Commun. Theor. Phys. **62** (2014) 33–40 Vol. 62, No. 1, July 1, 2014

Direct Proof of Determinant Representation for Scalar Products of the XXZ Gaudin Model with Generic Non-Diagonal Boundary Terms*

WANG Xu (王煦), HAO Kun (郝昆),† and YANG Wen-Li (杨文力)

Institute of Modern Physics, Northwest University, Xi'an 710069, China

(Received December 11, 2013; revised manuscript received March 24, 2014)

Abstract *After constructing the Bethe state of the XXZ Gaudin model with generic non-diagonal boundary terms, we analyze the properties of this state and obtain the determinant representations of the scalar products for this XXZ Gaudin model.*

PACS numbers: 04.20.Jb, 03.65.Fd, 05.30.-d, 75.10.Jm

Key words: Gaudin models, open spin chains, scalar products, partition function

1 Introduction

The Gaudin models introduced in 1976 by Gaudin,[1] describe completely integrable classical and quantum long-range interacting spin chains, playing a distinguished role in many areas of modern physics,[2−3] in establishing the integrability of the Seiberg–Witten theory,[4−5] in constructing the integral representations of the solutions to the Knizhnik–Zamolodchikov (KZ) equation[6−9] and in using as a testing ground for ideas such as the functional Bethe ansatz and general procedure of separation of variables .[10−12] It has been found that they have closely related to the BCS model, which describes the metal superconductivity in the theory of condensed matter physics.[13−15] In particular, the results in [16] obtained by Richardson show that the XXZ Gaudin model has played an essential role in the study of the reduced BCS model.

The main problem in quantum integrable models, after having found the eigenvalue and the eigenfunctions of the system, is calculating correlation functions (vacuum expectation value of some local operators) explicitly.[17] Correlation functions, contained a large number of quantum properties of the system such as quantum correlation and entanglement, have an important significance in the realm of quantum field theory[18] and quantum information field.[19] Breakthrough in this theory will greatly expand the application in the range of integrable model theory and in some of related areas and have important application value. Unfortunately, the analytical expressions (non perturbative calculation) for correlation functions of a quantum integrable system is still few. Because with increasing lattice number N, the commutation relation between the operators will become very complex, this causes the efficient analytical expressions for correlation functions becomes very difficult.[17] In the beginning of 80's, due mainly breakthrough work of Korepin,[20] it was realized that the correlation functions (the scalar product) of XXZ spin with periodic boundary conditions can be expressed in terms of some determinants. In the series of Maillet *et al.*'s work,[21] they made the computation of the correlation functions for XXX and XXZ spin chain simplified dramatically by choosing a set of special base (the so-called F-basis) with helps of the Drinfeld twists, and strictly proved the result of Korepin. By extend Maillet's work to $gl(m|n)$ super algebra related to the spin chain with periodic boundary conditions,[22−23] Yang *et al.* successfully solved the determinant representation for the correlation functions of supersymmetry t-J models (corresponding to graded su(3) spin chain) which has great importance in the condensed physics.[24] Recently, the determinant representation of the correlation functions for various integrable spin chain models have been obtained.[25−27]

As for the Gaudin type models with long-range interaction, the XXX Gaudin model was studied by Sklyanin[28] and Jurčo[29] from the point of view of the quantum inverse scattering method. They obtained the eigenvalue and the eigenfunctions of the model. In 1992 Hikami used the transfer matrix of the periodic chain, calculated and yielded a variety of conserved quantities of Gaudin model (or Gaudin operators).[30] Sklyanin's work[31] leads to systematic studies the integrable model with integrable boundary conditions. In particular, Hikami constructed the energy spectrum (the eigenvalue and eigenfunctions) for the Gaudin model with special open boundary condition which is associated with diagonal K-matrices (which are the solutions to the reflection equations[32]).[7] This method was then used to get the exact solution of the related BCS model,[13] the integrable XXZ Gaudin model with general boundary condition,[33] and the boundary

*Supported by the National Natural Science Foundation of China under Grant Nos. 11075126, 11031005, 11375141, and the State Education Ministry of China under Grant No. 20116101110017 and SRF for ROCS

†Corresponding author, E-mail: haoke72@163.com

integrable Gaudin model associated with higher rank algebraic.[34–35] On the other hand, to obtain correlation functions, we need to calculate the scalar product of Bethe states.[17] In 2002, Zhou *et al.*,[36] by using the relationship between periodic Gaudin model and spin chain, constructed the determinant representation of scalar product of Bethe state with periodic Gaudin model. However, the exact solution for other integrable boundary conditions of the integrable model is very difficult[33,37] even if the energy spectrum have obtained. Recently, based on the determinant representation hypothesis of the scalar product for $S^{1,2}$ (see formula (28)), the author successfully formed the determinant representation of scalar product of Bethe state for Gaudin model with general non-diagonal boundary interactions.[38] In this paper we shall give a direct proof of this determinant representation, giving a complete proof the determinant representations of correlation functions for the XXZ Gaudin model with general non-diagonal boundary interactions.

This paper is organized as follows. In Sec. 2 we provide some preliminaries about the properties of XXZ Gaudin model's Bethe state (the eigenfunctions of Gaudin operators), discuss Bethe Ansatz method in this case. In Sec. 3 we obtain the determinant representations for correlation functions of the model. In Sec. 4, we summarize our results and give some discussions.

2 Eigenstates and the Corresponding Eigenvalues

Let V be a two-dimensional linear space and $\sigma^{\pm}$, σ^z be the Pauli matrices. The well-known six-vertex model R-matrix $R(u) \in \mathrm{End}(V \otimes V)$[17] given by

$$R(u) = \begin{pmatrix} 1 & & & \\ & \frac{\sin u}{\sin(u+\eta)} & \frac{\sin \eta}{\sin(u+\eta)} & \\ & \frac{\sin \eta}{\sin(u+\eta)} & \frac{\sin u}{\sin(u+\eta)} & \\ & & & 1 \end{pmatrix}. \tag{1}$$

Here we assume η is a generic complex number. The R-matrix satisfies the quantum Yang–Baxter equation (QYBE),

$$R_{1,2}(u_1-u_2)R_{1,3}(u_1-u_3)R_{2,3}(u_2-u_3) = R_{2,3}(u_2-u_3)R_{1,3}(u_1-u_3)R_{1,2}(u_1-u_2)\,. \tag{2}$$

Following [31] let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$R_{1,2}(u_1-u_2)K_1^-(u_1)R_{2,1}(u_1+u_2)K_2^-(u_2) = K_2^-(u_2)R_{1,2}(u_1+u_2)K_1^-(u_1)R_{2,1}(u_1-u_2)\,, \tag{3}$$

and the latter satisfies the dual RE

$$R_{1,2}(u_2-u_1)K_1^+(u_1)R_{2,1}(-u_1-u_2-2\eta)K_2^+(u_2) = K_2^+(u_2)R_{1,2}(-u_1-u_2-2\eta)K_1^+(u_1)R_{2,1}(u_2-u_1)\,. \tag{4}$$

Here we consider the K-matrix $K^-(u)$ which is a generic solution to the RE (3) associated the six-vertex model R-matrix[39–40]

$$K^-(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix} \equiv K(u)\,. \tag{5}$$

The coefficient functions are

$$\begin{aligned} k_1^1(u) &= \frac{\cos(\lambda_1-\lambda_2)-\cos(\lambda_1+\lambda_2+2\xi)\,\mathrm{e}^{-2\mathrm{i}u}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}\,, \quad k_2^1(u) = \frac{-\mathrm{i}\sin(2u)\,\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}\,\mathrm{e}^{-\mathrm{i}u}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}\,, \\ k_1^2(u) &= \frac{\mathrm{i}\sin(2u)\,\mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)}\,\mathrm{e}^{-\mathrm{i}u}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}\,, \quad k_2^2(u) = \frac{\cos(\lambda_1-\lambda_2)\,\mathrm{e}^{-2\mathrm{i}u}-\cos(\lambda_1+\lambda_2+2\xi)}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}\,. \end{aligned} \tag{6}$$

At the same time, we introduce the corresponding dual K-matrix $K^+(u)$ which is a generic solution to the dual reflection equation (4) with a particular choice of the free boundary parameters:

$$K^+(u) = \begin{pmatrix} {k^+}_1^1(u) & {k^+}_2^1(u) \\ {k^+}_1^2(u) & {k^+}_2^2(u) \end{pmatrix}\,, \tag{7}$$

with the matrix elements

$$\begin{aligned} {k^+}_1^1(u) &= \frac{\cos(\lambda_1-\lambda_2)\,\mathrm{e}^{-\mathrm{i}\eta}-\cos(\lambda_1+\lambda_2+2\bar{\xi})\,\mathrm{e}^{2\mathrm{i}u+\mathrm{i}\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}\,, \\ {k^+}_2^1(u) &= \frac{\mathrm{i}\sin(2u+2\eta)\,\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)}\,\mathrm{e}^{\mathrm{i}u-\mathrm{i}\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}\,, \\ {k^+}_1^2(u) &= \frac{-\mathrm{i}\sin(2u+2\eta)\,\mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)}\,\mathrm{e}^{\mathrm{i}u+\mathrm{i}\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}\,, \\ {k^+}_2^2(u) &= \frac{\cos(\lambda_1-\lambda_2)\,\mathrm{e}^{2\mathrm{i}u+\mathrm{i}\eta}-\cos(\lambda_1+\lambda_2+2\bar{\xi})\,\mathrm{e}^{-\mathrm{i}\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}\,. \end{aligned} \tag{8}$$

The K-matrices depend on four free boundary parameters $\{\lambda_1,\ \lambda_2,\ \xi,\ \bar{\xi}\}$ which specify integrable boundary conditions.[40] We remark that $K^-(u)$ does not depend on the crossing parameter η but $K^+(u)$ does, and it will become $K^-(u)^{-1}$ as η tends to 0. The parameter $\bar{\xi}$ is required to have the following expansion:

$$\bar{\xi} = \xi + \eta\Delta + \cdots\,, \quad \eta \to 0\,. \tag{9}$$

Let us introduce the generalized XXZ Gaudin operators[1] $\{H_j|j=1,2,\ldots,N\}$ associated with generic boundaries specified by the boundary K-matrices in (5) and (7):

$$H_j=\Gamma_j(z_j)+\sum_{k\neq j}^{2M}\frac{1}{\sin(z_j-z_k)}\Big\{\sigma_k^+\sigma_j^-+\sigma_k^-\sigma_j^++\cos(z_j-z_k)\frac{\sigma_k^z\sigma_j^z-1}{2}\Big\}$$

$$+\sum_{k\neq j}^{2M}\frac{K_j^{-1}(z_j)}{\sin(z_j+z_k)}\Big\{\sigma_j^+\sigma_k^-+\sigma_j^-\sigma_k^++\cos(z_j+z_k)\frac{\sigma_j^z\sigma_k^z-1}{2}\Big\}K_j(z_j)\,, \tag{10}$$

in the Gaudin operator above, $\Gamma_j(u)=(\partial/\partial\eta)\{\bar{K}_j(u)\}|_{\eta=0}K_j(u)$, $j=1,\ldots,N$, with $\bar{K}_j(u)=\mathrm{tr}_0\,\{K_0^+(u)R_{0j}(2u)P_{0j}\}$,

$$\Gamma_j(z_j)=\mathrm{tr}_0\Big\{\frac{\partial K^+(z_j)}{\partial\eta}\Big|_{\eta=0}R_{0j}(2z_j)K^-(z_j)P_{j0}\Big\}+\mathrm{tr}_0\Big\{K^+(z_j)\frac{\partial R_{0j}(2z_j)}{\partial\eta}\Big|_{\eta=0}K^-(z_j)P_{j0}\Big\}\,, \tag{11}$$

in the above expression of $\Gamma_j(z_j)$, the two parts can be respectively expanded as follows,

$$\mathrm{tr}_0\Big\{\frac{\partial K^+(z_j)}{\partial\eta}\Big|_{\eta=0}R_{0j}(2z_j)K^-(z_j)P_{j0}\Big\}=\mathrm{tr}_0\Big\{\frac{\partial K^+(z_j)}{\partial\eta}\Big|_{\eta=0}K^-(z_j)P_{j0}\Big\}=\frac{\partial K_j^+(z_j)}{\partial\eta}\Big|_{\eta=0}K_j^-(z_j)\,, \tag{12}$$

$$\mathrm{tr}_0\Big\{K^+(z_j)\frac{\partial R_{0j}(2z_j)}{\partial\eta}\Big|_{\eta=0}K^-(z_j)P_{j0}\Big\}=\mathrm{tr}_0\Big\{K^+(z_j)\frac{1}{\sin(2z_j)}\Big[\sigma_0^+\sigma_j^-+\sigma_j^-\sigma_0^++\cos(2z_j)\frac{\sigma_j^z\sigma_0^z-1}{2}\Big]K^-(z_j)P_{j0}\Big\}\,, \tag{13}$$

with

$$\frac{\partial R(u)}{\partial\eta}\Big|_{\eta=0}=\begin{pmatrix}0&&&\\&-\frac{\cos u}{\sin u}&\frac{1}{\sin u}&\\&\frac{1}{\sin u}&-\frac{\cos u}{\sin u}&\\&&&0\end{pmatrix}. \tag{14}$$

And $\{z_j\}$ some general complex numbers which are usually called the inhomogeneous parameters. For a generic choice of the boundary parameters $\{\lambda_1,\,\lambda_2,\,\xi,\Delta\}$, $\Gamma_j(u)$ is a non-diagonal matrix, in contrast to that of [7].

A key step in applying the algebraic Bethe ansatz approach is to construct a suitable pseudo-vacuum state, which is simultaneously the common eigenstate of the operators $\mathcal{A}$, $\mathcal{D}$ and is annihilated by the operator $\mathcal{C}$. Compared with the case of the spin-1/2 XXZ open chain with diagonal $K^{\pm}(u)$,[31] for the open chain with generic non-diagonal K-matrices,[33] the usually highest-weight state $\binom{1}{0}\otimes\cdots\otimes\binom{1}{0}$ is no longer the pseudo-vacuum state.

The relation[38] $(H_j=(\partial/\partial\eta)\theta(z_j)|_{\eta=0})$ between $\{H_j\}$ and $\{\theta(z_j)\}$ and the fact that the first term of the double-row transfer matrix expanding form[33,38] $(\theta(z_j)=\mathrm{id}+\eta H_j+O(\eta^2),\ j=1,\ldots,N)$ is the identity operator enable us to extract the eigenstates of the Gaudin operators and the corresponding eigenvalues from those of the XXZ chain.

It has been shown[38] that the Gaudin operators given by (10) can be obtained by expanding the transfer matrix of the XXZ spin chain with boundary terms specified by the K-matrices in (5) and (7) in terms of parameter η as $\eta\to 0$. Such a relation enables us to extract the eigenstates of the Gaudin operators and the corresponding eigenvalues from those of the XXZ chain obtained in [33, 37, 41–48] as follows. For this purpose, let us introduce the states $|\Omega^{(1)}\rangle$ and $|\Omega^{(2)}\rangle$,

$$|\Omega^{(1)}\rangle=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}(z_1+2\lambda_1)}\\1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}\mathrm{e}^{-i(z_N+2\lambda_1)}\\1\end{pmatrix}, \tag{15}$$

$$|\Omega^{(2)}\rangle=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}(z_1+2\lambda_2)}\\1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}\mathrm{e}^{-\mathrm{i}(z_N+2\lambda_2)}\\1\end{pmatrix}, \tag{16}$$

and a matrix $g(u)\in\mathrm{End}(V)$ and an associated gauged Pauli operator $\sigma^{\pm}(u)\in\mathrm{End}(V)$

$$g(u)=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}(u+2\lambda_1)}&\mathrm{e}^{-\mathrm{i}(u+2\lambda_2)}\\1&1\end{pmatrix}, \tag{17}$$

$$\sigma^{\pm}(u)=g(u)\sigma^{\pm}g(u)^{-1}\,. \tag{18}$$

Then we define the states,

$$|\{v_i^{(1)}\}\rangle^{(1)}=\prod_{i=1}^{M}B(v_i^{(1)})|\Omega^{(1)}\rangle\,, \tag{19}$$

$$|\{v_i^{(2)}\}\rangle^{(2)}=\prod_{i=1}^{M}C(v_i^{(2)})|\Omega^{(2)}\rangle\,. \tag{20}$$

The associated operators $B(u)$ and $C(u)$ are

$$B(u)=\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi+z_i)\sin(\lambda_2+\xi-z_i)\sin(2u)}{\sin(\lambda_1+\xi-u)\sin(\lambda_2+\xi-u)\sin(u-z_i)\sin(u+z_i)}\sigma^-(z_i)\,, \tag{21}$$

$$C(u)=\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2u)}{\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)\sin(u-z_i)\sin(u+z_i)}\sigma^+(z_i)\,. \tag{22}$$

Using the same procedure as in [7], we can show that the above states (19) and (20) are the common eigenstates of the Gaudin operators $\{H_j\}$, parameters $\{v_i^{(j)}\}$ satisfy the following two sets of Bethe ansatz equations

$$\frac{1-\Delta}{\sin(\lambda_1+\xi+v_\alpha^{(1)})\sin(\lambda_1+\xi-v_\alpha^{(1)})}+\frac{1+\Delta}{\sin(\lambda_2+\xi+v_\alpha^{(1)})\sin(\lambda_2+\xi-v_\alpha^{(1)})}$$
$$=\sum_{k\neq\alpha}^{M}\frac{2}{\sin(v_\alpha^{(1)}-v_k^{(1)})\sin(v_\alpha^{(1)}+v_k^{(1)})}-\sum_{k=1}^{2M}\frac{1}{\sin(v_\alpha^{(1)}-z_k)\sin(v_\alpha^{(1)}+z_k)},\quad \alpha=1,\ldots,M\,, \tag{23}$$

or

$$\frac{1+\Delta}{\sin(\lambda_1+\xi+v_\alpha^{(2)})\sin(\lambda_1+\xi-v_\alpha^{(2)})}+\frac{1-\Delta}{\sin(\lambda_2+\xi+v_\alpha^{(2)})\sin(\lambda_2+\xi-v_\alpha^{(2)})}$$
$$=\sum_{k\neq\alpha}^{M}\frac{2}{\sin(v_\alpha^{(2)}-v_k^{(2)})\sin(v_\alpha^{(2)}+v_k^{(2)})}-\sum_{k=1}^{2M}\frac{1}{\sin(v_\alpha^{(2)}-z_k)\sin(v_\alpha^{(2)}+z_k)},\quad \alpha=1,\ldots,M\,. \tag{24}$$

Here Δ is the parameter of first order expansion of $\bar{\xi}$ in terms of η.

Namely,

$$H_j\,|\{v_\alpha^{(i)}\}\rangle^{(i)}=E_j^{(i)}|\{v_\alpha^{(i)}\}\rangle^{(i)},\quad i=1,2\,, \tag{25}$$

where the corresponding two sets of eigenvalues $E_j^{(i)}$ are given respectively by

$$E_j^{(1)}=\cot 2z_j+\sum_{j=1}^{2}\cot(\lambda_j+\xi-z_j)-\frac{\Delta\sin(2z_j)}{\sin(\lambda_1+\xi-z_j)\sin(\lambda_1+\xi+z_j)}+\sum_{k=1}^{M}\frac{\sin(2z_j)}{\sin(v_k^{(1)}-z_j)\sin(v_k^{(1)}+z_j)}\,, \tag{26}$$

$$E_j^{(2)}=\cot 2z_j+\sum_{j=1}^{2}\cot(\lambda_j+\xi-z_j)-\frac{\Delta\sin(2z_j)}{\sin(\lambda_2+\xi-z_j)\sin(\lambda_2+\xi+z_j)}+\sum_{k=1}^{M}\frac{\sin(2z_j)}{\sin(v_k^{(2)}-z_j)\sin(v_k^{(2)}+z_j)}\,. \tag{27}$$

3 Proof of Determinant Representation of Scalar Product $S^{1,2}$

We introduce two kinds of scalar product of Gaudin model: $S^{1,2}(\{u_\alpha\};\{v_i\})$ and $S^{2,1}(\{u_\alpha\};\{v_i\})$ (another two kinds are of the form $S^{1,1}(\{u_\alpha\};\{v_i^{(1)}\})$ and $S^{2,2}(\{u_\alpha\};\{v_i^{(2)}\})$). And in fact, the calculations of $S^{1,2}(\{u_\alpha\};\{v_i\})$ and $S^{2,1}(\{u_\alpha\};\{v_i\})$ (this paper) are the basis of calculating procedure [27,38] of the scalar products $S^{1,1}(\{u_\alpha\};\{v_i^{(1)}\})$ and $S^{2,2}(\{u_\alpha\};\{v_i^{(2)}\})$. To obtain correlation functions, it suffices to calculate the scalar products of on-shell Bethe states with general off-shell Bethe states[17] (see also [25–26] for the open XXZ chain with diagonal boundaries). The aim of this paper is to give the explicit proof procedure to induct the determinant form of the following scalar products of the XXZ Gaudin model with non-diagonal boundary terms. We remark that they have quite similar algebraic structures compared with the partition functions of N (here $N=2M$) lattice sites with domain wall boundary conditions.[27]

We introduce:

$$Z_N^{(1)}(\{\bar{u}_J\})\equiv S^{1,2}(\{u_\alpha\};\{v_i\})=\langle\Uparrow|\tilde{C}(u_1)\cdots\tilde{C}(u_M)\,\tilde{C}(v_1)\cdots\tilde{C}(v_M)|\Downarrow\rangle\,, \tag{28}$$

$$Z_N^{(2)}(\{\bar{u}_J\})\equiv S^{2,1}(\{u_\alpha\};\{v_i\})=\langle\Downarrow|\tilde{B}(u_1)\cdots\tilde{B}(u_M)\,\tilde{B}(v_1)\cdots\tilde{B}(v_M)|\Uparrow\rangle\,, \tag{29}$$

here the states $|\Uparrow\rangle$ and $\langle\Uparrow|$ (resp. $|\Downarrow\rangle$ and $\langle\Downarrow|$) are the all spin-up state and its dual (resp. all spin-down and its dual), where N free parameters $\{\bar{u}_J|J=1,\ldots,N\}$ are defined by:

$$\bar{u}_i=u_i\quad(i=1,\ldots,M)\ \text{and}\ \bar{u}_{M+i}=v_i\quad(i=1,\ldots,M)\,. \tag{30}$$

In the XXZ spin-chain case, the quantum monodromy operator is a 2×2 matrix with entries $A,\ B,\ C,\ D$ which are obtained as sums of 2^{N-1} operators which themselves are products of N local operators on the quantum chain. As an example, the B operator is given as

$$B_{1\cdots N}(\mu)=\sum_{i=1}^{N}\sigma_i^-\Omega_i+\sum_{i\neq j\neq k}\sigma_i^-(\sigma_j^-\sigma_k^+)\ \Omega_{ijk}+\text{higher terms}\,, \tag{31}$$

where σ^+ and σ^-, are the standard Pauli matrices and the matrices Ω_i, Ω_{ijk}, are diagonal operators acting respectively on all sites but i, on all sites but i,j,k, and the higher order terms involve more and more exchange spin terms like $\sigma_j^-\sigma_k^+$. It means that the B operator returns one spin somewhere on the chain, this operation being however dressed non-locally and with non-diagonal operators by multiple exchange terms of the type $\sigma_j^-\sigma_k^+$.

Under F-basis[21] and after a gauge (face-vertex) transformation, the associated operators $\tilde{B}(u)$ and $\tilde{C}(u)$ simultaneously become diagonalized at most sites.

$$\tilde{B}(u)=\tilde{\mathcal{T}}_F^{+}(\lambda|u)_2^1=\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}\sum_{i=1}^{N}\frac{\sin(\lambda_2+\xi-z_i)\sin(\lambda_1+\xi+z_i)\sin(2u+2\eta)\sin\eta}{\sin(\lambda_1+\xi-u-\eta)\sin(\lambda_2+\xi-u-\eta)\sin(u+z_i)\sin(u-z_i+\eta)}$$
$$\times E_{21}^i\otimes_{j\neq i}\begin{pmatrix}\frac{\sin(u-z_j)\sin(u+z_j+\eta)}{\sin(u-z_j+\eta)\sin(u+z_j)} & \\ & \frac{\sin(z_j-z_i+\eta)}{\sin(z_j-z_i)}\end{pmatrix}_{(j)},\tag{32}$$

$$\tilde{C}(u)=\tilde{\mathcal{T}}_F^{-}(\lambda|u)_1^2=\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin 2u\sin\eta}{\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)\sin(u-z_i+\eta)\sin(u+z_i)}$$
$$\times E_{12}^i\otimes_{j\neq i}\begin{pmatrix}\frac{\sin(u-z_j)\sin(u+z_j+\eta)\sin(z_i-z_j+\eta)}{\sin(u-z_j+\eta)\sin(u+z_j)\sin(z_i-z_j)} & \\ & 1\end{pmatrix}_{(j)}.\tag{33}$$

Here, in Gaudin case, we make the parameter take the limit $\eta\to 0$, the operators can be reduced to the following form:

$$\tilde{B}(u)=\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi+z_i)\sin(\lambda_2+\xi-z_i)\sin(2u)}{\sin(\lambda_1+\xi-u)\sin(\lambda_2+\xi-u)\sin(u-z_i)\sin(u+z_i)}\times\sigma_i^{-},\tag{34}$$

$$\tilde{C}(u)=\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2u)}{\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)\sin(u-z_i)\sin(u+z_i)}\times\sigma_i^{+}.\tag{35}$$

Then $\tilde{B}(u)$ and $\tilde{C}(u)$ become the operators only depend on lattice position. Now we come to the concrete proof procedure of the determinant representation of scalar product $S^{1,2}(\{u_\alpha\};\{v_i\})$. Firstly, insert in the scalar product a complete set of states $|j_1,\ldots,j_N\rangle$ beyond each operator $\tilde{C}$,

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\})=\sum_{i=1}^{N}\langle j_1,\ldots,j_N|\tilde{C}(v_N)|j_1,j_2,\ldots,\downarrow_i,\ldots,j_N\rangle\times\langle j_1,j_2,\ldots,\downarrow_i,\ldots,j_N|\tilde{C}(v_{N-1})\cdots\tilde{C}(v_1)|\Downarrow\rangle,\tag{36}$$

where we denote by the state $|j_1,j_2,\ldots,\downarrow_i,\ldots,j_N\rangle$ with one spin down in the site i and with $N-1$ spins up in the other sites. We make the summation over all terms in the complete set, taking into account the property of operater $\tilde{C}$, and leaving only one non-vanish for each. We are thus led to considering the intermediate functions

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\})=\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)\sin(v_N-z_i)\sin(v_N+z_i)}Z_{N-1}^{(1)}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i}).\tag{37}$$

This suggests that for any positive integer site N, the scalar product $Z_N^{(1)}(\{v_\alpha\};\{z_i\})$ can be uniquely determined by the initial condition $Z_0^{(1)}(\{v_\alpha\};\{z_i\})=1$ and the recurrence relation (37) completely.

Now suppose a series of function $\{K_I(\{v_\alpha\};\{z_i\})|I=1,\ldots,N\}$ possess the following form:

$$K_N(\{v_\alpha\};\{z_i\})=\frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)}{\prod_{\alpha>\beta}\sin(v_\alpha-v_\beta)\sin(v_\alpha+v_\beta)\prod_{k<j}\sin(z_k-z_j)\sin(z_k+z_j)}$$
$$\times\det\Big|\frac{\sin(\lambda_1+\xi-z_j)\sin(\lambda_2+\xi+z_j)\sin(2v_\alpha)}{\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_2+\xi+v_\alpha)\sin^2(v_\alpha-z_j)\sin^2(v_\alpha+z_j)}\Big|.\tag{38}$$

We prove that for any positive integer I, the $Z_I^{(1)}(\{v_\alpha\}$ and the function $K_I(\{v_\alpha\};\{z_i\}$ are equivalent at all times, that is to say:

$$Z_I^{(1)}(\{v_\alpha\};\{z_i\})=K_I(\{v_\alpha\};\{z_i\}).\tag{39}$$

We shall prove this equation by induction.

(i) For the case of $N=1$, they are obviously equal:

$$Z_1^{(1)}(v_1;z_1)=K_1(v_1;z_1)=\frac{\sin(\lambda_1+\xi-z_1)\sin(\lambda_2+\xi+z_1)\sin(2v_1)}{\sin(\lambda_1+\xi+v_1)\sin(\lambda_2+\xi+v_1)\sin(v_1-z_1)\sin(v_1+z_1)}.\tag{40}$$

(ii) Suppose the relation (39) holds for the case $I\leqslant N-1$, We prove that (38) also holds for $I=N$. We make the following transformation of (38):

$$K_N(\{v_\alpha\};\{z_i\})=\frac{\prod_{j\neq i}^{N}\sin(v_N+z_j)\sin(v_N-z_j)\sin(v_N+z_i)\sin(v_N-z_i)}{\prod_{j=i+1}^{N}\sin(z_i-z_j)\sin(z_i+z_j)\prod_{k=1}^{i-1}\sin(z_k-z_i)\sin(z_k+z_i)}\frac{\prod_{\alpha=1}^{N-1}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)}{\prod_{\beta=1}^{N-1}\sin(v_N-v_\beta)\sin(v_N+v_\beta)}$$

$$\times \frac{\prod_{\alpha=1}^{N-1}\prod_{j\neq i}^{N}\sin(v_\alpha+z_j)\sin(v_\alpha-z_j)}{\prod_{\alpha>\beta;\ \alpha,\beta\neq N}\sin(v_\alpha-v_\beta)\sin(v_\alpha+v_\beta)\prod_{k<j;\ k,j\neq i}\sin(z_k-z_j)\sin(z_k+z_j)}$$
$$\times \det\Big|\frac{\sin(\lambda_1+\xi-z_j)\sin(\lambda_2+\xi+z_j)\sin(2v_\alpha)}{\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_2+\xi+v_\alpha)\sin^2(v_\alpha-z_j)\sin^2(v_\alpha+z_j)}\Big|\,. \tag{41}$$

We then expand the determinant along the N-th row:

$$K_N(\{v_\alpha\};\{z_i\}) = \frac{\prod_{j\neq i}^{N}\sin(v_N+z_j)\sin(v_N-z_j)\sin(v_N+z_i)\sin(v_N-z_i)}{\prod_{j=i+1}^{N}\sin(z_i-z_j)\sin(z_i+z_j)\prod_{k=1}^{i-1}\sin(z_k-z_i)\sin(z_k+z_i)}$$
$$\times \frac{\prod_{\alpha=1}^{N-1}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)}{\prod_{\beta=1}^{N-1}\sin(v_N-v_\beta)\sin(v_N+v_\beta)}$$
$$\times \frac{\prod_{\alpha=1}^{N-1}\prod_{j\neq i}^{N}\sin(v_\alpha+z_j)\sin(v_\alpha-z_j)}{\prod_{\alpha>\beta;\ \alpha,\beta\neq N}\sin(v_\alpha-v_\beta)\sin(v_\alpha+v_\beta)\prod_{k<j;\ k,j\neq i}\sin(z_k-z_j)\sin(z_k+z_j)}$$
$$\times (-1)^{N+i}\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)\sin^2(v_N-z_i)\sin^2(v_N+z_i)}$$
$$\times \det\Big|\frac{\sin(\lambda_1+\xi-z_j)\sin(\lambda_2+\xi+z_j)\sin(2v_\alpha)}{\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_2+\xi+v_\alpha)\sin^2(v_\alpha-z_j)\sin^2(v_\alpha+z_j)}\Big|_{(\alpha\neq N,\ j\neq i)}\,. \tag{42}$$

Obviously, the second order terms in the expansion $\sin^2(v_N-z_i)\sin^2(v_N+z_i)$ are indeed the first order terms, then rewrite the $N-1$ determinant and absorb the relative coefficients into the K_{N-1} form

$$K_N(\{v_\alpha\};\{z_i\}) = \frac{\prod_{j\neq i}^{N}\sin(v_N+z_j)\sin(v_N-z_j)}{\prod_{j=i+1}^{N}\sin(z_i-z_j)\sin(z_i+z_j)\prod_{k=1}^{i-1}\sin(z_k-z_i)\sin(z_k+z_i)}$$
$$\times \frac{\prod_{\alpha=1}^{N-1}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)}{\prod_{\beta=1}^{N-1}\sin(v_N-v_\beta)\sin(v_N+v_\beta)}$$
$$\times (-1)^{N+i}\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)\sin(v_N-z_i)\sin(v_N+z_i)}$$
$$\times K_{N-1}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,, \tag{43}$$

after simplification, $K_N(\{v_\alpha\};\{z_i\})$ satisfy:

$$K_N(\{v_\alpha\};\{z_i\}) = \sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)\sin(v_N-z_i)\sin(v_N+z_i)}$$
$$\times \prod_{l=1}^{N-1}\frac{\sin(v_l-z_i)\sin(v_l+z_i)}{\sin(v_N-v_l)\sin(v_N+v_l)}\prod_{j\neq i}\frac{\sin(v_N-z_j)\sin(v_N+z_j)}{\sin(z_j-z_i)\sin(z_j+z_i)}K_{N-1}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,. \tag{44}$$

(iii) In the following, we are to prove that (37) is equal to (44):

Firstly, according to the form of determinant (37) and the recursive relation (44), we can conclude that both $K_N(\{v_\alpha\};\{z_i\})$ and $Z_N^{(1)}(\{v_\alpha\};\{z_i\})$ are polynomial of v_N, and they have the same simple poles with the same residues located at:

$$\pm z_i, -(\lambda_1+\xi), -(\lambda_2+\xi) \quad i=1,\ldots,N\,. \tag{45}$$

We take out the coefficient which has nothing to do with the recursive relation, the (37) and (44) would become:

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\}) = \frac{\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)}\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)}{\sin(v_N-z_i)\sin(v_N+z_i)}$$
$$\times Z_{N-1}^{(1)}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,, \tag{46}$$

$$K_N(\{v_\alpha\};\{z_i\}) = \frac{\sin(2v_N)}{\sin(\lambda_1+\xi+v_N)\sin(\lambda_2+\xi+v_N)}\sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)}{\sin(v_N-z_i)\sin(v_N+z_i)}$$
$$\times \prod_{l=1}^{N-1}\frac{\sin(v_l-z_i)\sin(v_l+z_i)}{\sin(v_N-v_l)\sin(v_N+v_l)}\prod_{j\neq i}\frac{\sin(v_N-z_j)\sin(v_N+z_j)}{\sin(z_j-z_i)\sin(z_j+z_i)}$$

$$\times K_{N-1}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,. \tag{47}$$

These relations tell us only simple poles $\{\pm z_i\}$ take effect on the recursive relation.

Take a concrete z_i as an example, (so does $-z_i$), then the residues are:

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\}) = \sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}{\sin(\lambda_1+\xi+z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}Z_{N-1}^{(1)}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,, \tag{48}$$

$$\begin{aligned}K_N(\{v_\alpha\};\{z_i\}) &= \sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}{\sin(\lambda_1+\xi+z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}\\ &\quad\times\prod_{l=1}^{N-1}\frac{\sin(v_l-z_i)\sin(v_l+z_i)}{\sin(z_i-v_l)\sin(z_i+v_l)}\prod_{j\neq i}\frac{\sin(z_i-z_j)\sin(z_i+z_j)}{\sin(z_j-z_i)\sin(z_j+z_i)}K_{N-1}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\\ &= \sum_{i=1}^{N}\frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}{\sin(\lambda_1+\xi+z_i)\sin(\lambda_2+\xi+z_i)\sin(2z_i)}K_{N-1}(\{v_\alpha\}_{\alpha\neq N};\{z_j\}_{j\neq i})\,.\end{aligned} \tag{49}$$

Therefore the residues of (37) and (44) for each simple pole, are equivalent.

Besides, we denote $f(x) = Z_N^{(1)}(\{v_\alpha\};\{z_i\}) - K_N(\{v_\alpha\};\{z_i\})$, when $v_N\to\infty$ the limitation of $f(x)$ tends to 0, say:

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\})|_{v_N\to\infty} = K_N(\{v_\alpha\};\{z_i\})|_{v_N\to\infty}\,. \tag{50}$$

(iv) Thus the equation (39) also holds for $I = N$.

In conclusion, $K_I(\{v_\alpha\};\{z_i\})$ is the determinant form of $Z_I^{(1)}(\{v_\alpha\};\{z_i\})$, namely, we get the determinant representation of the scalar product $S^{1,2}(\{u_\alpha\};\{v_i\})$. Consequently we also obtain the determinant representation of the partition function for $N = 2M$ lattice with domain wall boundary conditions $Z_N^{(1)}(\{v_\alpha\};\{z_i\})$:

$$Z_N^{(1)}(\{v_\alpha\};\{z_i\}) = \frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)\det\mathscr{N}^{(1)}}{\prod_{\alpha>\beta}\sin(v_\alpha-v_\beta)\sin(v_\alpha+v_\beta)\prod_{k<j}\sin(z_k-z_j)\sin(z_k+z_j)}\,, \tag{51}$$

in which, the $N\times N$ Matrices $\mathscr{N}^{(1)}(\{v_\alpha\};\{z_j\})$ are

$$\mathscr{N}^{(1)}(\{v_\alpha\};\{z_j\})_{\alpha,j} = \frac{\sin(\lambda_1+\xi-z_j)\sin(\lambda_2+\xi+z_j)\sin(2v_\alpha)}{\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_2+\xi+v_\alpha)\sin^2(v_\alpha-z_j)\sin^2(v_\alpha+z_j)}\,. \tag{52}$$

Reference [38] provided the form of $Z_N^{(2)}$:

$$Z_N^{(2)}(\{v_\alpha\};\{z_i\}) = \frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(v_\alpha+z_i)\sin(v_\alpha-z_i)\det\mathscr{N}^{(2)}}{\prod_{\alpha>\beta}\sin(v_\alpha-v_\beta)\sin(v_\alpha+v_\beta)\prod_{k<j}\sin(z_k-z_j)\sin(z_k+z_j)}\,, \tag{53}$$

where $\mathscr{N}^{(2)}(\{v_\alpha\};\{z_j\})$ is of the form

$$\mathscr{N}^{(2)}(\{v_\alpha\};\{z_j\})_{\alpha,j} = \frac{\sin(\lambda_1+\xi+z_j)\sin(\lambda_2+\xi-z_j)\sin(2v_\alpha)}{\sin(\lambda_1+\xi-v_\alpha)\sin(\lambda_2+\xi-v_\alpha)\sin^2(v_\alpha-z_j)\sin^2(v_\alpha+z_j)}\,. \tag{54}$$

Actually, $Z_N^{(1)}$ and $Z_N^{(2)}$ possess the same algebraic structures, the difference between them are spin flip at each corresponding site.

4 Conclusions

We have studied the XXZ Gaudin model with generic boundaries specified by the non-diagonal K-matrices $K^{\pm}(u)$, constructed the corresponding pseudo-vacuum state (15) and (16) and applied the algebraic Bethe ansatz method to derive their common eigenstates (19) and (20) and eigenvalues (26) and (27) as well as the associated Bethe ansatz equations (23) and (24). We obtain the determinant representations for the scalar products of the form $S^{1,2}$. It constitutes a complete proof of the determinant representations of the scalar products of the XXZ Gaudin model with non-diagonal boundary terms.

References

[1] M. Gaudin, J. Phys. (Paris) **37** (1976) 1087.

[2] B.M. Garraway, Phil. Trans. R. Soc. A **369** (2011) 1137.

[3] N.M. Bogoliubov and P.P. Kulish, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) (Russian) **398** (2012) 26.

[4] N. Seiberg and E. Witten, Nucl. Phys. B **426** (1994) 19.

[5] H. Braden, A. Marshakov, A. Mirohov, and A. Morozov, *The Ruijsenaars–Schneider Model in the Context of Seiberg–Witten Theory*, arXiv:hep-th/9902205.

[6] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B **247** (1984) 83.

[7] K. Hikami, J. Phys. A **28** (1995) 4997.

[8] H.M. Babujian, J. Phys. A **26** (1993) 6981.

[9] B. Feign, E. Frenkel, and N. Reshetikhin, Commun. Math. Phys. **166** (1994) 27.

[10] E.K. Sklyanin, J. Sov. Math. **47** (1989) 2473.

[11] E.K. Sklyanin and T. Takebe, Phys. Lett. A **219** (1996) 217.

[12] E.K. Sklyanin, Lett. Math. Phys. **47** (1999) 275.

[13] L. Amico, A. Di Lorenzo, and A. Osterloh, Phys. Rev. Lett. **86** (2001) 5759; Nucl. Phys. B **614** (2001) 449.

[14] J.M. Roman, G. Sierra, and J. Dukelski, Nucl. Phys. B **634** (2002) 483.

[15] J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. **76** (2004) 643.

[16] R.W. Richardson, Phys. Lett. A **3** (1963) 277; Phys. Lett. B **5** (1963) 82.

[17] V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Function*, Cambridge Univ. Press, Cambridge (1993).

[18] Bogoliubov and Shirkov, *Introduction to the Theory of Quantized Fields*, Interscience Publ., New York (1959).

[19] S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press, Cambridge (2001).

[20] V.E. Korepin, Commun. Math. Phys. **86** (1982) 391.

[21] N. Kitanine, J.M. Maillet, and V. Terras, Nucl. Phys. B **554** (1999) 647.

[22] W.L. Yang, Y.Z. Zhang, and S.Y. Zhao, J. High Energy Phys. **12** (2004) 038.

[23] W.L. Yang, Y.Z. Zhang, and S.Y. Zhao, Commun. Math. Phys. **264** (2006) 87.

[24] S.Y. Zhao, W.L. Yang, and Y.Z. Zhang, Commun. Math. Phys. **268** (2006) 505.

[25] Y.S. Wang, Nucl. Phys. B **622** (2002) 633.

[26] N. Kitanine, K.K. Kozlowski, J.M. Maillet, G. Niccoli, N.A. Slavnov, and V. Terras, J. Stat. Mech. **10** (2007) 10009.

[27] W.L. Yang, X. Chen, J. Feng, K. Hao, B.Y. Hou, K.J. Shi, and Y.Z. Zhang, J. High Energy Phys. **01** (2011) 006.

[28] E.K. Sklyanin, J. Sov. Math. **47** (1989) 2473.

[29] B. Jurčo, J. Math. Phys. **30** (1989) 1289.

[30] K. Hikami, P.P. Kulish, and M. Wadati, J. Phys. Soc. Jpn. **61** (1992) 3071.

[31] E.K. Sklyanin, J. Phys. A **21** (1988) 2375.

[32] I.V. Cherednik, Theor. Math. Fiz. **61** (1984) 35.

[33] W.L. Yang, Y.Z. Zhang, and M. Gould, Nucl. Phys. B **698** (2004) 312.

[34] W.L. Yang, R. Sasaki, and Y.Z. Zhang, J. High Energy Phys. **09** (2004) 046.

[35] W.L. Yang, Y.Z. Zhang, and R. Sasaki, Nucl. Phys. B **729** (2005) 594.

[36] H.Q. Zhou, J.R. Links, R. Mackenzie, and M.D. Gould, Phys. Rev. B **65** (2002) 060502.

[37] J. Cao, H.Q. Lin, K.J. Shi, and Y. Wang, Nucl. Phys. B **663** (2003) 487.

[38] K. Hao, W.L. Yang, H. Fan, S.Y. Liu, K. Wu, Z.Y. Yang, and Y.Z. Zhang, Nucl. Phys. B **862** (2012) 835.

[39] H.J. de Vega and A. Gonzalez-Ruiz, J. Phys. A **26** (1993) L519.

[40] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A **9** (1994) 3841.

[41] R.I. Nepomechie, J. Stat. Phys. **111** (2003) 1363; J. Phys. A **37** (2004) 433.

[42] W. Galleas and M.J. Martins, Phys. Lett. A **335** (2005) 167; C.S. Melo, G.A.P. Ribeiro, and M.J. Martins, Nucl. Phys. B **711** (2005) 565.

[43] J. de Gier and P. Pyatov, J. Stat. Mech. **03** (2004) 002; A. Nichols, V. Rittenberg, and J. de Gier, J. Stat. Mech. **03** (2005) 03003; J. de Gier, A.Nichols, P. Pyatov, and V. Rittenberg, Nucl. Phys. B **729** (2005) 387.

[44] J. de Gier and F.H.L. Essler, Phys. Rev. Lett. **95** (2005) 240601; J. Stat. Mech. **12** (2006) 12011.

[45] Z. Bajnok, J. Stat. Mech. **06** (2006) 06010.

[46] A. Doikou and P.P. Martin, J. Stat. Mech. **06** (2006) 06004; A. Dikou, J. Stat. Mech. **09** (2006) 09010.

[47] R. Murgan, R.I. Nepomechie, and C. Shi, J. Stat. Mech. **08** (2006) 08006.

[48] W.L. Yang and Y.Z. Zhang, J. High Energy Phys. **04** (2007) 044.