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Exact Solutions and Their Asymptotic Behaviors for the Averaged Generalized Fractional Elastic Models*

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Abstract *The generalized fractional elastic models govern the stochastic motion of several many-body systems, e.g., polymers, membranes, and growing interfaces. This paper focuses on the exact formulations and their asymptotic behaviors of the average of the solutions of the generalized fractional elastic models. So we directly analyze the Cauchy problem of the averaged generalized elastic model involving time fractional derivative and the convolution integral of a radially symmetric friction kernel with space fractional Laplacian.*

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1 Introduction

Many experiments show that a broad range of non-classical phenomena in the applied sciences and engineering can be described by fractional calculus.[1−4] It has been a powerful tool in depicting the anomalous diffusion kinetics which arise in physics, chemistry, biology, and other complex dynamics. Recently there has been growing interest in investigating the solutions of fractional differential equations. The generalized fractional elastic (GFE) models have been extensively used in statistical mechanics to study the dynamics of many-body physical systems including polymers, membranes, single-file systems, and rough interfaces.[5−6] In this paper, we focus on the exact formulations and their asymptotic behaviors of the average of the solutions of GFE model. After taking the ensemble average on both sides of the GFE model, we get the following averaged generalized fractional elastic equation

$$\mathcal{D}_t^\alpha u(\boldsymbol{x},t)=\int_{R^d}\mathcal{K}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^\mu}{\partial|\boldsymbol{x}|^\mu}u(\boldsymbol{y},t)\,\mathrm{d}^d\boldsymbol{y}\,,\quad \boldsymbol{x}\in R^d\,,\quad t>0\,,\tag{1}$$

where $\mathcal{D}_t^\alpha$ is the Caputo (or Riemann–Liouville) fractional derivative of order $\alpha\,(0<\alpha<2)$; $\mathcal{K}$ is radially symmetric friction kernel, and it describes the fluid-mediated interaction between different sites $\boldsymbol{x}$ and $\boldsymbol{y}$. For example, if taking $\mathcal{K}(\boldsymbol{x})=1/|\boldsymbol{x}|^\beta$, which is a hydrodynamic friction kernel, its Fourier transform is[5−6]

$$\hat{\mathcal{K}}(\boldsymbol{k})=\frac{(4\pi)^{d/2}}{2^\beta}\frac{\Gamma((d-\beta)/2)}{\Gamma(\beta/2)}|\boldsymbol{k}|^{\beta-d}\,,\ 0<\beta<d\,,$$

and the fractional Laplacian $\partial^\mu/\partial|\boldsymbol{x}|^\mu$ is usual defined in the form of Fourier transform[3,7]

$$\mathcal{F}\Big(\frac{\partial^\mu}{\partial|\boldsymbol{x}|^\mu}u(\cdot,t)\Big)=-|\boldsymbol{k}|^\mu\mathcal{F}(u(\boldsymbol{x}))\,,\quad 0<\mu\le 2\,,\tag{2}$$

where

$$\mathcal{F}(u(\boldsymbol{x}))=\hat{u}(\boldsymbol{k})=\frac{1}{(2\pi)^d}\int_{R^d}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,,\quad \boldsymbol{k}=(k_1,k_2,\ldots,k_d)\in R^d\,.$$

If taking the friction kernel $\mathcal{K}(\boldsymbol{x}-\boldsymbol{y})=\delta^d(\boldsymbol{x}-\boldsymbol{y})$, Eq. (1) reduces to the time-space fractional diffusion equation in terms of Caputo or Riemann–Liouville time-fractional derivative[8−9]

$$\mathcal{D}_t^\alpha u(\boldsymbol{x},t)=\frac{\partial^\mu}{\partial|\boldsymbol{x}|^\mu}u(\boldsymbol{x},t)\,,\quad \boldsymbol{x}\in R^d\,,\quad t>0\,.\tag{3}$$

Much effort has been dedicated to study the solutions of the fractional diffusion-type equations by many authors. The main tools to find the exact solutions of the fractional differential equations are the integral transforms, such as Laplace transform, Fourier transform, and Mellin transform. Schneider and Wyss[10] first considered an n-dimensional time fractional diffusion-wave equation in the form of integro-differential equation and found the corresponding Green's functions in terms of Fox's functions. Using the method of the Laplace transform, Gorenflo *et al.*[11] considered a map in the form

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of a linear integral operator between solutions of fractional diffusion-wave equation. An integral transformation which maps a Gaussian type of diffusion onto fractional diffusion for fractional Fokker–Planck equation was discussed by Barkai.[12] Jiang *et al.*[13] discussed the analytical solutions of the multi-term time-space Caputo–Riesz fractional advection diffusion equations with nonhomogeneous Dirichlet boundary conditions. Hanyga[14] discussed the Green's functions and the propagators for the multi-dimensional anisotropic time-space fractional diffusion equation. Anh and Leonenko[15] presented the Green's functions and spectral representations of an n-dimensional fractional diffusion-wave equations with random initial conditions. Kilbas *et al.*[16] investigated the Cauchy-type problem for diffusion-wave equation with Riemann–Liouville time-fractional derivative in R^m. Recently, Camargo *et al.*[17] considered the modified generalized Mittag–Leffler function and used it to solve the n-dimensional fractional telegraph equation. For more analytical methods of solving the fractional kinetic equations and some physical applications, one may refer to the (review) works.[1–2,9–10,12,15,18–19]

More specifically, our goal in this paper is to discuss the analytical solutions of the averaged GFE equation (1) with the friction kernel $\mathcal{K}$ with power-law form[5–6]

$$\mathcal{K}(\boldsymbol{x})=\frac{1}{|\boldsymbol{x}|^{\beta}},\quad \beta\neq d+2l,\ \beta\neq -2l\quad l\in N_0,\ \ \boldsymbol{x}\in R^d, \tag{4}$$

where N_0 denotes the collection of nonnegative integers. Applying the integral transform techniques, we present the fundamental solutions for fractional elastic model (1) with Caputo and Riemann–Liouville fractional derivatives, respectively; and the fundamental solutions of the proposed equations are formulated by some special functions. Furthermore, the moments of Green's functions are examined by means of Mellin transform. And the asymptotic behaviour of the presented solutions are discussed by means of the asymptotic expansions of the Fox's H and Mittag–Leffler functions.

The paper is organized as follows. In Sec. 2, we introduce the definitions and some useful lemmas. In Sec. 3, we present the main results, including the exact solutions, the moments of the Green's functions, and the asymptotic behaviours of the solutions. The paper is concluded with some remarks in Sec. 4.

2 Preliminary Results

In this section we first give some definitions and useful lemmas that will be used in the next section.

Definition 1[1] The two-parameter Mittag–Leffler function is defined by

$$E_{\tilde\alpha,\tilde\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\tilde\alpha+\tilde\beta)}\,,\quad \tilde\alpha>0\,,\quad \tilde\beta, z\in\boldsymbol{R}\,. \tag{5}$$

The m-th derivatives of the Mittag–Leffler function gives

$$E^{(m)}_{\tilde\alpha,\tilde\beta}(z)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}(E_{\tilde\alpha,\tilde\beta}(z))=\sum_{n=0}^{\infty}\frac{(n+m)!z^n}{n!\Gamma(\tilde\alpha(n+m)+\tilde\beta)}\,. \tag{6}$$

For the Fourier transform of a radial function, we have the following Bochner's formula

Lemma 1[3,20] If $\psi(\boldsymbol{x})\in L^1(R^d)\cap C(R^d)$ is radial with $\psi(\boldsymbol{x})=\varphi(|\boldsymbol{x}|)$, then its Fourier transform is also radial and is given by

$$\frac{1}{(2\pi)^d}\int_{R^d}\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\varphi(|\boldsymbol{k}|)\mathrm{d}\boldsymbol{k}=\frac{1}{(2\pi)^{d/2}}|\boldsymbol{x}|^{1-d/2}\times\int_0^{\infty}|\boldsymbol{k}|^{d/2}\varphi(|\boldsymbol{k}|)\mathrm{J}_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|)\,\mathrm{d}|\boldsymbol{k}|\,. \tag{7}$$

Here, $|\cdot|$ denotes the modulus of the vectors $\boldsymbol{x},\boldsymbol{k}$; $\mathrm{J}_{d/2-1}(\cdot)$ represents the Bessel function of the first kind with order $d/2-1$.

Particularly, for the power function $|\boldsymbol{x}|^{-\delta}$, we have

Lemma 2[3,21] For an arbitrary real δ, the Fourier transform of the function $|\boldsymbol{x}|^{-\delta}$, gives

$$\frac{1}{(2\pi)^d}\int_{R^d}\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}|\boldsymbol{x}|^{-\delta}\mathrm{d}\boldsymbol{k}=c_d(\delta)|\boldsymbol{k}|^{\delta-d}\,,\quad \text{if } \delta-d\neq 2l\,,\ \delta\neq -2l\,,\ l\in N_0\,, \tag{8}$$

where

$$c_d(\delta)=(4\pi)^{d/2}\Gamma\Big(\frac{d-\delta}{2}\Big)\Big/2^{\delta}\Gamma\Big(\frac{\delta}{2}\Big)\,.$$

For the Mittag–Leffler function $E^{(n)}_{\alpha,\beta}(z)$, the following holds

Lemma 3 For $\tilde\alpha>0$, if $\tilde\beta$ is an arbitrary complex number, and $a\in R$, $\mu>0$, then we have

$$\int_{R^d}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}E^{(n)}_{\tilde\alpha,\tilde\beta}(-a|\boldsymbol{k}|^{\mu})\mathrm{d}\boldsymbol{k}=(2\pi)^{d/2}|\boldsymbol{x}|^{1-d/2}\times\int_0^{\infty}|\boldsymbol{k}|^{d/2}E^{(n)}_{\tilde\alpha,\tilde\beta}(-a|\boldsymbol{k}|^{\mu})\mathrm{J}_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|)\,\mathrm{d}|\boldsymbol{k}|\,. \tag{9}$$

Proof In fact, we just need to check that the integral in the right-hand of Eq. (9) is convergent. Noticing that the m-th derivatives of the Mittag–Leffler function can be expressed as the following Fox's H-function[22]

$$E^{(n)}_{\alpha,\beta}(z)=H^{1,1}_{1,2}\Big[-z\Big|^{(-n,1)}_{(0,1),(1-(\alpha n+\beta),\alpha)}\Big], \tag{10}$$

and combining the asymptotic expressions of the Fox's H-function $H^{1,1}_{1,2}(z)$[22] and the Bessel function $\mathrm{J}_{d/2-1}(z)$ near zero and infinity,[3,20] the conclusion is obtained immediately.

3 Fundamental Solutions and Related Properties

3.1 *Exact Solution*

In this subsection, we discuss the exact solution of Eq. (1) with Caputo and Riemann–Liouville fractional derivatives, respectively. The main results are presented in the following two theorems.

Theorem 1 Consider the Cauchy problem of the following fractional elastic equation

$${}_0^CD_t^{\alpha}u(\boldsymbol{x},t)=\int_{R^d}\mathcal{K}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^{\mu}}{\partial|\boldsymbol{x}|^{\mu}}u(\boldsymbol{y},t)\mathrm{d}^d\boldsymbol{y}\,,\quad \boldsymbol{x}\in R^d\,,\ t>0\,,$$

$$\frac{\partial^{j-1}u(\boldsymbol{x},t)}{\partial t^{j-1}}\Big|_{t=0} = \phi_j(\boldsymbol{x})\,,\ j=1,2\,,$$
$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x},t)=0\,,\ t>0\,, \tag{11}$$

where ${}_0^CD_t^\alpha$ is the Caputo fractional derivative of order $\alpha\in(1,2)$ defined as

$${}_0^CD_t^\alpha u(\boldsymbol{x},t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(\boldsymbol{x},s)}{\partial s^2}\mathrm{d}s\,. \tag{12}$$

Then, for $\beta+\mu>d$ and $\phi_j(\boldsymbol{x})\in L^1(\boldsymbol{R}^d)$, the solution of problem (11) gives

$$u(\boldsymbol{x},t)=\sum_{j=1}^2\int_{R^d} G(\boldsymbol{x}-\boldsymbol{y},t;j)\phi_j(\boldsymbol{y})\mathrm{d}^d\boldsymbol{y}\,, \tag{13}$$

where the Green's functions $G_j(\boldsymbol{x},t;j)$, $j=1,2$ are given by

$$G(\boldsymbol{x},t;j)=\frac{t^{j-1}}{(\mu+\beta-d)\pi^{d/2}|\boldsymbol{x}|^d}H^{2,1}_{2,3}\times\left[\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/(\mu+\beta-d)}}\Bigg|^{(1,\frac{1}{\mu+\beta-d}),(j,\frac{\alpha}{\mu+\beta-d})}_{(\frac{d}{2},\frac12),(1,\frac{1}{\mu+\beta-d}),(1,\frac12)}\right]. \tag{14}$$

Proof Applying the Laplace transform to the first equation in problem (11) with respect to the time variable t and using the Laplace transform of Caputo fractional derivative[1]

$$\mathcal{L}\{{}_0^CD_t^\alpha u(\boldsymbol{x},t);s\} = s^\alpha\tilde{u}(\boldsymbol{x},s) - \sum_{j=1}^2 s^{\alpha-j}\frac{\partial^{j-1}u(\boldsymbol{x},t)}{\partial t^{j-1}}\Big|_{t=0}\,, \tag{15}$$

we have

$$s^\alpha\tilde{u}(\boldsymbol{x},s)-s^{\alpha-1}\phi_1(\boldsymbol{x})-s^{\alpha-2}\phi_2(\boldsymbol{x}) = \int_{R^d}\mathcal{K}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^\mu}{\partial|\boldsymbol{x}|^\mu}u(\boldsymbol{y},s)\mathrm{d}^d\boldsymbol{y}\,. \tag{16}$$

Using the Fourier transform with respect to the spatial variable $\boldsymbol{x}$ and the convolution theorem of the Fourier transform, we obtain

$$s^\alpha\hat{\tilde{u}}(\boldsymbol{k},s)-s^{\alpha-1}\hat{\phi}_1(\boldsymbol{x})-s^{\alpha-2}\hat{\phi}_2(\boldsymbol{x}) = -|\boldsymbol{k}|^\mu\hat{\mathcal{K}}(\boldsymbol{k})\hat{\tilde{u}}(\boldsymbol{k},s)\,, \tag{17}$$

where the d-dimensional Fourier transform of fractional Laplacian operator (2) is used.

Using Lemma 3, from Eq. (17), we get

$$\hat{\tilde{u}}(\boldsymbol{k},s)=\sum_{j=1}^2\frac{s^{\alpha-j}}{s^\alpha+c_d(\beta)|\boldsymbol{k}|^{\mu+\beta-d}}\hat{\phi}_j(\boldsymbol{k}) = \sum_{j=1}^2\hat{\tilde{G}}(\boldsymbol{k},s;j)\hat{\phi}_j(\boldsymbol{k})\,. \tag{18}$$

Employing the Laplace transform involving the derivative of the Mittag–Leffler function[1,22]

$$\int_0^\infty \mathrm{e}^{-st}t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(-at^\alpha)\mathrm{d}t=\frac{m!s^{\alpha-\beta}}{(s^\alpha+a)^{m+1}}\,,\quad \mathrm{Re}(s)>|a|^{1/\alpha}\,, \tag{19}$$

we have

$$\hat{G}(\boldsymbol{k},t;j)=t^{j-1}E_{\alpha,j}(-|\boldsymbol{k}|^\sigma c_d(\beta)t^\alpha)\,,\quad \mathrm{Re}(s)>|c_d(\beta)|\boldsymbol{k}|^\sigma t^\alpha|^{1/\alpha}\,, \tag{20}$$

where $\sigma=\mu+\beta-d$.

Furthermore, from Eq. (10), there exists

$$\hat{G}(\boldsymbol{k},t;j)=t^{j-1}H^{1,1}_{1,2}\left[|\boldsymbol{k}|^\sigma c_d(\beta)t^\alpha\Bigg|^{(0,1)}_{(0,1),(-j+1,\alpha)}\right]. \tag{21}$$

Using Lemma 3 and the property of the Fox's H-functions,[23] we get the exact representation

$$G(\boldsymbol{x},t;j)=\frac{t^{j-1}}{(2\pi)^d}\int_{R^d}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left\{H^{1,1}_{1,2}\left[|\boldsymbol{k}|^\sigma c_d(\beta)t^\alpha\Bigg|^{(0,1)}_{(0,1),(-j+1,\alpha)}\right]\right\}\mathrm{d}^d\boldsymbol{k}$$
$$=t^{j-1}\frac{|\boldsymbol{x}|^{1-d/2}}{(2\pi)^{d/2}}\int_0^\infty|\boldsymbol{k}|^{d/2}H^{1,1}_{1,2}\left[|\boldsymbol{k}|^\sigma c_d(\beta)t^\alpha\Bigg|^{(0,1)}_{(0,1),(-j+1,\alpha)}\right]\mathrm{J}_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|)\mathrm{d}|\boldsymbol{k}|\,. \tag{22}$$

Applying the Hankel transform of the Fox's H-functions,[24] we get

$$G(\boldsymbol{x},t;j)=\frac{t^{j-1}}{\pi^{d/2}|\boldsymbol{x}|^d}H^{1,2}_{3,2}\left[c_d(\beta)t^\alpha\Big(\frac{2}{|\boldsymbol{x}|}\Big)^\sigma\Bigg|^{(1-\frac{d}{2},\frac{\sigma}{2}),(0,1),(0,\frac{\sigma}{2})}_{(0,1),(-j+1,\alpha)}\right]. \tag{23}$$

Thanks to the properties of the Fox H-Functions,[22,24] we obtain

$$G(\boldsymbol{x},t;j)=\frac{t^{j-1}}{\sigma\pi^{d/2}|\boldsymbol{x}|^d}H^{1,2}_{3,2}\left[(c_d(\beta)t^\alpha)^{1/\sigma}\frac{2}{|\boldsymbol{x}|}\Bigg|^{(1-\frac{d}{2},\frac12),(0,\frac{1}{\sigma}),(0,\frac12)}_{(0,\frac{1}{\sigma}),(-j+1,\frac{\alpha}{\sigma})}\right] = \frac{t^{j-1}}{\sigma\pi^{d/2}|\boldsymbol{x}|^d}H^{2,1}_{2,3}\left[\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\Bigg|^{(1,\frac{1}{\sigma}),(j,\frac{\alpha}{\sigma})}_{(\frac{d}{2},\frac12),(1,\frac{1}{\sigma}),(1,\frac12)}\right]. \tag{24}$$

Combining Eqs. (21) and (24) leads to the general expression of $u(\boldsymbol{x},t)$. □

Remark 1 According to the properties of the Fox's H-functions,[22–24] the Green's functions $G(\boldsymbol{x},t;j)$ can also be expressed as the following series

$$G(\boldsymbol{x},t;j)=\frac{t^{j-1-\alpha d/\sigma}}{2\sigma\pi^{d/2}(c_d(\beta))^{d/\sigma}}\left[2\sum_{n=0}^\infty\frac{\Gamma(1-d/\sigma-2n/\sigma)\Gamma(d/\sigma+2n/\sigma)}{\Gamma(d/2+n)\Gamma(j-d\alpha/\sigma-2\alpha n/\sigma)}\frac{(-1)^n}{n!}\Big(\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\Big)^{2n}\right.$$
$$\left.+\sigma\sum_{n=0}^\infty\frac{\Gamma(d/2-\sigma/2-\sigma n/2)\Gamma(1+n)}{\Gamma(\sigma/2+\sigma n/2)\Gamma(j-\alpha-\alpha n)}\frac{(-1)^n}{n!}\Big(\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\Big)^{\sigma(1+n)-d}\right].$$

Furthermore, for

$$\left|\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\right| \ll 1\,,$$

we have

$$G(\boldsymbol{x},t;j) \sim \frac{t^{j-1-\alpha d/\sigma}}{2\sigma\pi^{d/2}(\gamma_d(\beta))^{d/\sigma}}\left[\frac{2\Gamma(1-d/\sigma)\Gamma(d/\sigma)}{\Gamma(d/2)\Gamma(j-d\alpha/\sigma)} + \frac{\sigma\Gamma(d/2-\sigma/2)}{\Gamma(\sigma/2)\Gamma(j-\alpha)}\Big(\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\Big)^{\sigma-d}\right].$$

For

$$\left|\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\right| \gg 1\,,$$

there exists

$$G(\boldsymbol{x},t;j) = \frac{t^{j-1}}{\sigma\pi^{d/2}|\boldsymbol{x}|^d}H^{1,2}_{3,2}\left[\frac{2(c_d(\beta)t^\alpha)^{1/\sigma}}{|\boldsymbol{x}|}\,\middle|\,\begin{matrix}(1-\frac{d}{2},\frac{1}{2}),(0,\frac{1}{\sigma}),(0,\frac{1}{2})\\(0,\frac{1}{\sigma}),(-j+1,\frac{\alpha}{\sigma})\end{matrix}\right]$$

$$= \frac{t^{j-1}}{\sigma\pi^{d/2}|\boldsymbol{x}|^d}\left[\sum_{n=0}^{\infty}\frac{\Gamma((d+\sigma n)/2)\Gamma(1+n)}{\Gamma(j+\alpha n)\Gamma(-\sigma n/2)}\frac{(-1)^n}{n!}\left(\frac{2(c_d(\beta)t^\alpha)^{1/\sigma}}{|\boldsymbol{x}|}\right)^{n\sigma}\right],$$

which follows the power-law asymptotics behavior

$$G(\boldsymbol{x},t;j) \sim \frac{t^{j-1+\alpha}}{\sigma\pi^{d/2}}\left[\frac{2\Gamma((d+\sigma)/2)}{\Gamma(j+\alpha)\Gamma(-\sigma/2)}\right]\Big(2(c_d(\beta))^{1/\sigma}\Big)^{\sigma}|\boldsymbol{x}|^{-\mu-\beta}\,. \tag{25}$$

If $\alpha = 1$ in Eq. (25), we observe the known result typical for Lévy distributions.[2] In a similar way, we can get the Green's function of fractional diffusion equation (3) with the Caputo time fractional derivative and the same initial-boundary conditions, being given as

$$u(\boldsymbol{x},t) = \sum_{j=1}^{2}\int_{R^d} G(\boldsymbol{x}-\boldsymbol{y},t;j)\phi_j(\boldsymbol{y})\,\mathrm{d}^d\boldsymbol{y}\,, \tag{26}$$

where the Green's functions $G(\boldsymbol{x},t;j)$ are given by

$$G(\boldsymbol{x},t;j) = \frac{t^{j-1}}{\mu\pi^{d/2}|\boldsymbol{x}|^d}H^{2,1}_{2,3}\left[\frac{|\boldsymbol{x}|}{2t^{\alpha/\mu}}\,\middle|\,\begin{matrix}(1,\frac{1}{\mu}),(j,\frac{\alpha}{\mu+\beta-d})\\(\frac{d}{2},\frac{1}{2}),(1,\frac{1}{\mu}),(1,\frac{1}{2})\end{matrix}\right]. \tag{27}$$

And for $||\boldsymbol{x}|/2t^{\alpha/\mu}| \ll 1$, there exists

$$G(\boldsymbol{x},t;j) \sim \frac{t^{j-1-\alpha d/\mu}}{2\mu\pi^{d/2}}\left[\frac{2\Gamma(1-d/\mu)\Gamma(d/\mu)}{\Gamma(d/2)\Gamma(j-d\alpha/\mu)} + \frac{\mu\Gamma(d/2-\mu/2)}{\Gamma(\mu/2)\Gamma(j-\alpha)}\Big(\frac{|\boldsymbol{x}|}{2t^{\alpha/\mu}}\Big)^{\mu-d}\right].$$

For $||\boldsymbol{x}|/2t^{\alpha/\mu}| \gg 1$, there is

$$G(\boldsymbol{x},t;j) \sim \frac{t^{j-1+\alpha}}{\mu\pi^{d/2}}\left[\frac{2^{\mu+1}\Gamma((d+\mu)/2)}{\Gamma(j+\alpha)\Gamma(-\mu/2)}\right]|\boldsymbol{x}|^{-\mu-d}\,. \tag{28}$$

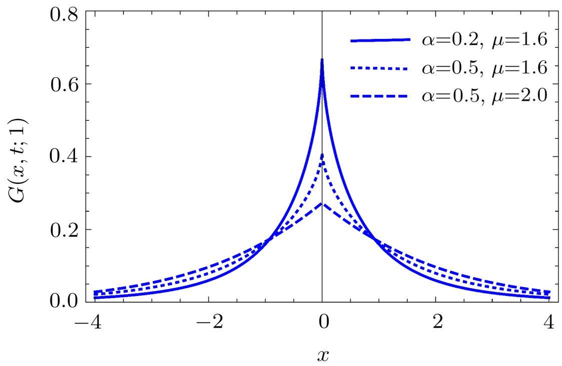

Fig. 1 Green function $G(x,t;1)$ given by Eq. (27) with different α, μ at $t=1$ in one dimension.

If the Caputo fractional derivative ${}^C_0D^\alpha_t$ is replaced by the Riemann–Liouville fractional derivative in problem (11), we have the following result.

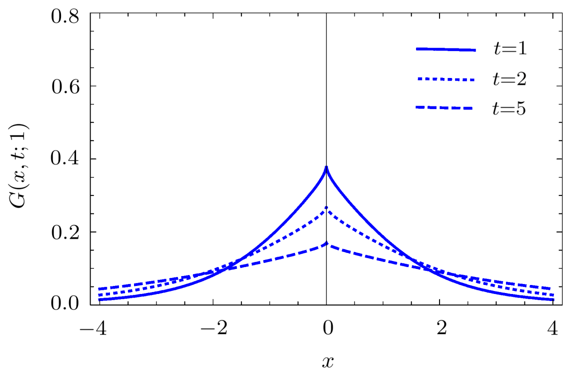

Fig. 2 Evolution of the Green function $G(x,t;1)$ given by Eq. (27) with $\alpha = 0.65$, $\mu = 1.6$ at different times.

Theorem 2 Consider the Cauchy problem of fractional elastic equation

$${}_0D^\alpha_t u(\boldsymbol{x},t) = \int_{R^d}\mathcal{K}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial^\mu}{\partial|\boldsymbol{x}|^\mu}u(\boldsymbol{y},t)\,\mathrm{d}^d\boldsymbol{y}\,,$$

$$\boldsymbol{x}\in R^d\,,\ t>0\,,$$

$$[{}_0D^{\alpha-j}_t u(\boldsymbol{x},t)]|_{t=0} = \varphi_j(\boldsymbol{x})\,,\ j=1,2\,,$$

$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x},t) = 0\,,\ t>0\,, \tag{29}$$

where the Riemann–Liouville fractional derivative ${}_0D_t^\alpha$ with $\alpha \in (1,2)$ is defined by

$${}_0D_t^\alpha u(\boldsymbol{x},t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial t^2}\int_0^t (t-s)^{1-\alpha}u(\boldsymbol{x},s)\,\mathrm{d}s\,. \tag{30}$$

Then, for $\beta+\mu>d$ and $\varphi_j(\boldsymbol{x}) \in L^1(\boldsymbol{R}^d)$, the solution of problem (29) reads

$$u(\boldsymbol{x},t) = \sum_{j=1}^{2}\int_{R^d}\mathcal{G}(\boldsymbol{x}-\boldsymbol{y},t;j)\varphi_j(\boldsymbol{y})\,\mathrm{d}^d\boldsymbol{y}\,, \tag{31}$$

where the Green's functions $\mathcal{G}(\boldsymbol{x},t;j)$ are given by

$$\mathcal{G}(\boldsymbol{x},t;j) = \frac{t^{\alpha-j}}{(\mu+\beta-d)\pi^{d/2}|\boldsymbol{x}|^d}H_{2,3}^{2,1}\times\left[\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/(\mu+\beta-d)}}\middle|^{(1,\frac{1}{\mu+\beta-d}),(1-j+\alpha,\frac{\alpha}{\mu+\beta-d})}_{(\frac{d}{2},\frac{1}{2}),(1,\frac{1}{\mu+\beta-d}),(1,\frac{1}{2})}\right]. \tag{32}$$

Proof Thanks to the Laplace transform of a Riemann–Liouville fractional derivative[1]

$$\mathcal{L}\{{}_0D_t^\alpha u(\boldsymbol{x},t);s\} = s^\alpha\tilde{u}(\boldsymbol{x},s) - \sum_{j=1}^{2}s^{j-1}[{}_0D_t^{\alpha-j}u(\boldsymbol{x},t)]|_{t=0}, \tag{33}$$

we obtain in Laplace–Fourier space

$$s^\alpha\hat{\tilde{u}}(\boldsymbol{k},s) - \sum_{j=1}^{2}s^{j-1}\varphi_j(\boldsymbol{k}) = -c_d(\beta)|\boldsymbol{k}|^{\beta+\mu-d}\hat{\tilde{u}}(\boldsymbol{k},s). \tag{34}$$

From Eq. (34), we have

$$\hat{\tilde{u}}(\boldsymbol{k},s) = \sum_{j=1}^{2}\frac{s^{j-1}\hat{\varphi}_j(\boldsymbol{k})}{s^\alpha + c_d(\beta)|\boldsymbol{k}|^{\beta+\mu-d}}\hat{\phi}_j(\boldsymbol{k}) = \sum_{j=1}^{2}\hat{\tilde{\mathcal{G}}}(\boldsymbol{k},s;j)\hat{\phi}_j(\boldsymbol{k}). \tag{35}$$

Employing the Laplace transform involving the derivative of the Mittag–Leffler function (19), for $\mathrm{Re}(s) > |\kappa_\mu|\boldsymbol{k}|^{\mu+\beta-d}|^{1/\alpha}$, we have

$$\hat{\mathcal{G}}(\boldsymbol{k},t;j)\hat{\phi}_j(\boldsymbol{k}) = t^{\alpha-j}E_{\alpha,\alpha-j+1}(-c_d(\beta)|\boldsymbol{k}|^{\beta+\mu-d}t^\alpha)\hat{\varphi}_j(\boldsymbol{k}). \tag{36}$$

With the similar procedure in deriving (14), Eq. (32) is obtained. □

Remark 2 Denoting $\sigma = \mu+\beta-d$, using the properties of Fox's H-function,[22–24] the Green's functions $\mathcal{G}(\boldsymbol{x},t;j)$ can be represented as the following series expansion

$$\mathcal{G}(\boldsymbol{x},t;j) = \frac{t^{\alpha-j-\alpha d/\sigma}}{2\sigma\pi^{d/2}(c_d(\beta))^{d/\sigma}}\Bigg[2\sum_{n=0}^{\infty}\frac{\Gamma(1-d/\sigma-2n/\sigma)\Gamma(d/\sigma+2n/\sigma)}{\Gamma(d/2+n)\Gamma(1-j+\alpha-d\alpha/\sigma-2\alpha n/\sigma)}\frac{(-1)^n}{n!}\Big(\frac{|\boldsymbol{x}|}{2(\gamma_d(\beta)t^\alpha)^{1/\sigma}}\Big)^{2n} + \sigma\sum_{n=0}^{\infty}\frac{\Gamma(d/2-\sigma/2-\sigma n/2)\Gamma(1+n)}{\Gamma(\mu/2+\sigma n/2)\Gamma(1-j-\alpha n)}\frac{(-1)^n}{n!}\Big(\frac{|\boldsymbol{x}|}{2(c_d(\beta)t^\alpha)^{1/\sigma}}\Big)^{\sigma(1+n)-d}\Bigg]. \tag{37}$$

In a similar way, we can get the Green's function of time-space fractional diffusion equation (3) with the Riemann–Liouville time fractional derivative and the same initial-boundary conditions, being given as

$$u(\boldsymbol{x},t) = \sum_{j=1}^{2}\int_{R^d}\mathcal{G}(\boldsymbol{x}-\boldsymbol{y},t;j)\varphi_j(\boldsymbol{y})\,\mathrm{d}^d\boldsymbol{y}\,, \tag{38}$$

where the Green's functions $G(\boldsymbol{x},t;j)$ are given by

$$\mathcal{G}(\boldsymbol{x},t;j) = \frac{t^{\alpha-j}}{\mu\pi^{d/2}|\boldsymbol{x}|^d}H_{2,3}^{2,1}\left[\frac{|\boldsymbol{x}|}{2t^{\alpha/\mu}}\middle|^{(1,\frac{1}{\mu}),(1-j+\alpha,\frac{\alpha}{\mu})}_{(\frac{d}{2},\frac{1}{2}),(1,\frac{1}{\mu}),(1,\frac{1}{2})}\right]. \tag{39}$$

For $0<\alpha<1$, with the similar procedure presented in Theorems 1 and 2, we can get the Green function of Eqs. (1) and (3) in the form of Eq. (27) and (39).[25–26] To see the difference, we plot Green functions of Eq. (3) in Figs. 1–4.

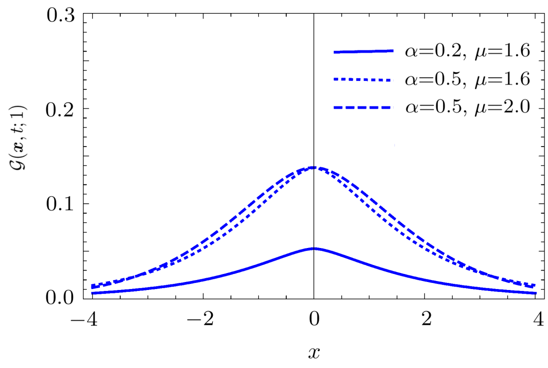

Fig. 3 Green function $\mathcal{G}(\boldsymbol{x},t;1)$ given by Eq. (39) with different α, μ at $t=1$ in one dimension.

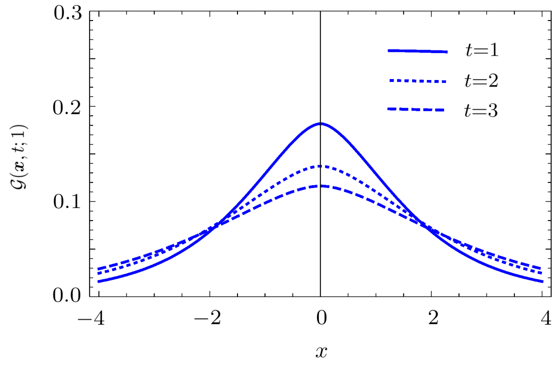

Fig. 4 Evolution of the Green function $\mathcal{G}(x,t;1)$ given by Eq. (39) with $\alpha=0.65$ and $\mu=1.6$ at different times.

It is worth noting that the initial value condition given in problem (29) is mainly under mathematical consideration. In practical application, it seems that the ones of Eq. (11) is more popular.[1,27] Hence, we focus on the moments of $G(\boldsymbol{x},t;j)$ for the Cauchy problem (11) in the following parts.

3.2 *Moments of Green Function*

In this subsection, we focus on calculating the fractional order moments of the Green's function $G(\boldsymbol{x},t;j)$, defined through

$$\langle r_j^\gamma(t)\rangle = \int_{-\infty}^{\infty} r^\gamma G(r,t;j)\,\mathrm{d}r = 2\int_0^{\infty} r^\gamma G(r,t;j)\,\mathrm{d}r\,,\quad \gamma>0\,,\quad j=1,2\,, \tag{40}$$

where $r=|\boldsymbol{x}|$ represents the axially symmetric radial coordinate. In view of the Mellin formula

$$\mathcal{M}\{f(r),\gamma\} = \int_0^{\infty} r^{\gamma-1} f(r)\,\mathrm{d}r\,,\quad \gamma>0\,, \tag{41}$$

after straightforward calculations, for $0<\gamma<\mu+\beta+1$, we have

$$\langle r_j^\gamma(t)\rangle = \int_0^{\infty} r^\gamma G(r,t;j)\,\mathrm{d}r = \frac{2^{\gamma+1} t^{j-1+\alpha((\gamma+1-d))/(\mu+\beta-d})}{(\mu+\beta-d)\pi^{d/2}(c_d(\beta))^{(d-\gamma-1)/(\mu+\beta-d)}} \times \frac{\Gamma((\gamma+1)/2)\Gamma(1+(\gamma+1-d)/(\mu+\beta-d))\Gamma((d-\gamma-1)/(\mu+\beta-d))}{\Gamma((d-\gamma-1)/2)\Gamma(j+(\gamma+1-d)\alpha/(\mu+\beta-d))}\,. \tag{42}$$

Particularly, if $\gamma\to 0$, there exists

$$\lim_{\gamma\to 0}\langle r_j^\gamma(t)\rangle = \frac{2t^{j-1+\alpha(1-d)/(\mu+\beta-d})}{(\mu+\beta-d)\pi^{(d-1)/2}(c_d(\beta))^{(d-1)/(\mu+\beta-d)}}\frac{\Gamma(1+(1-d)/(\mu+\beta-d))\Gamma((d-1)/(\mu+\beta-d))}{\Gamma((d-1)/2)\Gamma(j+(1-d)\alpha/(\mu+\beta-d))}\,, \tag{43}$$

which means that the Green function is not normalized. And, if $\gamma\to 2$, the mean squared displacement satisfies

$$\lim_{\gamma\to 2}\langle r_j^\gamma(t)\rangle = \frac{2^3 t^{j-1+\alpha(3-d)/(\mu+\beta-d)}}{(\mu+\beta-d)\pi^{d/2}(c_d(\beta))^{(d-3)/(\mu+\beta-d)}}\frac{\Gamma(3/2)\Gamma(1+(3-d)/(\mu+\beta-d))\Gamma((d-3)/(\mu+\beta-d))}{\Gamma((d-3)/2)\Gamma(j+(3-d)\alpha/(\mu+\beta-d))}\,. \tag{44}$$

The mean squared displacements $\langle r_1^2(t)\rangle$ are plotted; see Fig. 5.

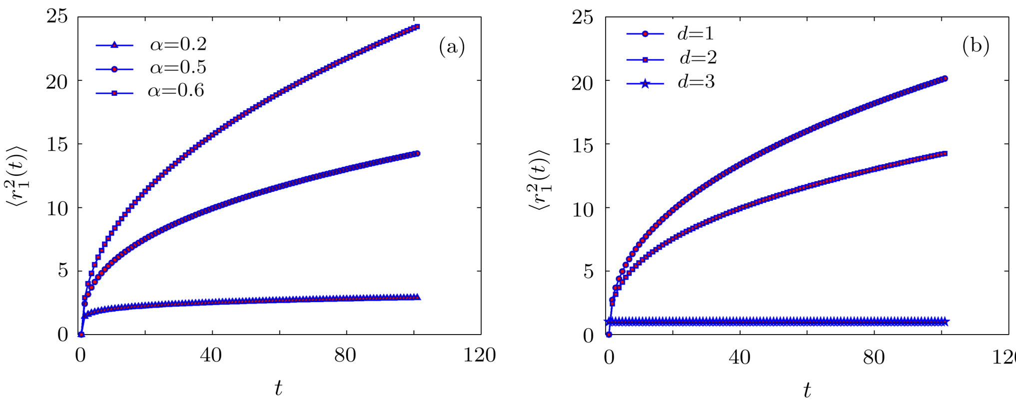

Fig. 5 The mean squared displacements $\langle r_1^2(t)\rangle$ with different d (b) and α (a); here for (a) we choose $\alpha=0.5$, $\mu=1.2$, $\beta=2.1$, $t=1000$, and for (b) we choose $\mu=1.2$, $\beta=2.1$, $t=1000$, $d=2$.

Next we calculate the Cartesian moments of the Green's function $G(\boldsymbol{x},t;j)$ given by Schneider and Wyss[10]

$$M_j(\varrho_1,\varrho_2\cdots\varrho_d) = \int_{R^d} x_1^{\varrho_1}x_2^{\varrho_2}\cdots x_d^{\varrho_d} G(\boldsymbol{x},t;j)\,\mathrm{d}^d\boldsymbol{x}\,,\quad j=1,2\,,$$

where $\varrho=(\varrho_1,\varrho_2\cdots\varrho_d)$ and $\{\varrho_m\}_{m=1}^d$ are non-negative integers. Owing to the symmetry of $G(\boldsymbol{x},t;j)$, if ϱ_m is an odd, denoting $\varrho_m=2\delta_m+1, \delta=\sum_{m=1}^d \delta_m$, we have $M_j(\varrho_1,\varrho_2\cdots\varrho_d)=0$. If ϱ_m is an even, i.e., $\varrho_m=2\delta_m$, there is

$$M_j(2\delta_1,2\delta_2\cdots 2\delta_d) = \frac{1}{2}M^0(2\delta_1,2\delta_2\cdots 2\delta_d)\mathcal{M}\big\{G(\boldsymbol{x},t;j),d+2\delta-1\big\} = \frac{1}{2}M^0(2\delta_1,2\delta_2\cdots 2\delta_d)\langle r_j^{d+2\delta-1}(t)\rangle\,,$$

where the spherical moments M_0 are given by[10,12]

$$M^0(2\delta_1,2\delta_2\cdots 2\delta_d) = \int_{S^{d-1}} e_1^{2\delta_1}e_2^{2\delta_2}\cdots e_d^{2\delta_d}\,d\Omega(\boldsymbol{e}) = 2\frac{\prod_{m=1}^d\Gamma(\delta_m+1/2)}{\Gamma(\delta+d/2)}\,.$$

Therefore,

$$M_j(2\delta_1,2\delta_2\cdots 2\delta_d) = \frac{2^{d+2\delta}t^{j-1+2\delta\alpha/(\mu+\beta-d)}}{(\mu+\beta-d)\pi^{d/2}(\gamma_d(\beta))^{-2\delta/(\mu+\beta-d)}}$$

$$\times \frac{\Gamma((d+2\delta)/2)\Gamma(1+2\delta/(\mu+\beta-d))\Gamma(-2\delta/(\mu+\beta-d))}{\Gamma(-\delta)\Gamma(j+2\delta\alpha/(\mu+\beta-d))} \frac{\prod_{m=1}^{d}\Gamma(\delta_m+1/2)}{\Gamma(\delta+d/2)}\,.$$

Remark 3 The fractional moments of the Green function (27) can also be calculated. In this case, due to Eq. (28), we know that the integral (40) converges when $\gamma < \mu + d - 1$.

3.3 *Asymptotic Behavior of Solutions as $t \to \infty$*

Theorem 3 If the initial data $\phi_j(\boldsymbol{x}) \in L^1(\boldsymbol{R}^d)$, then the solution of the Cauchy problem (11) satisfies the following time decay estimate

$$\|u(\boldsymbol{x},t)\|_\infty \le C t^{\max\{j-1-\alpha d/(\beta+\mu-d)\}}\,, \quad \text{as} \quad t \to \infty\,. \tag{45}$$

Proof Using the property of Fox's H-functions, we may rewrite the Green functions (22) as

$$G(\boldsymbol{x},t;j) = \frac{t^{j-1}}{(2\pi)^d} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,j}(-|\boldsymbol{k}|^{\beta+\mu-d} c_d(\beta) t^\alpha)\} \mathrm{d}\boldsymbol{k}\,, \quad j=1,2\,. \tag{46}$$

Denoting

$$\tilde{\boldsymbol{k}} = \gamma_d(\beta)^{1/(\beta+\mu-d)} t^{\alpha/(\beta+\mu-d)} (k_1, k_2, \ldots, k_d)\,,$$

we have

$$G(\boldsymbol{x},t;j) = \frac{t^{j-1}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} t^{d\alpha/(\beta+\mu-d)}} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,j}(-|\tilde{\boldsymbol{k}}|^{\beta+\mu-d})\} \mathrm{d}\tilde{\boldsymbol{k}}\,, \tag{47}$$

Using the asymptotic behaviour presented in formula

$$E_{\tilde{\alpha},\tilde{\beta}}(-z) \sim \frac{1}{z\,\Gamma(\tilde{\beta}-\tilde{\alpha})}\,, \quad z \to \infty\,, \quad 0 < \tilde{\alpha} < 2\,, \quad \tilde{\beta} > 0\,, \tag{48}$$

Eq. (47) yields

$$\begin{aligned} \lim_{t\to\infty} G(\boldsymbol{x},t;j) &= \lim_{t\to\infty} \frac{t^{j-1}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} t^{d\alpha/(\beta+\mu-d)}} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,j}(-|\tilde{\boldsymbol{k}}|^{\beta+\mu-d})\} \mathrm{d}\tilde{\boldsymbol{k}} \\ &= \frac{t^{j-1-d\alpha/(\beta+\mu-d)}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} \Gamma(j-\alpha)} \Upsilon_d(\boldsymbol{x}, \beta+\mu-d)\,. \end{aligned} \tag{49}$$

In the second equality, the interchange of the limit and the integral is justified by the uniform convergence of both. The last equality uses the inverse Fourier transform of the power function[3,28]

$$\mathcal{F}^{-1}\{|\boldsymbol{k}|^{-\delta}\} = \Upsilon_d(\boldsymbol{x},\delta) = \frac{1}{\gamma_d(\delta)} \begin{cases} |\boldsymbol{x}|^{\delta-d}\,, & \text{if } \delta+d \ne 2l\,,\ \delta \ne -2l\,,\ l \in N_0\,, \\ |\boldsymbol{x}|^{\delta-d} \ln \dfrac{1}{|\boldsymbol{x}|}\,, & \text{if } \delta+d = 2l\,,\ l \in N_0\,, \end{cases} \tag{50}$$

where

$$\gamma_d(\delta) = \begin{cases} 2^\delta \pi^{d/2} \Gamma\Big(\dfrac{\delta}{2}\Big) \Big/ \Gamma\Big(\dfrac{d-\delta}{2}\Big)\,, & \text{if } \delta \ne d+2l\,,\ l \in N_0\,, \\ 1\,, & \text{if } \delta = d+2l\,,\ l \in N_0\,. \end{cases} \tag{51}$$

□

Theorem 4 Suppose that the functions $\phi_j(\boldsymbol{x}) \in L^1(\boldsymbol{R}^d)$, $j = 1, 2$, for $\beta + \mu > d$, then the solution of problem (29) has the following asymptotic estimate

$$\|u(\boldsymbol{x},t)\|_\infty \le C t^{\max\{\alpha-j-\alpha d/(\beta+\mu-d)\}}\,, \quad \text{as} \quad t \to \infty\,. \tag{52}$$

Proof Using the property of Fox's H-functions, we can rewrite Eq. (22) as

$$\mathcal{G}(\boldsymbol{x},t;j) = \frac{t^{\alpha-j}}{(2\pi)^d} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,\alpha-j+1}(-|\boldsymbol{k}|^{\beta+\mu-d} c_d(\beta) t^\alpha)\} \mathrm{d}\boldsymbol{k}\,, \quad j=1,2\,. \tag{53}$$

Denoting $\tilde{\boldsymbol{k}} = \gamma_d(\beta)^{1/(\beta+\mu-d)} t^{\alpha/(\beta+\mu-d)} (k_1, k_2, \ldots, k_d)$, there exists

$$\mathcal{G}(\boldsymbol{x},t;j) = \frac{t^{\alpha-j}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} t^{d\alpha/(\beta+\mu-d)}} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,\alpha-j+1}(-|\tilde{\boldsymbol{k}}|^{\beta+\mu-d})\} \mathrm{d}\tilde{\boldsymbol{k}}\,. \tag{54}$$

Combining the asymptotic behaviour of Eqs. (48) and (50), for $j = 1$, and passing the limit inside the integral in Eq. (54), we get

$$\begin{aligned} \lim_{t\to\infty} \mathcal{G}(\boldsymbol{x},t;1) &= \lim_{t\to\infty} \frac{t^{\alpha-1}}{(2\pi)^d \gamma_d(\beta)^{d/(\beta+\mu-d)} t^{d\alpha/(\beta+\mu-d)}} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,\alpha}(-|\tilde{\boldsymbol{k}}|^{\beta+\mu-d})\} \mathrm{d}\tilde{\boldsymbol{k}} \\ &= \frac{t^{\alpha-1-d\alpha/(\beta+\mu-d)}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} \Gamma(-\alpha)} \Upsilon_d(\boldsymbol{x}, 2(\beta+\mu-d))\,, \end{aligned} \tag{55}$$

where the following formula is used

$$E_{\tilde{\alpha},\tilde{\alpha}}(-z) \sim \frac{1}{z^2\Gamma(-\tilde{\alpha})}\,, \quad z \to \infty\,, \quad 0 < \tilde{\alpha} < 2\,. \tag{56}$$

And for $j = 2$, we get

$$\begin{aligned}\lim_{t\to\infty} \mathcal{G}(\boldsymbol{x}, t; 2) &= \lim_{t\to\infty} \frac{t^{\alpha-2}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)} t^{d\alpha/(\beta+\mu-d)}} \int_{R^d} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \{E_{\alpha,\alpha-1}(-|\tilde{\boldsymbol{k}}|^{\beta+\mu-d})\}\,\mathrm{d}\tilde{\boldsymbol{k}} \\ &= \frac{t^{\alpha-2-d\alpha/(\beta+\mu-d)}}{(2\pi)^d c_d(\beta)^{d/(\beta+\mu-d)}\Gamma(3-\alpha)} \Upsilon_d(\boldsymbol{x}, \beta+\mu-d)\,,\end{aligned} \tag{57}$$

which agrees with the conclusion given in Eq. (37). □

Remark 4 For the solution of problem (3) with the Caputo fractional derivative, the following asymptotic estimate holds

$$\|u(\boldsymbol{x}, t)\|_\infty \le C t^{-\alpha d/\mu+1}\,, \quad \text{as} \quad t \to \infty\,, \tag{58}$$

and for the solution of problem (3) with Riemann–Liouville fractional derivative, there is the following asymptotic estimate

$$\|u(\boldsymbol{x}, t)\|_\infty \le C t^{-\alpha d/\mu+\alpha-1}\,, \quad \text{as} \quad t \to \infty\,, \tag{59}$$

which recover the results presented in Ref. [29].

4 Conclusions

We discuss the exact formulations and asymptotic behaviors of the average of the solutions of generalized fractionalelastic models, which is a stochastic model being used to describe the stochastic motion of many-body system. The basic strategy is first to take ensemble average on both sides of the equation, then use the techniques of integral transformations, including Fourier transform, Laplace transform, to derive the exact solutions and analyze the asymptotic behaviors of the averaged generalized fractional elastic models. The presented methods in this paper can be used to find the exact solutions of multi-term time-space fractional equations in multidimensional cases, such as the fractional Cattaneo equation, modified fractional diffusion equation and multi-term time fractional diffusion-wave equation.

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