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Integrability and Transition Coefficients Related to Jack Polynomials*

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Abstract *Integrability plays a central role in solving many body problems in physics. The explicit construction of the Jack polynomials is an essential ingredient in solving the Calogero–Sutherland model, which is a one-dimensional integrable system. Starting from a special class of the Jack polynomials associated to the hook Young diagram, we find a systematic way in the explicit construction of the transition coefficients in the power-sum basis, which is closely related to a set of mutually commuting operators, i.e. the conserved charges.*

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1 Introduction

Integrability plays a central role in solving many body problem in physics. In particular, one-dimensional integrable systems provide a playground for either toy models or approximations for real physics. In this paper, we shall consider a particular sample 1d N-body integrable system, the Calogero–Sutherland model,[1−3] which is exactly solvable. Its wave functions are composed of the Jack polynomials,[4] which have attracted much attention from the societies of both physics and mathematics[5−7] and found their applications in research areas such as matrix models,[8] fractional quantum Hall effects,[9] refined topological vertex,[10] 2d conformal blocks,[11−12] etc., see also references therein.

The Jack polynomials $\mathrm{J}_\lambda[z_1,\ldots,z_N,\alpha]$ form a basis for the space of symmetric homogeneous polynomials in N variables $\{z_i\}$, $i=1,\ldots,N$. Each $\mathrm{J}_\lambda[z_1,\ldots,z_N,\alpha]$ is related to a partition λ, which are sequences of non-negative integers in weakly decreasing ordering, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $n \equiv l\ (\lambda)$, here $l(\lambda)$ is the number of non-zero parts of the partition λ. In cases without confusion, the parameter α in the Jack function will be omitted safely but implied. In this paper we use the dominance ordering on partition, $\lambda' \leq \lambda \Rightarrow \sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \lambda_i$, $\forall i$. We also use another notation for partition $\lambda = (\{i^{m_i}\})$, $i=1,2,\ldots$, and m_i is the number of times i appears as a part. A partition λ can be represented graphically in the form of Young diagram, and its conjugate λ' corresponds to the transposed Young diagram. All our notation is consistent with Macdonald's convention in [13].

One of the challenges is to find the analytical expressions for the Jack polynomials. An operator formalim has been found in [14–15]. The transition coefficients to the power-sum basis has been investigated in [16–17]. In our previous paper,[18] we found a recursive way in the construction of the Jack functions. The basic ingredients for that construction are the Jack functions for the rectangular Young graphs, which are, however, involved in the calculation of the multiple integrations of the Selberg type, and lack of analytical solutions in general. In [19], we also found a way to expand Jack polynomials in the basis of Schur functions.

Here we approach the problem of the construction of the Jack functions in a different way, which depends on the integrability essentially. Our new strategy is best illustrated by considering the case of the Jack polynomials for the hook Young graphs, which can be parameterized as $(N_1|N_2)$. The generating function for the Jack functions for the hooks can be obtained by considering the following vertex operators[20] in two variables z_1 and z_2,

$$V_{k,-k^{-1}}(z_1,z_2) = \exp\Big[\sum_{n>0} \frac{a_{-n}}{n}(kz_1^n - k^{-1}z_2^n)\Big], \qquad (1)$$

where $z_i = \mathrm{e}^{x_i}$ and $\partial_{x_i} z_i = z_i$. In fact, the above vertex operator is a product of the two others, one for Jacks for the row Young graphs ($l(\lambda)=1$), and the other for the column-type ($l(\lambda')=1$). Let us first consider the generating function for the row-type,

$$\begin{aligned} V_k(z_1) &= \exp\Big[\sum_{n>0} k\frac{a_{-n}}{n} z_1^n\Big] \\ &= \sum_{\sum i m_i = N_1} \prod_{i, m_i>0} \frac{\beta^{m_i} z_1^{i m_i}}{m_i!\, i^{m_i}} \Big(\frac{a_{-i}}{k}\Big)^{m_i} \\ &= \sum_{N_1>0} \frac{\beta^{N_1} z_1^{N_1}}{N_1!} \mathrm{J}_{N_1}\Big(\frac{a_-}{k}\Big). \end{aligned} \qquad (2)$$

Here we identify $\beta = \alpha^{-1} = k^2$.

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In this paper we shall mainly concern ourselves with the Jack polynomials in the power-sum basis. $\mathrm{J}_\lambda[z_1,\ldots,z_N,\alpha] = \mathrm{J}_\lambda(\{p_n\})$, with $p_n \equiv \sum_i z_i^n$. There is a simple rule relating power-sum basis to the bosonic oscillator modes a_n. Here, our vertex operator contains only the creation modes, so we may identify $p_n \sim a_{-n}/k$, and leave the variable z_i as the parameter for the generating function. We can then represent the Jack polynomial for the row type Young graphs in the second quantized form as:

$$\mathrm{J}_\lambda\Big(\Big\{\frac{a_{-n}}{k}\Big\}\Big) = \Big(\frac{a_{-1}}{k}\Big)^n + \cdots$$

and $|\lambda| = n$. The coefficient for the term $(a_{-1}/k)^n$ in Jack functions normalized to 1 is just the standard normalization adopted in [21]. In what follows we shall simplify our notation as

$$\mathrm{J}_\lambda\Big(\frac{a_-}{k}\Big) \equiv \mathrm{J}_\lambda\Big(\Big\{\frac{a_{-n}}{k}\Big\}\Big)\,.$$

The generating function for the Jack polynomial of column-type can be treated in a similar fashion,

$$V_{-k^{-1}}(z_2) = \exp\Big[\sum_{n>0} -k^{-1}\frac{a_{-n}}{n}z_2^n\Big] = \sum_{N_1\geq 0}\frac{(-1)^{N_2}z_2^{N_2}}{N_2!}J_{1^{N_2}}\Big(\frac{a_-}{k}\Big)\,.$$

The generating function for the hook-type Jack polynomial can be obtained by multiplying the above two generating functions together, $V_{k,-k^{-1}}(z_1,\ z_2) = V_k(z_1)V_{-k^{-1}}(z_2)$. The hook-type Jack polynomial is a non-trivial combination of the row-type and column-type Jack polynomials. To see this, consider the Pieri's formula,[21]

$$J_{N_1}J_{1^{N_2}} = \frac{N_1\alpha J_{(N_1-1|N_2)}}{N_1\alpha+N_2} + \frac{N_2 J_{(N_1|N_2-1)}}{N_1\alpha+N_2}\,. \tag{3}$$

Here $J_{(N_1-1|N_2)}$ corresponds to the Young diagram with the partition $(N_1, 1^{N_2})$. Substituting Eq. (3) into our generating function for the hook type, we get

$$V_{k,-k^{-1}}(z_1,z_2) = 1 + \sum_{N_1,N_2\geq 0}\frac{\beta^{N_1}(-1)^{N_2}}{N_1!N_2!}\Big(\frac{z_1^{N_1+1}z_2^{N_2}}{(N_1+1)\alpha+N_2} - \frac{z_1^{N_1}z_2^{N_2+1}}{N_1\alpha+N_2+1}\Big)J_{(N_1|N_2)}\Big(\frac{a_-}{k}\Big)\,.$$

Define

$$\Phi_{(N_1|N_2)}(z_1,z_2) = \frac{z_1^{N_1+1}z_2^{N_2}}{(N_1+1)\alpha+N_2} - \frac{z_1^{N_1}z_2^{N_2+1}}{N_1\alpha+N_2+1}\,. \tag{4}$$

We get

$$\begin{aligned} V_{k,-k^{-1}}(z_1,z_2) - 1 &= \sum_{N_1,N_2\geq 0}\frac{\beta^{N_1}(-1)^{N_2}}{N_1!N_2!}\Phi_{(N_1|N_2)}(z_1,z_2)J_{\{N_1|N_2\}}\Big(\frac{a_-}{k}\Big) \\ &= \frac{1}{\alpha\partial_{x_1}+\partial_{x_2}}(z_1-z_2)\sum_{N_1,N_2\geq 0}\frac{\beta^{N_1}(-1)^{N_2}}{N_1!N_2!}z_1^{N_1}z_2^{N_2}J_{(N_1|N_2)}\Big(\frac{a_-}{k}\Big)\,. \end{aligned} \tag{5}$$

As presented above, $V_{k,-k^{-1}}(z_1,z_2)$ is the generating function of the hook-type Jack polynomials, and $J_{(N_1|N_2)}(a_-/k)$ can be obtained by the orthogonality of the basis function $\Phi_{(N_1|N_2)}(z_1,z_2)$. When normalization is not a concern, we shall introduce the following conjugate of $\Phi_{(N_1|N_2)}(z_1,z_2)$ as

$$\bar{\Phi}_{(N_1|N_2)}(z_1,z_2) = -\int(\alpha\partial_{x_1}+\partial_{x_2})\frac{z_1^{-N_1}z_2^{-N_2}}{z_1-z_2} \times \mathrm{d}x_1\mathrm{d}x_2\,. \tag{6}$$

And they are orthogonal,

$$\int\bar{\Phi}_{(M_1|M_2)}(z_1,z_2)\Phi_{(N_1|N_2)}(z_1,z_2)\mathrm{d}x_1\mathrm{d}x_2 \sim \delta_{N_1,M_1}\delta_{N_2,M_2}\,. \tag{7}$$

Then

$$J_{(N_1|N_2)}\Big(\frac{a_-}{k}\Big) \sim \int\bar{\Phi}_{(N_1|N_2)}(z_1,z_2)V_{k,-k^{-1}}(z_1,z_2) \times \mathrm{d}x_1\mathrm{d}x_2\,. \tag{8}$$

In [21], starting from Pieri's formula, Eq. (3), an analytical expression for the hook-type Jack function has been obtained. However, here our derivation from the vertex operators will lead to a solution, which can not be derived from Stanley's results in a straightforward way, although numerically we have checked that the two results agree. Starting with the eigenfunction $\Phi_{(N_1|N_2)}(z_1,z_2)$, later we shall find all the commuting operators with their eigenvalues identified with the transition coefficients of the Jack function to the power-sum basis. The advantage of the construction of the mutually commuting operators is that the results may be generalized to Jack functions beyond hook type. However, we shall emphasize here that the eigenfunction $\Phi_{(N_1|N_2)}(z_1,z_2)$ presented in this way is not a symmetric function in z_1, z_2, and can not be identified with the Jack function, and the Jack function considered here is composed of the bosonic creators which can be identified with the power-sum basis not related to z_1, z_2.

2 Jack Polynomial for the Hook Young Diagram

In the previous section, we have found a systematic way to get the hook-type Jack polynomial. In this section, we shall calculate the explicit form of the transition coefficients, which turn out to have some geometric interpretation. T he ultimate goal of our study is to identify $P^{\tilde\lambda}_{(N_1|N_2)}$ as the eigenvalues of the mutually commuting operators $P^{\tilde\lambda}$, which we call higher order conserved charges. In general, finding the exact form of the operators $P^{\tilde\lambda}$ is a challenging problem. However, a necessary condition is that those operators $P^{\tilde\lambda}$, acting on the Jack polynomials for the hook Young diagrams, $J_{(N_1|N_2)}(a_-/k)$, shall produce the eigenvalue $P^{\tilde\lambda}_{(N_1|N_2)}$, which has a simple interpretation in terms of $\tilde\lambda$ content cycles. In this paper, we shall concentrate ourselves on the Jack polynomials associated with the hook Young diagrams, verifying by example that such higher order conserved charges exist indeed. In what follows we shall explain what the $\tilde\lambda$ content cycle means. First, for each square s with coordinate (i,j) in the Young diagram, we shall associate it with a content,

$$c(i,j)=(j-1)\alpha-(i-1)\,. \tag{9}$$

Now if we join the two ends of the hook together to form a circle, the contents in the hook Young diagrams are evenly distributed along the circle, see Figs. 1 and 2. In what follows we shall consider the products of n consecutive contents. So it is convenient to introduce the downward n-factorial $(x)_n$ defined as $(x)_n\equiv\prod_{i=0}^{n-1}(x-i)$. We have the following useful relations, which are needed in our further calculations,

$$\sum_{j=1}^{N}(j-1)_m=\frac{(N)_{m+1}}{m+1}\,,$$
$$\sum_{j_2=1}^{j_1-m_1-1}(j_2-1)_{m_2}(j_1-1)_{m_1}=\frac{1}{m_2+1}(j_1-1)_{m_1+m_2+1}\,. \tag{10}$$

Parameterizing λ as $\{i^{m_i}\}$, a λ content cycle is a disconnected product of the i-content-cycles, each with the multiplicity m_i. Each i-content-cycle is associated with the oscillator mode a_{-i-1} and is formed by summing over all the distinct products of i consecutive contents. We can represent i-content-cycle as c^i, and similarly λ content cycle as c^λ. Here, the disconnected production means the consecutive contents from one factor i-content-cycle are separated at least by one square from the adjacent consecutive contents from the other factor j-content-cycle (j can be equal to i if $m_i>1$). For any partition $\lambda=(\{i^{m_i}\})$, $i=1,2,\ldots$, there is a relation $\lambda_1^t=\sum m_i=l(\lambda)$, $y_1=\max\{i|m_i>0\}=l(\lambda^t)$. Depicting the first column of length λ_1^t from the partition λ, we obtain another related partition $\tilde\lambda\equiv\lambda-1^{l(\lambda)}=(\{\tilde i^{m_i}\})=(\{(i-1)^{m_i}\})$, and $|\tilde\lambda|=|\lambda|-l(\lambda)$, $m_1=l(\lambda)-l(\tilde\lambda)=|\lambda|-\sum_{i>1}im_i$. We have found it is more convenient to use the partition $\tilde\lambda$ to label the transition coefficients in the power-sum basis. To see how the transition coefficient can be interpreted this way let us start by considering some special cases.

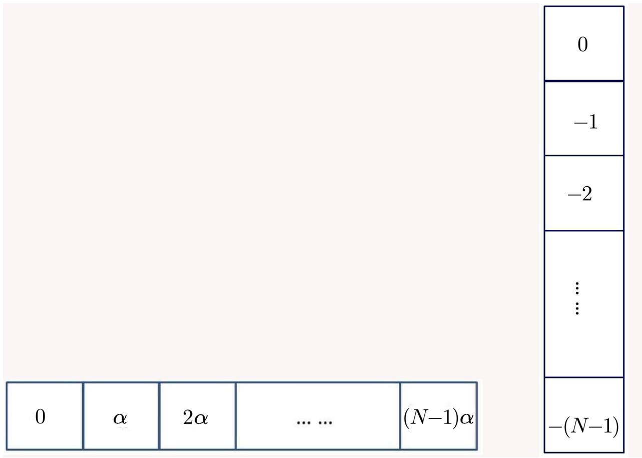

Fig. 1 The contents of the Young diagram λ with $l(\lambda)=1$, $l(\lambda^t)=1$.

2.1 *Row-type Jack Polynomial* $\lambda=\{\lambda_1\}=N$

Let us first consider the two special kind of hooks, the row ($l(\lambda)=1$) and the column ($l(\lambda^t)=1$) Young diagram. The contents of these two cases are illustrated as in Fig. 1. For the row cases, from Eq. (2), it can be read,

$$\mathrm{J}_N\Bigl(\frac{a_-}{k}\Bigr)=\sum_{|\lambda|=N}\Bigl(\prod_{i\ge1}\Bigl(\frac{a_{-i}}{k}\Bigr)^{m_i}\Bigr)P_N^{\tilde\lambda}\,,$$

with

$$P_N^{\tilde\lambda}=(N)_{|\tilde y|+l(\tilde\lambda)}\alpha^{|\tilde\lambda|}\prod\frac{1}{(\tilde i+1)^{m_i}m_i!}\,. \tag{11}$$

The first a few terms look like,

$$\mathrm{J}_N\Bigl(\frac{a_-}{k}\Bigr)=\Bigl(\frac{a_{-1}}{k}\Bigr)^N+\Bigl(\frac{a_{-1}}{k}\Bigr)^{N-2}\Bigl(\frac{a_{-2}}{k}\Bigr)P_N^1+\Bigl(\frac{a_{-1}}{k}\Bigr)^{N-3}\Bigl(\frac{a_{-3}}{k}\Bigr)P_N^2+\Bigl(\frac{a_{-1}}{k}\Bigr)^{N-4}\Bigl(\frac{a_{-4}}{k}\Bigr)P_N^3+\Bigl(\frac{a_{-1}}{k}\Bigr)^{N-4}\Bigl(\frac{a_{-2}}{k}\Bigr)^2P_N^{(1,1)}+\cdots$$

There is a simple interpretation for $P_N^{\tilde y}$ in terms of the content cycles. For example, we have

$$P_N^m=c^m=\alpha^m\sum_{j=1}^{N}(j-1)_m=\alpha^m\frac{(N)_{m+1}}{m+1}\,,$$
$$P_N^{(m_1,m_2)}=c^{m_1,m_2}=\sum_{\{j_1-i|i=0,\ldots,m_1+1\}\cap\{j_2-i|i=1,\ldots,m_2\}=\emptyset}(j_1-1)_{m_1}(j_2-1)_{m_2}\alpha^{m_1+m_2}$$

$$
\begin{aligned}
&= \alpha^{m_1+m_2} \sum_{j_1=m_1+1}^{N} \sum_{j_2=1}^{j_1-m_1-1} (j_2-1)_{m_2}(j_1-1)_{m_1} \\
&\quad + \alpha^{m_1+m_2} \sum_{j_2=m_2+1}^{N} \sum_{j_1=1}^{j_2-m_2-1} (j_1-1)_{m_1}(j_2-1)_{m_2} = \alpha^{m_1+m_2} \frac{(N)_{m_1+m_2+2}}{(m_1+1)(m_2+1)}\,, \\
P_N^{(m^2)} &= \sum_{j_1<j_2,\{j_1-i|i=0,\ldots,m+1\}\cap\{j_2-i|i=1,\ldots,m\}=\emptyset} (j_1-1)_m(j_2-1)_m\alpha^{2m} \\
&= \alpha^{2m} \sum_{j_1=m+1}^{N} \sum_{j_2=1}^{j_1-m-1} (j_2-1)_m(j_1-1)_m = \alpha^{2m} \frac{(N)_{2(m+1)}}{2(m+1)^2}\,.
\end{aligned}
$$

In a similar manner, one can deduce, in general,

$$
P_N^{(\{i^{m_i}\})} = \prod_{i>0,\ \text{disconnected}} (\alpha^i c^i)^{m_i} = (N)_{\sum_{i>0}(i+1)m_i} \prod_{i>0} \frac{\alpha^{im_i}}{m_i!(i+1)^{m_i}} = N!\alpha^{\sum_{i>1} im_i} \prod_{i>0} \frac{1}{m_i! i^{m_i}}\,. \tag{12}
$$

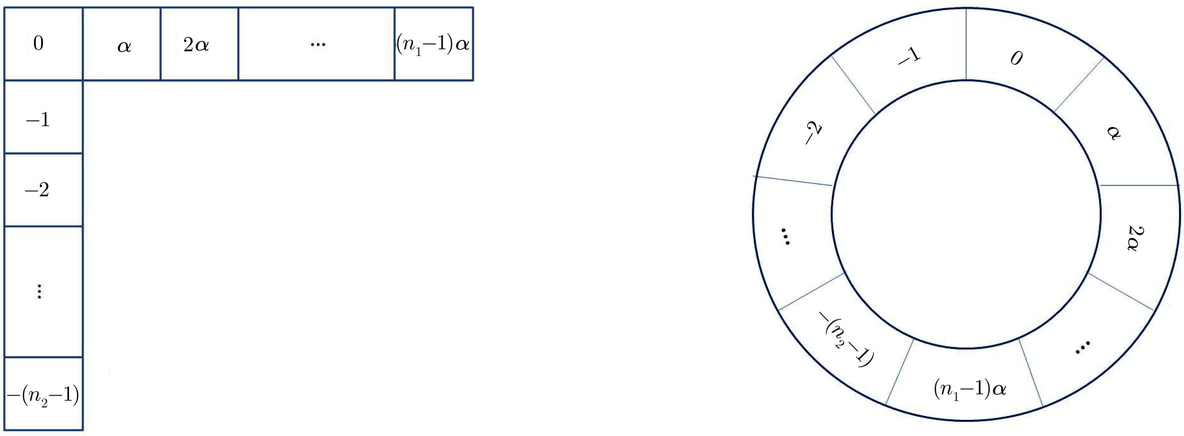

Fig. 2 The contents of the hook Young diagram $\lambda = (n_1 - 1|n_2 - 1)$.

Let us summarize our results. P_N^m is the sum of the product of m consecutive contents related to the row Young diagram of N squares. $P_N^{(m_1,m_2)}$ is the sum of the product of the two consecutive contents, with length m_1 and m_2 respectively and seperated at least by one square. In a more general case, $\tilde{\lambda} = (\{\tilde{i}^{m_{\tilde{i}}}\})$, $\tilde{i} = 1, 2, \ldots$, then $P_N^{\tilde{\lambda}}$ is the sum of the product of $\sum_{\tilde{i}\geq 1} m_{\tilde{i}}$ sequences of consecutive contents, each with length $\tilde{i} = i - 1$ and multiplicity $m_{\tilde{i}}$. Any adjacent sequences must be separated at least by one square in the row type Young diagram of N squares.

For the column-type Jack polynomial $\mathrm{J}_{1^N}(a_-/k)$, the discussion parallels with the replacement $\alpha \to (-1)$.

$$
\mathrm{J}_{1^N}\Big(\frac{a_-}{k}\Big) = \sum_{\sum im_i = N} \Big(\prod_{i\geq 1}\Big(\frac{a_{-i}}{k}\Big)^{m_i}\Big) P_{1^N}^{\tilde{\lambda}}\,, \tag{13}
$$

where

$$
P_{1^N}^{\tilde{\lambda}} = (N)_{|\tilde{\lambda}|+l(\tilde{\lambda})}(-1)^{|\tilde{\lambda}|} \prod_{\tilde{i}>0} \frac{1}{(\tilde{i}+1)^{m_i} m_i!} = (N)!(-1)^{|\lambda|-l(\lambda)} \prod_{i>0} \frac{1}{i^{m_i} m_i!}\,.
$$

2.2 *Jack Polynomial for the Hook Young Diagram*

In this subsection, we shall consider the case of the hook Young diagram. We give a systematic way to construct the hook-type Jack polynomial in the introduction. Now from (8), we obtain,

$$
\begin{aligned}
\mathrm{J}_{(N_1|N_2)}\Big(\frac{a_-}{k}\Big) &= \frac{N_1!N_2!}{\beta^{N_1}(-1)^{N_2}} \int \frac{z_1^{-N_1} z_2^{-N_2}}{z_1 - z_2} (\alpha\partial_{x_1} + \partial_{x_2}) \exp\Big[\sum_{n>0} \frac{a_{-n}}{n}(kz_1^n - k^{-1}z_2^n)\Big] \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \sum_{m_1+n_1=N_1, m_2+n_2=N_2} \frac{a_{-m_1-m_2-1}}{k} \mathrm{J}_{n_1}\Big(\frac{a_-}{k}\Big) \mathrm{J}_{1^{n_2}}\Big(\frac{a_-}{k}\Big) \alpha^{m_1}(-1)^{m_2}(N_1)_{m_1}(N_2)_{m_2} \\
&= \sum \frac{a_{-\lambda}}{k}\Big|_{|\lambda|=N_1+N_2+1} P_{(N_1|N_2)}^{\tilde{\lambda}}\,,
\end{aligned}
$$

where

$$\frac{a_{-\lambda}}{k}\Big|_{|\lambda|=N_1+N_2+1} = \frac{a_{-m_1-m_2-1}}{k}\frac{a_{-\lambda^1}}{k}\Big|_{|\lambda^1|=N_1-m_1}\frac{a_{-\lambda^2}}{k}\Big|_{|\lambda^2|=N_2-m_2},$$
$$P^{\tilde\lambda}_{(N_1|N_2)} = \alpha^{m_1}(-1)^{m_2}(N_1)_{m_1}(N_2)_{m_2}P^{\tilde\lambda^1}_{N_1-m_1}P^{\tilde\lambda^2}_{N_2-m_2}. \tag{14}$$

Joining the two ends of the hook Young diagram together to form a circle, we can generalize the concept of the content cycle introduced previously to the case of the hook Young diagram. By reading the explicit form of $P^{\tilde\lambda}_{(N_1|N_2)}$ in Eq. (14), we can see that $\tilde\lambda$ is partitioned into three parts: $\tilde\lambda^1$ counts those sequences distributed along the row, and $\tilde\lambda^2$ those along the column, and there is one sequence crossing the row and the column with the length m_1+m_2 and its value $\alpha^{m_1}(-1)^{m_2}(N_1)_{m_1}(N_2)_{m_2}$. Now let us give an example of calculating the coefficient $P^X_{(N_1|N_2)}$ for one of the expansion term of $\mathrm{J}_{(N_1|N_2)}(a_-/k)$, with X an integer,

$$\mathrm{J}_{(N_1|N_2)}\Big(\frac{a_-}{k}\Big) = \cdots + \Big(\frac{a_{-1}}{k}\Big)^{N_1+N_2-X}\Big(\frac{a_{-X-1}}{k}\Big)P^X_{(N_1|N_2)} + \cdots \tag{15}$$

It is easy to see that there are three different contributions to this coefficient,

- $m_1+m_2=X, \quad P^{\tilde\lambda^1}_{N_1-m_1} = P^{\tilde\lambda^2}_{N_2-m_2} = 1,$
- $m_1=m_2=0, \quad P^{\tilde\lambda^1}_{N_1-m_1} = P^X_{N_1-m_1}, \quad P^{\tilde\lambda^2}_{N_2-m_2} = 1,$
- $m_1=m_2=0, \quad P^{\tilde\lambda^1}_{N_1-m_1} = 1, \quad P^{\tilde\lambda^2}_{N_2-m_2} = P^X_{N_2}.$

And the final result is,

$$\begin{aligned} P^X_{(N_1|N_2)} &= \sum_{m_1+m_2=X}\alpha^{m_1}(-1)^{m_2}(N_1)_{m_1}(N_2)_{m_2} + P^X_{N_1-m_1} + P^X_{N_2} \\ &= \sum_{j=1}^{i}\alpha^{(i-j)}(N_1)_{i-j}(-1)^j(N_2)_j + \sum_{j=1}^{N_1+1}\alpha^i(j-1)_i + \sum_{j=1}^{N_2+1}(-1)^i(j-1)_i\,. \end{aligned} \tag{16}$$

We can explain the above result in this way. The second and the third term of (16) come from the X-length consecutive content products of the row and the column part of the hook respectively. If we join the two ends of the hook together to form a circle, the first term is just the X-length consecutive content products, which contain the blocks acrossing both edges. In this way, we shall identify $P^{\tilde\lambda}_{(N_1|N_2)}$ with $c^{\tilde\lambda}$.

3 Higher Order of Conserved Quantities

The conserved quantities play a central role for the integrable physical systems. It is important to get all the conserved quantities for the system being integrable. For reasons explained below, we may identify the transition coefficients $P^{\tilde\lambda}_{(N_1|N_2)}$ as the eigenvalues of the mutually commuting operators $\hat P^{\tilde\lambda}$. For $\tilde\lambda=(n)$, $n\in\mathbb{N}$, the corresponding operators $\hat P^n$ constitute a complete set of conserved charges for the Calogero–Sutherland model. The reason is that $\hat P^1$ can be taken as the Hamiltonian, which commutes with all the other $\hat P^n$'s. And $\hat P^{\tilde\lambda}$ can be constructed as polynomials of $\hat P^n$'s. What makes the set of conserved charges $\{\hat P^n\}$ special here is that they only contain the connected summations in terms of the oscillating modes, as will be illustrated in our examples. The construction of the $\hat P^n$'s can be started as the following,

$$\begin{aligned} \hat P^n V_{k,-k^{-1}}(z_1,z_2) &= (\hat P^n J_{(N_1|N_2)})\Phi_{(N_1|N_2)}(z_1,z_2) = J_{(N_1|N_2)}(\tilde P^n\Phi_{(N_1|N_2)}(z_1,z_2)) \\ &= J_{(N_1|N_2)}(P^n_{(N_1|N_2)}\Phi_{(N_1|N_2)}(z_1,z_2))\,. \end{aligned}$$

Here, $\hat P^n$ is the second quantized form of the conserved quantity, and $\tilde P^n$ the associated differential operator. As the differential calculation is easier to perform, the next step is to find a way to obtain $\tilde P^n$. We have the following eigenfunction,

$$\phi_{(N_1|N_2)}(z_1,z_2) = \frac{z_1^{N_1+1}z_2^{N_2}}{(N_1+1)+N_2\beta} - \frac{z_1^{N_1}z_2^{N_2+1}}{N_1+(N_2+1)\beta} = \frac{1}{\partial_{x_1}+\beta\partial_{x_2}}(z_1-z_2)z_1^{N_1}z_2^{N_2}\,.$$

$P^m_{(N_1|N_2)}$ can be read from (16),

$$\begin{aligned} P^m_{(N_1|N_2)} = \frac{\alpha^m}{m+1}\frac{(N_1+1)!}{(N_1-m)!} + \frac{(-1)^m}{m+1}\frac{(N_2+1)!}{(N_2-m)!} + \sum_{i=1}^{m-1}N_1(N_1-1)\cdots \\ (N_1-m+i+1)N_2(N_2-1)\cdots(N_2-i+1)\alpha^{m-i}(-1)^i \equiv \frac{\alpha^m}{m+1}\tilde P^m_{(N_1|N_2)}\,. \end{aligned}$$

Here we identify $\tilde{P}^m_{(N_1|N_2)}$ as the eigenvalue of the differential operator $P^m(\partial_{x_1},\partial_{x_2})$ which satisfies $P^m(\partial_{x_1},\partial_{x_2})z_1^{N_1}z_2^{N_2} = \tilde{P}^m_{(N_1|N_2)}z_1^{N_1}z_2^{N_2}$. Normalizing the coefficient for the term $(\partial_{x_1}+1)_{m+1}$ to 1, we get

$$P^m(\partial_{x_1},\partial_{x_2}) = (\partial_{x_1}+1)_{m+1} + (-\beta)^m(\partial_{x_2}+1)_{m+1} + \sum_{i=1}^{m-1}(\partial_{x_1})_{m-i}(\partial_{x_2})_i(-\beta)^i(m+1)\,.$$

The real conserved charges are the operators $\tilde{f}$, which satisfy $\tilde{f}\phi_{(N_1|N_2)}(z_1,z_2) = f(N_1,N_2)\phi_{(N_1|N_2)}(z_1,z_2)$. Such kind of operators can be obtained by just a similarity transformation,

$$\tilde{f} = \frac{1}{\partial_{x_1}+\beta\partial_{x_2}}(z_1-z_2)f(\partial_{x_1},\partial_{x_2})\frac{1}{z_1-z_2}(\partial_{x_1}+\beta\partial_{x_2})\,. \tag{17}$$

The question is now in which condition the differential operator $\tilde{f}$ remains a polynomial in ∂_{x_1} and ∂_{x_2}, when $f(\partial_{x_1},\partial_{x_2})$ is a polynomial. First, we shall arrange $\tilde{f}$ in (17) into the following form,

$$\tilde{f} = \frac{1}{2}(f(\partial_{x_1}-1,\partial_{x_2}) - f(\partial_{x_1},\partial_{x_2}-1))\frac{1}{\partial_{x_1}+\beta\partial_{x_2}}\frac{z_1+z_2}{z_1-z_2}(\partial_{x_1}+\beta\partial_{x_2}) + \frac{1}{2}(f(\partial_{x_1}-1,\partial_{x_2}) + f(\partial_{x_1},\partial_{x_2}-1))\,.$$

So $\tilde{f}$ remains a polynomial requires

$$\frac{1}{2}(f(\partial_{x_1}-1,\partial_{x_2}) - f(\partial_{x_1},\partial_{x_2}-1)) \propto (\partial_{x_1}+\beta\partial_{x_2})\,.$$

Make the following substitution, $f \Rightarrow P^m(\partial_{x_1},\partial_{x_2})$, $\tilde{f} \Rightarrow \tilde{P}^m(\partial_{x_1},\partial_{x_2})$,

$$\tilde{P}^m = S + \tilde{A}\frac{1}{\partial_{x_1}+\beta\partial_{x_2}}\frac{z_1+z_2}{z_1-z_2}(\partial_{x_1}+\beta\partial_{x_2})\,, \tag{18}$$

with

$$\begin{aligned}S &= \frac{1}{2}(P^m(\partial_{x_1}-1,\partial_{x_2}) + P^m(\partial_{x_1},\partial_{x_2}-1)) = (\partial_{x_1})_{m+1} + (-\beta)^m(\partial_{x_2})_{m+1}\\ &\quad + \frac{1}{2}\sum_{i=0}^{m-1}(\partial_{x_1}-1)_{m-i-1}(\partial_{x_2}-1)_i(-\beta)^i(m+1)(\partial_{x_1}-\beta\partial_{x_2})\,,\\ \tilde{A} &= \frac{1}{2}(P^m(\partial_{x_1}-1,\partial_{x_2}) - P^m(\partial_{x_1},\partial_{x_2}-1))\\ &= \frac{1}{2}\sum_{i=0}^{m-1}(-\partial_{x_1}-\beta\partial_{x_2})(\partial_{x_1}-1)_{m-i-1}(\partial_{x_2}-1)_i(-\beta)^i(m+1)\,.\end{aligned}$$

Here we have used the identity

$$\frac{(x)_{m+1}}{m+1} + (x)_m = \frac{(x+1)_{m+1}}{m+1}\,,$$

and the result shows $\tilde{A} \propto (\partial_{x_1}+\beta\partial_{x_2})$. $\tilde{P}^m$ is really a polynomial just needed.

$$\tilde{P}^m = (\partial_{x_1})_{m+1} + (-\beta)^m(\partial_{x_2})_{m+1} - \sum_{i=0}^{m-1}(\partial_{x_1}-1)_{m-i-1}(\partial_{x_2}-1)_i(-\beta)^i(m+1)\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\,. \tag{19}$$

We list some examples here

$$\begin{aligned}\tilde{P}^0 &= \partial_{x_1}+\partial_{x_2}\,,\\ \tilde{P}^1 &= (\partial_{x_1})_2 - \beta(\partial_{x_2})_2 - 2\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\,,\\ \tilde{P}^2 &= (\partial_{x_1})_3 + \beta^2(\partial_{x_2})_3 - 3[(\partial_{x_1}-1) - \beta(\partial_{x_2}-1)]\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\,,\\ \tilde{P}^3 &= (\partial_{x_1})_4 - \beta^3(\partial_{x_2})_4 - 4[(\partial_{x_1}-1)_2 - \beta(\partial_{x_1}-1)(\partial_{x_2}-1) + \beta^2(\partial_{x_2}-1)_2]\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\,,\\ \tilde{P}^4 &= (\partial_{x_1})_5 + \beta^4(\partial_{x_2})_5 - 5[(\partial_{x_1}-1)_3 - \beta(\partial_{x_1}-1)_2(\partial_{x_2}-1) + \beta^2(\partial_{x_1}-1)(\partial_{x_2}-1)_2\\ &\quad - \beta^3(\partial_{x_2}-1)_3]\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\,.\end{aligned}$$

Next we would like to derive the second quantized form of $\tilde{P}^m$, which can be obtained by acting $\tilde{P}^n$ on the state created by the vertex operator $V_{k,-k^{-1}}(z_1,z_2)$. The advantage is that the second quantized form of $\tilde{P}^m$ are composed of bosonic oscillating modes and can act on the Jack functions for arbitrary Young graphs and the corresponding eigenvalue determines the transition coefficients in power-sum basis. The problem is that (19) is got from the study of the hook-type Jack polynomial, and it only has two free variables. When we consider the second quantized form of $\tilde{P}^m$, some terms, which have zero eigenvalue on hook-type Jack functions, may not appear. But combined with properties such as symmetry and hermiticity, we manage to get the correct second quantized form of $\tilde{P}^m$ for the first a few examples. Define $a_{n_1}^{x_1} = ka_{-n_1}z_1^{n_1}$, $a_{n_1}^{x_2} = -k^{-1}a_{-n_1}z_2^{n_1}$, and making use of the following identity, $\partial_{x_1}z_1^n = z_1^n(\partial_{x_1}+n)$. We get

$$\begin{aligned}(\partial_{x_1})_{m+1} &\sim V_k(z_1)^{-1}(\partial_{x_1})_{m+1}V_k(z_1) = \sum_{n_1>0} a_{-n_1}a_{n_1}^{x}(\partial_{x_1}-1+n_1)_m \\ &= \sum_{n_1>0}\left(a_{-n_1}a_{n_1}^{x}(n_1-1)_m + \sum_{l_1>0} a_{-n_1}a_{n_1}^{x}\binom{m}{l_1}(n_1-1)_{m-l_1}(\partial_{x_1})_{l_1}\right). \end{aligned} \tag{20}$$

Observe that the last term of (20) has the similar expansion as $(\partial_{x_1})_{m+1}$, which means we could do this kind of operations recursively. We give the result directly here without further elaboration

$$\begin{aligned}(\partial_{x_1})_{m+1} \sim \sum_{l_i>0} k^{l-1}\Big(\prod_{i=1}^{l} a_{-n_i}\Big)a^{x}_{\sum_{i=1}^{l} n_i}\binom{m}{l_1}(n_1-1)_{m-l_1}\binom{l_1-1}{l_2}(n_2-1)_{l_1-1-l_2}\cdots \\ \binom{l_{l-2}-1}{l_{l-1}}(n_{l-1}-1)_{l_{l-2}-1-l_{l-1}}(n_l-1)_{l_{l-1}-1}\,. \end{aligned}$$

Some simple examples we used frequently in calculating the quantized form of $\tilde{P}^m$ are listed below

$$\begin{aligned} &\partial_{x_1} \sim a_{-n}a_n^{x_1}\,,\\ &(\partial_{x_1})_2 \sim a_{-n_1}a_{n_1}^{x_1}(n_1-1) + a_{-n_1}a_{-n_2}a_{n_1+n_2}^{x_1}k\,,\\ &(\partial_{x_1})_3 \sim a_{-n_1}a_{n_1}^{x_1}(n_1-1)_2 + a_{-n_1}a_{-n_2}ka_{n_1+n_2}^{x_1}(2(n_1-1)+(n_2-1)) + a_{-n_1}a_{-n_2}a_{-n_3}k^2a_{n_1+n_2+n_3}^{x_1}\,,\\ &(\partial_{x_1})_4 \sim a_{-n_1}a_{n_1}^{x_1}(n_1-1)_3 + a_{-n_1}a_{-n_2}ka_{n_1+n_2}^{x_1}(3(n_1-1)_2+3(n_1-1)(n_2-1)+(n_2-1)_2)\\ &\qquad + a_{-n_1}a_{-n_2}a_{-n_3}k^2a_{n_1+n_2+n_3}^{x_1}(3(n_1-1)+2(n_2-1)+(n_3-1))\\ &\qquad + a_{-n_1}a_{-n_2}a_{-n_3}a_{-n_4}k^3a_{n_1+n_2+n_3+n_4}^{x_1}\,, \end{aligned}$$

along with two useful identities

$$\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\exp\Big[\frac{a_{-n}}{n}(kz_1^n-k^{-1}z_2^n)\Big] = ka_{-n_1-n_2}z_1^{n_1}z_2^{n_2}\exp\Big[\frac{a_{-n}}{n}(kz_1^n-k^{-1}z_2^n)\Big]\,, \tag{21}$$

$$a_n\exp\Big[\frac{a_{-n}}{n}(kz_1^n-k^{-1}z_2^n)\Big] = (kz_1^n-k^{-1}z_2^n)\exp\Big[\frac{a_{-n}}{n}(kz_1^n-k^{-1}z_2^n)\Big]\,. \tag{22}$$

Making use of the above equalities, some second quantized form of $\tilde{P}^m$ are listed below for illustration purpose

$$\tilde{P}^0 = \partial_{x_1}+\partial_{x_2} \sim \sum_{n>0}(a_{-n}a_n^{x_1}+a_{-n}a_n^{x_2}) = \sum_{n>0}a_{-n}a_n\,. \tag{23}$$

$$\begin{aligned}\tilde{P}^1 &= (\partial_{x_1})_2 - \beta(\partial_{x_2})_2 - 2\frac{z_1z_2}{z_1-z_2}(\partial_{z_1}+\beta\partial_{z_2})\\ &\sim \sum_{n_1,n_2>0}(a_{-n_1}(n_1-1)(a_{n_1}^{x_1}-\beta a_{n_1}^{x_2}) + a_{-n_1}a_{-n_2}(ka_{n_1+n_2}^{x_1}+ka_{n_1+n_2}^{x_2}) - 2ka_{-n_1-n_2}z_1^{n_1}z_2^{n_2})\\ &= \sum_{n_1,n_2>0}(a_{-n_1}(n_1-1)(kz_1^{n_1}+kz_2^{n_1}) + a_{-n_1}a_{-n_2}(\beta z_1^{n_1+n_2}-z_2^{n_1+n_2})\\ &\quad + ka_{-n_1-n_2}(kz_1^{n_1}-k^{-1}z_2^{n_1})(kz_1^{n_2}-k^{-1}z_2^{n_2}) - k^3a_{-n_1-n_2}z_1^{n_1+n_2} - k^{-1}a_{-n_1-n_2}z_2^{n_1+n_2})\\ &= \sum_{n_1,n_2>0}(k(a_{-n_1}a_{-n_2}a_{n_1+n_2}+a_{-n_1-n_2}a_{n_1}a_{n_2}) + (1-\beta)(n_1-1)a_{-n_1}a_{n_1})\,. \end{aligned} \tag{24}$$

$\tilde{P}^2$ and $\tilde{P}^3$ are obtained in a similar way,

$$\begin{aligned}\tilde{P}^2 &= \frac{\beta}{4}\sum_{n_1+n_2+n_3+n_4=0} :a_{n_1}a_{n_2}a_{n_3}a_{n_4}: + \sum_{n_i>0}\Big[\frac{3}{2}k(1-\beta)(n_1+n_2-2)(a_{-n_1-n_2}a_{n_1}a_{n_1}\\ &\quad + a_{-n_1}a_{-n_2}a_{n_1+n_2}) - \frac{3}{2}\beta(a_{-n}a_n)^2 + \frac{1}{2}(n-1)_2(-3\beta+2\beta^2+2)a_{-n}a_n\Big]\,, \end{aligned} \tag{25}$$

$$\tilde{P}^3 = \sum_{n_i>0}[k^3(a_{-n_1}a_{-n_2}a_{-n_3}a_{-n_4}a_{n_1+n_2+n_3+n_4} + a_{-n_1-n_2-n_3-n_4}a_{n_1}a_{n_2}a_{n_3}a_{n_4})$$

$$
\begin{aligned}
&+ k^3 a_{-n_1-n_2} a_{-n_3} a_{-n_4} (4 a_{n_1} a_{n_2+n_3+n_4} + 2 a_{n_1+n_3} a_{n_2+n_4}) \\
&+ k^3 (4 a_{-n_1} a_{-n_2-n_3-n_4} + 2 a_{-n_1-n_3} a_{-n_2-n_4}) a_{n_1+n_2} a_{n_3} a_{n_4} \\
&+ 6\beta(1-\beta)(a_{-n_1-n_2} a_{-n_3} a_{-n_4} a_{n_1+n_2+n_3+n_4} + a_{-n_1-n_2-n_3-n_4} a_{n_1+n_2} a_{n_3} a_{n_4}) \\
&+ 8\beta(1-\beta) a_{-n_1-n_2-n_3} a_{-n_4} a_{n_1} a_{n_2+n_3+n_4} + \beta(1-\beta) a_{-n_1-n_2} a_{-n_3-n_4} a_{n_1+n_3} a_{n_2+n_4} \\
&+ 4\beta(1-\beta)(a_{-n_1-n_2-n_3} a_{-n_4} a_{n_1+n_3} a_{n_2+n_4} + a_{-n_1-n_3} a_{-n_2-n_4} a_{n_1+n_2+n_3} a_{n_4}) \\
&+ 4k(2\beta^2 - 3\beta + 2)(a_{-n_1-n_2-n_3} a_{-n_4} a_{n_1+n_2+n_3+n_4} + a_{-n_1-n_2-n_3-n_4} a_{n_1+n_2+n_3} a_{n_4}) \\
&+ k(3\beta^2 - 5\beta + 3)(a_{-n_1-n_2} a_{-n_3-n_4} a_{n_1+n_2+n_3+n_4} + a_{-n_1-n_2-n_3-n_4} a_{n_1+n_2} a_{n_3+n_4}) \\
&+ (-6\beta^3 + 11\beta^2 - 11\beta + 6) a_{-n_1-n_2-n_3-n_4} a_{n_1+n_2+n_3+n_4}] \,.
\end{aligned} \tag{26}
$$

We can see that the quantized form of $\tilde{P}^3$ is already very complicated and we need a more efficient mechanism to generate all the higher $\tilde{P}^m$. And how to read the eigenvalues $\tilde{P}^m_\lambda$ for arbitrary Young diagrams λ from the operator $\tilde{P}^m$ remains another problem. Such kind of problems can be solved to some extent by the other methods, which will not be listed here for lacking of space in introducing them. However, an important lesson learnt from here is that the summation in bosonic modes listed in our sample $\tilde{P}^m$s are all connected sums, by which we mean we cannot separate any two summations into a product of independent ones. For example, in Eq. (25), the term $(\sum_{n>0} a_{-n} a_n)^2$ is an unwanted term, so it must be canceled by the contributions from the other summation terms.

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