+1}}{a_{i_2^*+1}+a_{i_2^*}}\,.$$

So $v^* = (v_0,\ldots,v_N)$ puts probability p on state i_2^* and $1-p$ on state i_2^*+1.

When $2i_1^*+1 > i_2^*$, $2i_1^*-i_2^*+1 > 0$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} 1, & j = 0\,, \\ 0, & \text{otherwise}\,, \end{cases}$ v^* puts probability 1 on state 0.

When $2i_1^*+1 = i_2^*$, $2i_1^*-i_2^*+1 = 0$, $v_j = \lim_{\varepsilon\to 0} v_j^\varepsilon = \begin{cases} p_1, & j = 0\,, \\ p_2, & j = i_2^*\,, \\ 1-p_1-p_2, & j = i_2^*+1\,, \\ 0, & \text{otherwise}\,, \end{cases}$ with

$$p_1 = \frac{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1}}{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1} + a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1} + a_{i_1^*+1}\cdots a_{i_2^*}}\,,$$

$$p_2 = \frac{a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1}}{a_1\cdots a_{i_1^*}\cdot a_{i_2^*+1} + a_{i_1^*+1}\cdots a_{i_2^*-1}\cdot a_{i_2^*+1} + a_{i_1^*+1}\cdots a_{i_2^*}}\,,$$

v^* puts probability p_1 on state 0, p_2 on state i_2^* and $1-p_1-p_2$ on state i_2^*+1. □

Proof of Theorem

When $\beta+\gamma-\alpha > 0$ and $\alpha^2-4\beta\gamma > 0$,

$$x_1^* = \frac{-(\alpha-2\gamma)-\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}\,,$$

$$x_2^* = \frac{-(\alpha-2\gamma)+\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)} \quad (x_1^* < x_2^*)\,,$$

$x_1^* < 0 \Leftrightarrow \alpha > 2\gamma$ or $\alpha < 2\gamma$, $\quad \gamma < 0$

and $x_2^* > 1 \Leftrightarrow \alpha > 2\beta$ or $\alpha < 2\beta$, $\quad \beta < 0$,

$0 < x_1^*, x_2^* < 1 \Leftrightarrow \alpha < 0, \quad \beta > 0, \quad \gamma > 0$,

$0 < x_1^* < 1, \quad x_2^* > 1 \Leftrightarrow \alpha < 2\gamma, \quad \beta < 0, \quad \gamma > 0$,

$x_1^* < 0, \quad 0 < x_2^* < 1 \Leftrightarrow \alpha < 2\beta, \quad \beta > 0, \quad \gamma < 0$.

When $\beta+\gamma-\alpha < 0$ and $\alpha^2-\beta\gamma > 0$,

$$x_1^* = \frac{-(\alpha-2\gamma)+\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)}\,,$$

$$x_2^* = \frac{-(\alpha-2\gamma)-\sqrt{\alpha^2-4\beta\gamma}}{2(\beta+\gamma-\alpha)} \quad (x_1^* < x_2^*)\,,$$

$x_1^* < 0 \Leftrightarrow \alpha < 2\gamma$ or $\alpha > 2\gamma, \gamma > 0$

and $x_2^* > 1 \Leftrightarrow \alpha < 2\beta$ or $\alpha > 2\beta$, $\beta > 0$,

$0 < x_1^*, x_2^* < 1 \Leftrightarrow \alpha > 0, \quad \beta < 0, \quad \gamma < 0$,

$0 < x_1^* < 1, \quad x_2^* > 1 \Leftrightarrow \alpha > 2\gamma, \quad \beta > 0, \quad \gamma < 0$,

$x_1^* < 0, \quad 0 < x_2^* < 1 \Leftrightarrow \alpha > 2\beta, \quad \beta < 0, \quad \gamma > 0$.

(i) In the dominant-case games,

(a) If strategy A is strictly dominant than strategy B, $\beta > 0$, $\gamma > 0$.

Case 1: $\beta+\gamma = \alpha$, so $\alpha > \gamma > 0$.

$$x^* = \frac{\gamma}{2\gamma-\alpha} > 1 \text{ or } < 0\,,$$

$$h(0) = \gamma > 0, \quad h(1) = \alpha-\gamma > 0\,.$$

Case (1-ii-a) satisfies the condition.

Case 2: $\beta+\gamma > \alpha$.

When $\alpha^2-4\beta\gamma \le 0$, case (2-i) satisfies the condition.

When $\alpha^2-4\beta\gamma > 0$, as $\beta > 0$, $\gamma > 0$, only case (2-ii-d) satisfies the condition.

Case 3: $\beta+\gamma<\alpha$, so $\alpha^2>(\beta+\gamma)^2\geq 4\beta\gamma$.

As $\beta>0$, $\gamma>0$, only case (3-ii-a) satisfies the condition.

(b) If strategy B is strictly dominant than strategy A, $\beta<0$, $\gamma<0$.

Case 1: $\beta+\gamma=\alpha$, so $\alpha<\gamma<0$.

$x^*=\gamma/(2\gamma-\alpha)>1$ or <0. $h(0)=\gamma<0$, $h(1)=\alpha-\gamma<0$.

Case (1-ii-b) satisfies the condition.

Case 2: $\beta+\gamma>\alpha$, so $\alpha<0$, $\alpha^2>(\beta+\gamma)^2\geq 4\beta\gamma$.

As $\beta<0$, $\gamma<0$, only case (2-ii-a) satisfies the condition.

Case 3: $\beta+\gamma<\alpha$.

When $\alpha^2-4\beta\gamma\leq 0$, case (3-i) satisfies the condition.

When $\alpha^2-4\beta\gamma>0$, as $\beta<0$, $\gamma<0$, only case (3-ii-d) satisfies the condition.

(ii) In the coordinate-case games, $\beta>0$, $\gamma<0$, so $\alpha^2-4\beta\gamma>0$.

Case 1: $\beta+\gamma=\alpha$, so $\alpha>\gamma$.

$0<x^*=\gamma/(2\gamma-\alpha)<1, \gamma<0$, case (1-i-a) satisfies the condition.

Case 2: $\beta+\gamma>\alpha$, so $\alpha<\beta$.

As $\beta>0$, $\gamma<0$, only case (2-ii-b) satisfies the condition.

Case 3: $\beta+\gamma<\alpha$.

As $\beta>0$, $\gamma<0$, only case (3-ii-b) satisfies the condition.

(iii) In the coexistent-case games, $\beta<0$, $\gamma>0$, so $\alpha^2-4\beta\gamma>0$.

Case 1: $\beta+\gamma=\alpha$, so $\alpha<\gamma$.

$0<x^*=\gamma/(2\gamma-\alpha)<1$, $\gamma<0$, case (1-i-b) satisfies the condition.

Case 2: $\beta+\gamma>\alpha$. As $\beta<0$, $\gamma>0$, only case (2-ii-c) satisfies the condition.

Case 3: $\beta+\gamma<\alpha$.

As $\beta<0$, $\gamma>0$, only case (3-ii-c) satisfies the condition.

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