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Solving for the Fixed Points of 3-Cycle in the Logistic Map and Toward Realizing Chaos by the Theorems of Sharkovskii and Li–Yorke

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Abstract *Sharkovskii proved that, for continuous maps on intervals, the existence of 3-cycle implies the existence of all others. Li and Yorke proved that 3-cycle implies chaos. To establish a domain of uncountable cycles in the logistic map and to understand chaos in it, the fixed points of 3-cycle are obtained analytically by solving a sextic equation. At one parametric value, a fixed-point spectrum, resulted from the Sharkovskii limit, helps to realize chaos in the sense of Li and Yorke.*

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1 Introduction to One-Dimensional Continuous Maps on an Interval

Let x_i and x_j be two real numbers in (0,1). If $x_i \to x_j$ by an action f, we write the process as $x_j = f(x_i)$, calling it a map. If, for some value of x_i, say x^*, it goes on to itself and nowhere else, it is termed a fixed point of f, i.e., $f(x^*) = x^*$. Evidently a fixed point is a special value for f, to which f acts like an identity operation.

Let $x_1, x_2, x_3, \ldots$ be a set of points in (0,1), not necessarily an ordered set. By the action f, let us say: $f(x_1) = x_2, f(x_2) = x_3, \ldots, f(x_n) = x_{n+1}$, a process termed an iteration. Suppose $f(x_3) = x_1$, i.e., $x_4 = x_1$, the iteration stops in 3 steps and repeats thereafter like the dance of waltz. The three steps give the 3-cycle condition: $f(x_1) = x_2$, $f(x_2) = x_3$, and $f(x_3) = x_1$, which may be expressed as $f(f(f(x))) \equiv f^3(x) = x$, where $x = x_1, x_2$, or x_3, and $x_1 \neq x_2 \neq x_3$. These three values are prime fixed points of f^3, giving 3-cycle. Excluded are the fixed points of f, which are also fixed points of f^3, known as non-prime fixed points, since they do not satisfy the 3-cycle condition.

If $f(x_n) = x_1$, or $f^n(x) = x$, it defines n-cycle. It would be termed periodic since it returns to its initial value after n steps. If, however, $f(x_n) = x_{n+1}$, $n \to \infty$, never returning to its initial value, it would be termed chaotic, showing a trajectory that would never close. What metrical properties in f make a trajectory to be periodic or chaotic?

If f is a continuous map on an interval, there is a remarkable theorem proved by Sharkovskii[1] and later independently by Li and Yorke,[2] said to be the only rigorous theorem in chaos.[3] This theorem asserts that if 3-cycle exists in some domain of such a map, there exist all other cycles in said domain. Thus the existence of 3-cycle according to Li and Yorke implies chaos itself in that domain.

To demonstrate this theorem and possibly to realize chaos that is implied by the theorem, what seems to be required is to obtain 3 prime fixed points of 3-cycle for a $1d$ continuous map. It amounts to solving for three roots of $f^3(x) - x = 0$, probably a cubic equation, which should pose no special difficulty. Had the theorem called for 4-cycle, presenting a quartic equation, it is still no challenge. Had the theorem asked for 5-cycle, presenting a quintic equation, any attempt at an analytical solution would have to be abandoned. Fortuitously it is 3-cycle that the theorem asks for and this is why an analytical study seems feasible. But we cannot know for sure until the 3-cycle of an applicable map is constructed. We shall now turn to one such map to see what is to be realized.

2 Logistic Map

Perhaps one of the simplest and no doubt the most widely studied of all chaotic maps is the logistic map.[4] The logistic map is a one-dimensional continuous map of real numbers on the interval (0,1), to which the theorem of Sharkovskii and Li–Yorke applies. The usual way to define the map is:

$$x' = f(x)\,,$$

where

$$f(x) = ax(1-x)\,, \quad x = (0,1)\,, \tag{1}$$

where a is a control parameter, $0 < a \leq 4$.

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Unexpected perhaps, the logistic map has physical relevance, in particular to the local dynamics of a harmonic chain.[5–7] The discrete frequency spectrum of the chain becomes continuous in the thermodynamic limit, (which makes among others the local dynamical variables ergodic). Also see Appendix G. As we shall also see later, the fixed points of different cycles in the logistic map are much like these frequencies of the harmonic chain. If there are infinitely many, they can make the fixed point spectrum similarly continuous. The logistic map has been used in the studies of an antiferromagnetic Potts model on a recursive lattice,[8] an Ising–Heisenberg model on a diamond chain,[9] and three-site interaction Ising model.[10]

The prime fixed points of f and f^2 are obtained readily: By solving $f(x) - x = 0, x > 0$, we obtain $x = x_0 = 1 - a^{-1}$, $a > 1$. Next by solving $(f^2 - x)/(f - x) = 0$, we obtain a quadratic equation, from which $x = x_{1,2} = (a+1)/2a\{1 \pm \sqrt{(a-3)/(a+1)}\}$, $a > 3$. We anticipate that $(f^3 - x)/(f - x) = 0$ is a cubic equation, from which 3 roots are to be obtained.

We construct $Q(x) = (f^3 - x)/(f - x) = 0$ in a new variable $t = ax$, $t = (0, a)$, for simplification:[6]

$$\begin{aligned} Q(t) = t^6 &- (3a+1)t^5 + (3a^2+4a+1)t^4 \\ &- (a^3+5a^2+3a+1)t^3 + (2a^3+3a^2+3a+1)t^2 \\ &- (a^3+2a^2+2a+1)t + (a^2+a+1)\,. \end{aligned} \tag{2}$$

To our surprise it is not a cubic equation but a sextic equation. Since there are no known formulas for solving any equations of degrees equal to or greater than 5, at a first glance we seem to have arrived at an impasse.

Perhaps Q is not a general sextic equation. If it is not, it might be solvable. If E_k denotes an equation of degree k, Q as E_6 would be solvable if it could be expressed as: $E_2 \times E_2 \times E_2$ or $E_3 \times E_3$.

Looking at Eq. (2), we see at once that Q must be real if t is real since its coefficients are all real. It means that the roots can only be either 3 pairs of complex conjugates or two sets of 3 real roots, which would correspond to the two solvable forms for Q.

It should be noted that Shi and Yu[11] obtained the same equation (2), but expressed in x, given as $g(x) = 0$. See their Eq. (3) in Sec. 3 of Ref. [11]. They established the exact parameter range of the stability of 3-cycle trajectories. Also see Refs. [12–14]. This approach complements ours which is to obtain the roots themselves by solving the sextic equation.

Any signs that Q is not a general sextic equation? Express (2) as

$$Q(t) = \sum_{k=0}^{6} (-)^k \delta_k t^k\,, \tag{3}$$

where the δ's are the coefficients of Q. Observe that

$$\Delta \equiv \sum_{k=0}^{5} (-)^k \delta_k = 0 \tag{4}$$

referred to as the delta sum rule. This sum rule is a clear indication that Q is not a general sextic equation.

There are already several studies in the literature which prove the existence of 3-cycle in the logistic map.[11–14] They obtain the onset value of a for the real stable solutions of the sextic equation and the onset value of instability. Except showing that the solutions are real in the interval (0,1), they do not solve the sextic equation.

In what is to follow we do differently from the above by actually solving the sextic equation, therewith to obtain the fixed points exactly. That would be an independent proof that 3-cycle exists. Our greater goal, however, goes beyond this proof. It is to construct a spectrum of fixed points beginning with the fixed points of 3-cycle. A trajectory starting from a point in this spectrum may show e.g. mixing, said to be a sign of chaos.

3 Trigonal Relation

We are looking for real positive roots of Q in $a_\infty < a < 4$, where $a_\infty = 3.569\,946\,72\cdots$ at which stable bifurcation of the fixed points ends.[4] At least the three positive real roots must lie in the interval $t = (0, a)$, corresponding to $x = (0, 1)$.

If $a = 0$ in Eq. (2), the roots are: $\exp(\pm i\pi/7)$, $\exp(\pm i3\pi/7)$, and $\exp(\pm i5\pi/7)$, 3 pairs of complex conjugates. If the roots are to be real, none of them may lie on the negative real axis of t since the coefficients of t in odd power have $-$ signs. The six roots must all lie on the positive real axis. If $a \to \infty$, one of the roots goes as $1/a$ asymptotically.

Thus as a increases from zero, the three pairs of complex roots must move toward the positive real axis. At some value of $a = \tilde{a}$ say, the complex-conjugate pairs simultaneously merge to become three real roots. As a increases past $\tilde{a}$, each real root splits into two. This overall behavior is consistent with the two solvable forms for Q. We cannot yet know however whether $a_\infty < \tilde{a} < 4$.

If $a > \tilde{a}$, there must be two sets of 3 positive real roots, not one set as one might have anticipated from the theorem of Sharkovskii. Hence, we shall write that, for $a > \tilde{a}$,

$$Q = q \times q'\,, \tag{5}$$

where q and q' are cubic equations. With (as yet undetermined) roots t_k and t'_k, $k = 1, 2, 3$, for q and q' respectively, they may be formally expressed as:

$$q(t) = \Pi_k (t - t_k)\,, \quad q'(t) = \Pi_k (t - t'_k)\,. \tag{6}$$

Putting them in the standard cubic equation form:

$$q(t) = t^3 - \alpha t^2 + \beta t - \gamma\,, \tag{7a}$$

$$q'(t) = t^3 - \alpha' t^2 + \beta' t - \gamma' , \tag{7b}$$

where

$$\alpha = t_1 + t_2 + t_3, \quad \beta = t_1 t_2 + t_2 t_3 + t_3 t_1, \quad \gamma = t_1 t_2 t_3, \tag{8}$$

and α', β', and γ' are similarly with primes on Eq. (8). They (α, β, γ and the primed ones) will be referred to as trigonals.

We shall now see whether there are some structural relations for the trigonals, which could explain the delta sum rule. Indeed, as proved in Appendix A, we find the following:

$$\alpha - \beta + \gamma = 0 , \quad \alpha' - \beta' + \gamma' = 0 , \tag{9}$$

referred to henceforth as the trigonal relation. In addition we also prove subsidiary properties:

$$\beta = (a+1)\alpha - (a^2 + a + 1) ,$$
$$\beta' = (a+1)\alpha' - (a^2 + a + 1) , \tag{10}$$
$$\gamma = a\alpha - (a^2 + a + 1) ,$$
$$\gamma' = a\alpha' - (a^2 + a + 1) . \tag{11}$$

There are thus only two unknowns α and α' for q and q', respectively, to be determined.

4 Delta Sum Rule by Trigonal Relation

We now express the coefficients of Q (the $\delta' s$), see (3), in terms of the trigonals:

$$\delta_6 = 1 , \tag{12a}$$
$$\delta_5 = \alpha + \alpha' , \tag{12b}$$
$$\delta_4 = \alpha\alpha' + \beta + \beta' , \tag{12c}$$
$$\delta_3 = \alpha\beta' + \beta\alpha' + \gamma + \gamma' , \tag{12d}$$
$$\delta_2 = \beta\beta' + \alpha\gamma' + \gamma\alpha' , \tag{12e}$$
$$\delta_1 = \beta\gamma' + \gamma\beta' , \tag{12f}$$
$$\delta_0 = \gamma\gamma' . \tag{12g}$$

In Appendix B we prove that the delta-sum rule follows from the trigonal relation. It indicates, as we had hoped for, that Q is not a general sextic equation. Thus at least in principle it may be solvable as a product of two cubic equations.

5 Transition Value by Trigonal Relation

We now turn to the transition value $a = \tilde{a}$, where the complex roots turn real. At $a = \tilde{a}$, there must be a transition condition formally put as: $E_2 \times E_2 \times E_2 = E_3 \times E_3 = (E_3)^2$. That is, $q = q'$ so that $Q = q^2$. As shown in Appendix C, when the trigonal relation is applied to the transition condition, it yields: $\sigma = 0$, where

$$\sigma = (a^2 - 2a - 7)^{1/2} . \tag{13}$$

Hence,

$$\tilde{a} = 1 + \sqrt{8} = 3.872\ 281\ 323 , \tag{14}$$

as the transition vale. Since $a_\infty < \tilde{a} < 4$, it satisfies a necessary condition for 3-cycle in the logistic map. This value was first determined some years ago in a very different way,[11–14] also see Refs. [6, 15].

6 Internal Degree of Freedom

If $a > \tilde{a}$, there are two sets of 3 positive real roots, or two forms of 3-cycle, seemingly richer by two in the logistic map than is asked for by the theorem of Sharkovskii. This richness however presents a problem. Unless the relationship between the two is established, that is, how α and α' are related, the two cubic equations still cannot be solved.

If there are two forms of 3-cycle, they must be distinguishable by some internal degree of freedom in q and q'. It would not appear in Q, meaning that it cannot appear in the δ's the coefficients of Q, e.g. $\delta_5 = 3a + 1 = \alpha + \alpha'$. Since the two forms must be equivalent, they may be like two states of parity, which would be in this case represented by a double-valued function. For 3-cycle in the logistic map it happens to be σ (13). Thus, we shall take the simplest possible form to construct α and α' with σ:

$$\alpha = 1/2\delta_5 + K\sigma , \quad \alpha' = 1/2\delta_5 - K\sigma , \tag{15}$$

where K is a constant to be determined. By Eq. (15), β, γ, and the primed ones are also given by Eqs. (10) and (11). Together they are to reproduce all the coefficients of Q correctly. In Appendix D, we prove that $K = 1/2$, giving

$$\alpha = 1/2(3a + 1 + \sigma) , \tag{16}$$
$$\beta = 1/2(a^2 + 2a - 1 + (a+1)\sigma) , \tag{17}$$
$$\gamma = 1/2(a^2 - a - 2 + a\sigma) , \tag{18}$$

and α', β', and γ' by taking $-\sigma$ in Eqs. (16)–(18). Hence if $q = q(\sigma)$, $q' = q(-\sigma)$. It is sufficient to solve only one of them. Since all the coefficients are now explicitly given in terms of a, q or q' can be solved by the formula due Cardano–Tartaglia or its variant versions. See Sec. 9.

7 Boundness of Roots by Trigonal Relation

That t_k or t'_k, $k = 1, 2, 3$, lies in the interval $(0, a)$ is the second condition for the existence of 3-cycle in the logistic map. We can prove it without actually solving q or q' by means of the trignonal relation. The proof goes as follows: Let $t_k = a - \epsilon$ for any k. It is sufficient to prove that $0 < \epsilon < a$. If $\tilde{a} \le a \le 4$, it was already proved (see Sec. 2) that $t_k > 0$. Hence $a - \epsilon > 0$, or $\epsilon < a$, which gives an upper bound on ϵ.

To obtain a lower bound on ϵ, we can construct α, β, and γ in terms of ϵ. Shown in Appendix E is:

$$\alpha - \beta + \gamma = -R(\epsilon)/(a - \epsilon) , \quad a - \epsilon > 0 , \tag{19}$$

where

$$R(\epsilon) = \epsilon^3 - A\epsilon^2 + B\epsilon - 1\,, \quad (20)$$

$$A = 1/2(3a - 1 - \sigma)\,, \quad (21)$$

$$B = 1/2(a^2 - 1 + \sigma - a\sigma)\,. \quad (22)$$

By the trigonal relation, $R(\epsilon) = 0$. If $\tilde{a} \le a \le 4$, A and $B > 0$ (note that $|\sigma| < 1$). To satisfy the trigonal relation, ϵ may not be negative. We obtain $\epsilon > 0$, a lower bound on ϵ. This proves that t_k is in the interval $(0, a)$ for all a in the range of a bounded by $\tilde{a}$ and 4.

8 Reflection Symmetry in q

Without actually solving q we find another interesting property. Let t be the solution for q at $a = 4$ and $\sigma = +1$. Also let $\tilde{t}$ the solution for q for $a = \tilde{a}$ for which $\sigma = 0$. As proved in Appendix F,

$$\tilde{q}(\tilde{\gamma} - \tilde{t}) = -q(t)\,, \quad (23)$$

where $\tilde{\gamma} = \gamma$ evaluated at $a = \tilde{a}$. The above symmetry says that, on the positive parity branch, the fixed points of 3-cycle at the boundaries of the 3-cycle domain are simply connected.

This symmetry turns out to be a special case of a more general symmetry that exists in q for $+\sigma$ centered on at $a = 1+(8+1/4)^{1/2} = 3.872\,281\,323\ldots$, evidently a special feature of the logistic map.

9 Cubic Solutions

Since all trigonals are now explicitly given in terms of a, we can proceed to solve q, the cubic Eq. (7a).

Let $\tau = t - \alpha/3$ and put Eq. (7a) in a reduced form:

$$\tau^3 - u\tau + v = 0\,, \quad (24)$$

where

$$u = \alpha^2/3 - \beta\,, \quad (25)$$

$$v = -2(\alpha/3)^3 + (\alpha/3)\beta - \gamma\,. \quad (26)$$

The two coefficients u and v are greatly simplified by a new function r, defined by

$$r = (7 - \sigma + \sigma^2)^{1/2}\,. \quad (27)$$

If $\tilde{a} \le a \le 4$, σ is real and $|\sigma| \le 1$. Thus in this interval of a, r is real and bounded by $3(3/4)^{1/2} < |r| < 3$. Let us choose + sign for r. In terms of r,

$$u = r^2/3\,, \quad (28)$$

$$v = -(1 - 2\sigma)u/9\,. \quad (29)$$

For solving the cubic equation (24) we do not follow the route of Cardano and Tartaglia. It can be done more simply by means of a multiple angle relation for $\cos 3\phi$. Let us change the variable once more, this time, τ to θ by

$$\tau = (4u/3)^{1/2} \cos\theta/3\,. \quad (30)$$

By Eq. (24), Eq. (30) is transformed into

$$4\cos^3\theta/3 - 3\cos\theta/3 - A = 0\,, \quad (31)$$

where

$$A = -(v/2)/(u/3)^{3/2} = (1 - 2\sigma)/2r\,. \quad (32)$$

By the multiple angle relation, $A = \cos\theta$. For a from $\tilde{a}$ to 4, A is real and $|A| < 1$ meaning that θ is real throughout this entire interval of a.

Replacing t by $x = t/a$, for any a from $\tilde{a}$ to 4, we can give solutions for (x_1, x_2, x_3), the first three roots of Q:

$$x_1 = \alpha/3a + 2r/3a\cos\theta_1/3\,, \quad (33)$$

where

$$\theta_1 = \cos^{-1}(1 - 2\sigma)/2r\,. \quad (34)$$

The solutions x_2 and x_3 are obtained from Eq. (33) by replacing θ_1 by $\theta_{2,3}$ where

$$\theta_{2,3} = \theta_1 \pm 2\pi\,. \quad (35)$$

The solutions for x'_1, x'_2, x'_3 are obtained from x_1, x_2, x_3 respectively. by $\sigma \to -\sigma$ therein.

Therewith we have obtained 6 positive real roots of Q, valid in the interval from $\tilde{a}$ to 4. They are prime fixed points of $f^3(x)$. Our exact analytic solutions prove that 3-cycle exists continuously in this interval.

10 Fixed Point Spectra and Cyclic Solutions

On a 2d space of x^* (vertical axis) vs. a (horizontal axis), we place the six prime fixed points of 3-cycle at $a \gtrsim \tilde{a}$. As a increases, the 6 points delineate 6 continuous non-intersecting lines of evolution in a. At each value of a the vertical line gives an x^*-spectrum, a discrete spectrum. The 2d space is a continuous juxtaposition of the x^*-spectra, a spectral diagram.

The prime fixed points of cycles 4, 5 and 6 say would similarly delineate continuous non-intersecting lines of evolution and discrete x^*-spectra. If all the fixed points of all possible cycles were delineated, the 2d space could be filled and the x^*-spectra possibly continuous. One could construct an x^*-spectrum at one of special values of 3-cycle such as $\sigma = 0$ ($a = \tilde{a}$), $\sigma = +1/2$, or $\sigma = +1$ or -1 ($a = 4$) or even at superstability or instability. At any one of them, a spectrum would have infinitely many points by the theorem of Sharkovskii and Li–Yorke.

Let us begin with the cubic solutions (Sec. 9). If $\sigma = -1$ ($a = 4$), $r = 3$ and $\theta_1 = \pi/3$. Hence 3 prime fixed points are: $\sin^2(\pi/9)$, $\sin^2(2\pi/9)$ and $\sin^2(4\pi/9)$. There is one non-prime fixed point $\sin^2(3\pi/9)$, which is the prime fixed point of 1-cycle.

The above result can be extended to other cycles. If $a = 4$, x^*_p the prime fixed points of n-cycle x^* are given by the cyclic form:[7] Writing $F(y^*_p) = \sin^2(\pi y^*_p/2)$

$$x^*_p = F(y^*_p)\,, \quad 0 < y^*_p < 1\,, \quad (36)$$

where by induction the cyclic values (pre fixed points) are given by

$$y_p^*/2 = l/(2N+1)\,. \tag{37}$$

Here $l = 1, 2, \ldots, N$ but excluding those reducible, i.e. $(2N+1)/l \neq$ odd integers save 1 (prime condition). The excluded will be called reducibles and the allowed irreducibles. The reducibles (e.g., 3/9) correspond to non-prime fixed points. If reducibles are included in Eq. (37), non-prime fixed points are also obtained by Eq. (36).

For n-cycle,

$$N = dn + r\,, \tag{38}$$

where d is the number of equivalent isocycles[7] and r the number of reducibles (not to be confused with r of Eq. (27)). (For example., for 3-cycle of $\sigma = -1$, $n = 3$, $d = 1$, and $r = 1$, giving $N = 4$ and $l = 1, 2, 4$.)

Equation (37) correctly yields all the cyclic values for cycles 3, 5, 6, 4, 2, 1.[7] See Table 1. With labor one could test it further with a few higher cycles like 7 and 8. For N very large, Eq. (37) is indirectly verified. See Appendix G. We shall assume that Eq. (37) is valid for any cycles.

We draw the following observations from Eq. (37) and Table 1: (i) Each y_p^* is unique. (ii) Each lies in the interval (0,1). (iii) Together they form a spectrum of their own (y^*-spectrum). Also see Appendix H.

Table 1 n cycle number, d degree of equivalent isocycles, r number of reducibles, $N = \mathrm{d}n + r$, $y_p^*/2$ cyclic values.

n	d	r	N	$y_p^*/2$
3	1	0	3	1/7 2/7 3/7
	1	1	4	1/9 2/9 4/9
5	1	0	5	1/11 2/11 4/11 3/11 5/11
	3	0	15	1/31 2/31 4/31 8/31 15/51
				3/31 6/31 12/31 7/31 14/51
				5/31 10/31 11/31 9/31 13/31
	2	6	16	1/33 2/33 4/33 8/33 16/33
				5/33 10/33 13/33 7/33 14/33
6	1	0	6	1/13 2/13 4/13 5/13 3/13 6/13
	1	4	10	1/21 2/21 4/21 8/21 5/21 10/21
	3	13	31	1/63 2/63 4/63 8/63 16/63 31/63
				5/63 10/63 20/63 23/63 17/63 29/63
				11/63 22/63 19/63 25/63 13/63 26/63
	4	8	32	1/65 2/65 4/65 8/65 16/65 32/65
				3/65 6/65 12/65 24/65 17/65 31/65
				7/65 14/65 28/65 9/65 18/65 29/65
				11/65 22/65 21/65 23/65 19/65 27/65
4	1	3	7	1/15 2/15 4/15 7/15
	2	0	8	1/17 2/17 4/17 8/17
				3/17 6/17 5/17 7/17
2	1	0	2	1/5 2/5
1	1	0	1	1/3

11 Sharkovskii Limit

For the dynamical studies, both the prime and non-prime fixed points of a cycle must be considered since it is $f^n(x^*) = x^*$ which enters into it. As already shown, the irreducible set gives the prime fixed points and the reducible set the non-prime ones. We shall thus concentrate on the irreducibles and the reducibles rather on the fixed points themselves.

To further illustrate the reducibles, consider 3/9, a reducible in 3-cycle. It also occurs as 5/15 in 4-cycle, 11/33 in 5-cycle, 7/21 in 6-cycle, all of which reduce to 1/3, an irreducible in 1-cycle. A reducible is an irreducible in a lower-numbered cycle. But different reducibles of the same irreducible (e.g. 3/9 and 7/21) belong to different cycles and have different cycle characteristics.

As the number of cycles increases, the number of irrreducibles grows and even more the number of reducibles. The y^*-spectrum begins to be filled with irreducibles and at every irreducible point there is an accumulation of reducibles "stacking up" on top of it like a column.

If the cycle number n is finite, N is finite. For N finite, the irreducibles are rational numbers with odd denominators. For the purpose of this work we shall call them "proper" rationals, denoted by Q. (Also those with even denominators "improper" rationals, denoted by Q'.) Both Q and Q' lie in the interval (0,1).

A proper rational say Q_M can be expressed as a finite sequence of some other smaller proper rationals, Q_1, $Q_1 + Q_2$, $Q_1 + Q_2 + Q_3$, etc., such that

$$Q_M = Q_1 + Q_2 + \cdots + Q_m \,, \tag{39}$$

where $Q_M > Q_1$, and $Q_1 > Q_2 > \cdots > Q_m$, $m < \infty$. See Refs. [16–17]. Except Q_M they may be reducibles out of the columns of accumulation.

Since 3-cycle exists, all cycles $1, 2, \ldots, n$, $n \to \infty$, exist by Sharkovskii's theorem. We shall call $N \to \infty$ the Sharkovskii limit (SL). It is analogous to the thermodynamic limit (TL) in statistical mechanics. See Appendix G.

If $N \gg 1$ on the right hand side of Eq. (37), finite contributions come from terms of $l = \nu N$, where ν is a rational number in the interval (0,1). If $\nu = 1/2$, $y_p^*/2 = 1/4(1 - 0(1/N))$. If $\nu = 2/3, y_p^*/2 = 1/3(1 - 0(1/N))$. In SL, both proper and improper rationals are obtained but in a limit process of increasing N. That is, they are approached in an infinite number of steps due to SL. This is in contrast to when N is finite, in which case the proper rationals are enumerated, not approached.

For a fixed finite N, let Q_1 and Q_2 be a pair of proper rational points closest and $\Delta Q = Q_1 - Q_2 > 0$ the interval between the two points. By definition there are no proper rationals in ΔQ. There are columns of reducibles at Q_1 and Q_2 separated by ΔQ.

If $N \to \infty$, $\Delta Q \to 0$ and the two columns rise with N. At the base Q_1 and Q_2 coalesce and from the base upward the columns coalesce, creating coalescing points (coalescents). They are distributed across the interval ΔQ in numbers increasing with N making it dense.

Since Q_1 and Q_2 are an arbitrary pair, the interval (0,1) may be said to consist of small dense intervals. This limit process can be given by Eq. (39) if it is made into an infinite sequence.

Improper rationals Q' are obtained from infinite sequences. If, for example, a sequence is given by s, $s + s^2$, $s + s^2 + s^3$, etc., where s is a proper rational

$$Q \to Q' = s/(1 - s) \,. \tag{40}$$

If $s = 1/5$ and $3/11$, $Q' = 1/4$ and $3/8$, respectively, as obtained in a limit process.

If I denotes irrationals of the interval (0,1), they are also obtained from infinite sequences as in the limit process, illustrated below:

(i) Sequence: $1/3^2$, $1/3^2 + 1/5^2$, $1/3^2 + 1/5^2 + 1/7^2$, etc.

$$Q \to I = (\pi^2/8 - 1) = 0.233\ 700\ 550\ 1 \cdots \,, \tag{41}$$

a Riemann ζ function.

(ii) Sequence: $1/3$, $1/3 + 8/21$, $1/3 + 8/21 + 13/51$, etc.

$$Q \to I = 1/\sqrt{2} - 1/3 = 0.373\ 773\ 444\ 8 \cdots \,, \tag{42}$$

where $1/\sqrt{2} = 1/(1 + 1/(2 + 1/(2 + \cdots)))$, a continued fraction.

(iii) Sequence: $1/3$, $1/3 + 1/3^2$, $1/3 + 1/3^2 + 2/3^4$, $1/3 + 1/3^2 + 2/3^4 + 1/3^5$, $1/3 + 1/3^2 + 2/3^4 + 1/3^5 + 2/5.3^6$, etc.

$$Q \to I = (\,\mathrm{e}^{2/3} - 1) = 0.473\ 867\ 02 \cdots \,, \tag{43}$$

which has an exponential.

Irrationals of π, e and $\sqrt{2}$ are thus generated by infinite sequences of proper rationals.

Every interval between a pair of nearest proper rationals is filled with improper rationals and irrationals. We conclude that, by SL,

$$y_p^* \to y^* \,, \quad y^* = (0, 1) \,,$$

where

$$y^* = Q + Q' + I \,. \tag{44}$$

If μ is the measure of an interval, $\mu(y^*) = \mu(I) = 1$ since $\mu(Q) = \mu(Q') = 0$.

12 Aleph Cycle

Correspondingly, $x_p^* \to x^*$ by replacing y_p^* by y^* in Eq. (36),

$$x^* = F(y^*) \,, \quad y^* = (0, 1) \,, \tag{45}$$

where y^* is given by Eq. (44), hence $x^* = (0, 1)$. Now by Eq. (1), $x = (0, 1)$ also. Thus x and x^* coincide. A point x_1 in (0,1) is also a fixed point x_1^* say of cyclic value y_1^*.

In Sec. 1 we have termed a trajectory periodic or chaotic by the iterative behavior of $f(x_n) = x_{n+1}, n = 1, 2, \ldots$ We now find that this iterative behavior is governed by the metrical property of its initial value under f. If it belongs to a set of points of measure 0, a trajectory is periodic. If it belongs to a set of points of measure 1, a trajectory is chaotic, said to be of an aleph cycle. As every initial point is a fixed point, every trajectory is necessarily initial-value sensitive whether periodic or chaotic. We shall now show that there is mixing in an aleph cycle.

13 Mixing, Ergodicity and Invariant Density

Mixing is a property of chaos. If the autocorrelation function decays to zero in the infinite time limit, there is said to be mixing.[18]

We can construct the correlation function with the fixed points, initially for N finite and then for $N \to \infty$ (SL). Since all reducibles and irreudcibles are included in SL, we will relax the prime condition on l and remove the subscript p from x_p^* and y_p^* in Eqs. (36) and (37).

We first need to symmetrize the fixed point spectrum i.e. the interval going from (0,1) to (−1/2,1/2) by making $y^* \to Y^* = y^* - 1/2$ and $x^* \to X^* = x^* - 1/2$. Also indicating the cyclic values (pre fixed points),

$$X_L^* = 1/2 \sin(\pi Y_L^*) \,, \tag{46}$$

$$Y_L^* = (2L - 1/2)/(2N + 1) \,, \tag{47}$$

where $L = l - N/2$.

We now introduce the autocorrelation function $\psi_N(t)$:

$$\psi_N(t) = 1/N \sum_L \text{Re} \exp(\mathrm{i}tX_L^*) \,. \tag{48}$$

By using Eqs. (43) and (44) and by taking SL, $\Psi(t) = \psi_N(t)$, $N \to \infty$, we obtain:

$$\Psi(t) = 1/\pi \int_{-\pi}^{\pi} \cos(t/2 \sin\theta) \mathrm{d}\theta \tag{49}$$

$$= \mathrm{J}_0(t/2) \,, \tag{50}$$

where J_0 is a Bessel function. Thus $\Psi(t \to \infty) = 0$, by which mixing is proved.

An aleph cycle is ergodic. According to the ergometric theory of the ergodic hypothesis,[19−20] a variable is ergodic if its $W \neq 0$, ∞, where

$$W = \int_0^{\infty} \Psi(t) \mathrm{d}t \,. \tag{51}$$

By Eq. (46), $W \neq 0$, ∞. Thus variables of an aleph cycle is ergodic. For a continuous map ergodicity has a stronger requirement than chaoticiy. It is possible that a variable in a continuous map may be chaotic but not ergodic.

One can also obtain the power spectrum or invariant density of an aleph cycle. If $\tilde{\Psi}(z) = T\Psi(t)$, where T is the Laplace transform operator, by Eq. (46)

$$\tilde{\Psi}\ (z) = 1/\sqrt{1/4 + z^2} \,. \tag{52}$$

If one defines the invariant density $\rho(\omega) = 1/\pi\tilde{\Psi}\ (z = \mathrm{i}\omega)$, such that $\int_{-\infty}^{\infty} \rho(\omega) \mathrm{d}\omega = 1$,

$$\rho(\omega) = \begin{cases} 1/\pi\sqrt{(1/4 - \omega^2)} \,, & -\dfrac{1}{2} < \omega < \dfrac{1}{2} \,, \\ 0 \,, & \text{if otherwise} \,. \end{cases} \tag{53}$$

If $\omega = 1/2 - \omega'$,

$$\rho(\omega') = 1/\pi\sqrt{\omega'(1 - \omega')} \,, \quad 0 < \omega' < 1 \,. \tag{54}$$

If $\omega' = x^*$, it is the invariant density at $a = 4$.[7] From Eq. (48) we note that there is a branch cut from $z = -(1/2)\mathrm{i}$ to $z = +(1/2)\mathrm{i}$. One may view it as an impervious line of irrational points.

14 Concluding Remarks

14.1 *Li–Yorke Chaos*

Sharkovskii, and Li and Yorke proved that the existence of 3-cycle in continuous maps implies the existence of all other cycles. We have taken it to mean that there are infinitely many cycles possible, permitting what we have termed the Sharkovskii limit (SL), similar to the thermodynamic limit (TL) in statistical mechanics.

The general solutions of the fixed points of 3-cycle in the logistic map are in cyclic form, giving special parametric values. At one such value, we obtain a fixed-point spectrum by applying SL and deduce from it the aleph cycles which yield chaotic trajectories.

We further show that aleph cycles are mixing. The invariant density is isomorphic to the spectral function for a harmonic chain in TL, therewith proving ergodicity. These results are all made possible by SL.

Sharkovskii "does not establish chaos in and of itself" although it follows from SL as shown by our special solutions. By the existence of all cycles, Li and Yorke formally proved the mixing and divergence behavior:

Mixing: If x_1 and x_2 both belong to an uncountable set contained in the interval,

$$\lim\inf_{(n\to\infty)} |f^n(x_1) - f^n(x_2)| = 0 \,, \tag{55}$$

$$\lim\sup_{(n\to\infty)} |f^n(x_1) - f^n(x_2)| > 0 \,. \tag{56}$$

Divergence: If x_1 belongs to an uncountable set, but x_2 not, both contained in the interval, the trajectory starting from x_1 diverges from that starts from x_2.

The behavior of a map that satisfies the above two properties is known in the literature as chaotic or chaos in the sense of Li and Yorke.

Let us test them with aleph cycles. Since two aleph cycles can come arbitrarily close to each other, the greatest lower bound on the separation distance is 0. The least upper bound on the separation distance has to be greater than 0 since their trajectories are unique, thus they cannot cross. The mixing property is satisfied. In Sec. 13 it was proved in another way.

The trajectory of a periodic (non-aleph) cycle returns to its initial value whereas that of an aleph cycle does not. Thus the trajectory of an aleph cycle necessarily diverges from that of a periodic cycle. It can be thus said that our special solutions realize chaos in the sense of Li and Yorke.

To be sure, our results are limited to one special parametric value. But all the fixed points are in cyclic form. In addition there is a reflection symmetry connecting one parametric value to another. They suggest that the same uncountable set of fixed points exists in all other spectra. The uncountable set may be contained in the interval, not necessarily coincident with the interval itself as when $a = 4$. It all means that our results realize chaos in the sense of Li and Yorke.

14.2 *Definitions of Chaos*

In the math literature one finds many definitions of chaos given by proposition. See Refs. [21–22]. Why so many? What Yorke has said, perhaps in an ironic jest, "People can still define chaos as they please," seems to epitomize the math culture.[23]

If one definition turns out to contradict another, how is it taken? In the math world one seems to have the luxury of deferring such a thorny issue by calling it chaos in one person's sense and not in another person's sense, therewith apparently raising no hackles.

But chaos is also a physical phenomenon which demands unambiguous definitions albeit equivalent ones. For example, the λ temperature T_λ of liquid He^4, the onset temperature of superfluidity, can be defined as the temperature where the difference in latent heat vanishes, or where the specific heat diverges.[24] Both are equivalent, ascribing different facets, and they do not contradict. There is no superfluidity in the sense of Onnes or Landau, just superfluidity.

A common definition of chaos is by positive Lyapunov exponent.[25] Another is by initial-value sensitivity. Yet another by transitivity. Let us apply them to the 3-cycle window, see Appendix I.

As is well known, here the Lyaupunov exponent[25] is not positive and trajectories seemingly not initial-value sensitive. By these definitions, there is no chaos in the 3-cycle window,[25] contrary to the theorem of Li and Yorke. What the Lyapunov exponent "sees" is 3-cycle of -parity only. It does not see 3-cycle of +parity, nor any other cycles of which there are infinitely many.

Metropolis *et al.*[26] have calculated a number of superstable points ranging from $a = 3.831\ 874$ to $a = 3.990\ 267\ 0$. Wherever a superstable point, there is a window centered on it. In every window there is one stable isocycle. In the spectral diagram, a fixed-point line which crosses the half-line ($x = 1/2$) is at a superstable point,[15] and there are infinitely many fixed-point lines crossing the half-line. Along the a-space there are infinitely many superstable points and infinitely many windows of almost all cycles. By the Lyapunov exponent the a-space looks like a 1d lattice of chaos and no chaos in perfect order like a lattice of spin up and spin down for an antiferromagnet. By Li–Yorke the space has no such lattice. So there clearly is a contradiction if not a difference throughout, not just at the 3-cycle window.[23,26]

Over this issue, Ruelle[27] seems to argue that practical necessity demands simple definitions like the Lyapunov exponent, not those bedeviled with math fineries. To our knowledge this issue is yet to be resolved.[22]

14.3 *Symptoms of Chaos*

The definitions of chaos found in the math texts seem to be based on what we would say symptoms rather than causes of chaos. Perhaps the following example from physics may help shed light.

Years ago several "symptoms" of the hydrogen atom were discovered in its atomic spectra, called the Balmer formula, the Paschen formula, etc.[28] They could have been used to define the hydrogen atom in the sense of Balmer, of Paschen and of others. The "senseless" definition of the hydrogen atom came later by solving the Schröedinger equation. The complete set of wavefunctions gives a fundamental or first-principles definition since it comes from a higher principle, that of quantum law.

Like other physical phenomena, chaos also shows symptoms e.g. exponential separation, initial-value sensitivity, transitivity. If definitions are built on them by proposition, they are not fundamental since they are not derived from a higher principle. What may be considered a higher principle of chaos as in the allegorical example of the hydrogen atom?

We would contend that, for continuous or continuously differentiable maps, a higher principle of chaos comprises of Sharkovskii's theorem and Li–Yorke's theorem. If this principle is satisfied by a map, there is chaos. In our view the two theorems collectively may be likened to the Schrodinger equation in quantum mechanics.

In this sense our solutions are of first principles like the wavefunctions for the hydrogen atom. Like these wavefunctions which imply the Balmer and other formulas, our solutions imply symptoms of chaos through aleph cycles. That Li and Yorke's theorem is satisfied serves to reinforce our contention.

14.4 *Windows*

Let us return to the issue on the windows and view it through another perspective. At every window there are uncountable cycles and just one stable isocyle of a cycle. If we consider the fixed-point spectrum at one point in a window, say the superstable point, the fixed points of one stable isocycle are contained in, not isolated from, the uncountable set of other cycles.

This fixed-point distribution recalls the spectral distribution in a harmonic chain with one variable mass.[29] If it is lighter than its neighboring ones, the spectrum consists of one resonant point and a continuum below it. If it is heavier, the resonant point falls into the continuum. A similar spectrum is also observed in the many-body RPA dynamics of a Coulomb gas at the ground state.[30] The spectrum consists of the plasmon mode and a continuum. At short wavelengths the plasmon mode disappears into the continuum. Physical processes like inelastic scattering are ruled by the continuum when the resonant mode is subsumed.

The spectrum at a point in a window is much like that of a harmonic chain or of a Coulomb gas consisting of a continuum in which a resonance is subsumed. By comparison the dynamics at a window should be governed by an uncountable set, giving rise to ergodicity among others. Not to confuse, one should perhaps call the windows not chaotic by the Lyapunov exponent or in the sense of Devaney or others but chaotic otherwise.

In our view, definitions of chaos are not fundamental if they do not come from higher principles of chaos. If definitions are made of symptoms, they can misdiagnose causes.

Appendix A: Trigonal Relation

By using $t = ax$, 3-cycle means:

$$t_2 = t_1(a - t_1)\,, \tag{A1}$$

$$t_3 = t_2(a - t_2)\,, \tag{A2}$$

$$t_1 = t_3(a - t_3)\,. \tag{A3}$$

First by taking (A1) + (A2) + (A3):

$$(a-1)\alpha + 2\beta - \alpha^2 = 0\,. \tag{A4}$$

Second, by taking (A1) × (A2) × (A3):

$$a^3 - 1 = a^2\alpha - a\beta + \gamma\,. \tag{A5}$$

There are two other relations, less elementary. By taking $t_1 \times$ (A1) + $t_2 \times$ (A2) + $t_3 \times$ (A3), we obtain (see Note 1):

$$-\alpha + \beta - \gamma + \{a\alpha + \alpha\beta - 2a\beta - 2\gamma\} = 0\,. \tag{A6}$$

Next, by taking [(A1)−(A2)]×[(A2)−(A3)]×[(A3)−(A1)] we obtain (see Note 2):

$$a^3 - 1 = 2a^2\alpha - a\alpha^2 - a\beta + \alpha\beta - \gamma\,. \tag{A7}$$

By subtracting Eq. (A7) by Eq. (A5),

$$a^2\alpha + \alpha\beta - a\alpha^2 - 2\gamma = 0\,. \tag{A8}$$

By Eq. (A4), Eq. (A8) is put in the form:

$$a\alpha + \alpha\beta - 2a\beta - 2\gamma = 0\,. \tag{A9}$$

Now by comparing the above with Eq. (A6), we obtain:

$$\alpha - \beta + \gamma = 0\,. \tag{A10}$$

Of the 6 roots for Eq. (2), the second set (t_1', t_2', t_3') say must also satisfy the trigonal relation in the form:

$$\alpha' - \beta' + \gamma' = 0\,. \tag{A11}$$

If the right hand side of Eq. (A5) is subtracted by Eq. (A10), we obtain:

$$(a-1)[(a^2 + a + 1) - (a+1)\alpha + \beta] = 0\,. \tag{A12}$$

Since $a > 1$ for 3-cycle to occur

$$\beta = (a+1)\alpha - (a^2 + a + 1)\,. \tag{A13}$$

Hence, by the trigonal relation,

$$\gamma = a\alpha - (a^2 + a + 1)\,. \tag{A14}$$

Similarly, β' and γ' are obtained from α'.

Note 1

By the identities:

$$t_1^2 + t_2^2 + t_3^2 = \alpha^2 - 2\beta\,, \tag{A15}$$

$$t_1^3 + t_2^3 + t_3^3 = \alpha^3 - 3\alpha\beta + 3\gamma\,, \tag{A16}$$

$$-\beta + a\alpha^2 - 2a\beta - \alpha^3 + 3\alpha\beta - 3\gamma = 0\,. \tag{A17}$$

By using Eq. (A4) on it twice, Eq. (A6) is obtained.

Note 2

By this process, the result can be put in the form:

$$1 = a^3 - a^2X + aY - Z\,, \tag{A18}$$

where by $u = \tilde{t}_{12}$, $v = \tilde{t}_{23}$, $w = \tilde{t}_{31}$, where $\tilde{t}_{ij} = t_i + t_j, i, j = 1, 2, 3,$

$$X = u + v + w = 2\alpha\,, \tag{A19}$$

$$Y = uv + vw + wu = \alpha^2 - \beta\,, \tag{A20}$$

$$Z = uvw = \alpha\beta - \gamma\,. \tag{A21}$$

To obtain Z we use another identity: $t_1^2\tilde{t}_{23} + t_2^2\tilde{t}_{31} + t_3^2\tilde{t}_{12} = \alpha\beta - 3\gamma$. If X, Y and Z are substituted in Eq. (A18), Eq. (A7) is obtained.

Appendix B: Delta Sum Rule

We prove Eq. (4) by repeatedly using the trigonal relation (9).

Using Eqs. (12b), (12c), (12d),

$$\delta_5 - \delta_4 + \delta_3 = -\alpha\alpha' + \alpha\beta' + \beta\alpha'\,, \tag{B1}$$

$$(B1) - \delta_2 = \beta\alpha' - \beta\beta' - \gamma\alpha'\,, \tag{B2}$$

$$(B2) + \delta_1 = -\gamma\alpha' + \gamma\beta'\,, \tag{B3}$$

$$(B3) - \delta_0 = 0\,. \qquad \text{QED} \tag{B4}$$

Appendix C: Transition Value by Trigonal Relation

At $a = \tilde{a}$, $q' = q$. Consequently, $Q = q^2$, where

$$q = \tilde{t}^3 - \tilde{\alpha}\tilde{t}^2 + \tilde{\beta}\tilde{t} - \tilde{\gamma}\,, \tag{C1}$$

where $\tilde{t} = \tilde{a}x$ and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ are α, β, γ evaluated at the transition point $\tilde{a}$. Thus, $\tilde{\delta}_6 = 1$, $\tilde{\delta}_5 = 2\tilde{\alpha}$, $\tilde{\delta}_4 = \tilde{\alpha}^2 + 2\tilde{\beta}$, $\tilde{\delta}_3 = 2\tilde{\alpha}\tilde{\beta} + 2\tilde{\gamma}$. It follows that:

$$\tilde{\alpha} = 1/2(3\tilde{a} + 1)\,, \tag{C2}$$

$$\tilde{\beta} = 1/2(3/4\tilde{a}^2 + 5/2\tilde{a} + 3/4)\,, \tag{C3}$$

$$\tilde{\gamma} = 1/2(-1/8\tilde{a}^3 + 7/8\,\tilde{a}^2 + 5/8\tilde{a} + 5/8)\,, \tag{C4}$$

The trigonal relation $\tilde{\alpha} - \tilde{\beta} + \tilde{\gamma} = 0$ implies that $(\tilde{a}+1)\tilde{\sigma}^2 = 0$, where

$$\sigma = (a^2 - 2a - 7)^{1/2}\,. \tag{C5}$$

Since $\tilde{a} \neq -1$, $\tilde{\sigma}^2 = \tilde{\sigma} = 0$ gives:

$$\tilde{a} = 1 + \sqrt{8}\,. \tag{C6}$$

Appendix D: Proof that $K = 1/2$

To see how may $\delta's$ be structured in α and α', consider $\delta_5 = \alpha + \alpha'$. Since the left hand side must be free of σ, we assume:

$$\alpha = 1/2\delta_5 + K\sigma\,, \tag{D1}$$

$$\alpha' = 1/2\delta_5 - K\sigma\,, \tag{D2}$$

where K is a constant to be determined for $\sigma \neq 0$. Evidently our assumed forms are the simplest possible by keeping σ linear.

Let us next turn to $\delta_4 = \alpha\alpha' + \beta + \beta'$. By Eq. (10), β and β' are obtained from α and α' respectively, given below

$$\beta = (a+1)(1/2\delta_5 + K\sigma) - (a^2 + a + 1)\,, \tag{D3}$$

$$\beta' = (a+1)(1/2\delta_5 - K\sigma) - (a^2 + a + 1)\,. \tag{D4}$$

We obtain

$$\delta_4 = 1/4\delta_5^2 - K^2\sigma^2 + (a+1)\delta_5 - 2(a^2+a+1) \\ = (1/4 - K^2)\sigma^2 + (a+1)\delta_5 \,. \quad \text{(D5)}$$

For the second term on the right hand side of Eq. (D5), $(a+1)\delta_5 = \delta_4$ by Eq. (2). Thus if $\sigma \neq 0$, $K = \pm 1/2$.

We choose + sign, which is sufficient since K and σ are conjugated. Our assumptions for α and α' with $K = +1/2$ can be tested on $\delta_3 = \alpha\beta' + \beta\alpha' + \gamma + \gamma'$.

By Eqs. (11), γ and γ' can be obtained from α and α', respectively

$$\gamma = a(1/2\delta_5 + K\sigma) - (a^2+a+1)\,, \quad \text{(D6)}$$

$$\gamma' = a(1/2\delta_5 - K\sigma) - (a^2+a+1)\,. \quad \text{(D7)}$$

We obtain $\delta_3 = a^3 + 5a^2 + 3a + 1$, which is exactly the same as given in Eq. (2). Similarly δ_2, δ_1 an0d δ_0 are all reproduced exactly.

Appendix E: Derivation of $R(\epsilon)$

Let $t_3 = a - \epsilon > 0$. Then,

$$\alpha = (a-\epsilon) + \tilde{t}_{12}\,, \quad \tilde{t}_{12} = t_1 + t_2\,, \quad \text{(E1)}$$

$$\beta = (a-\epsilon)\tilde{t}_{12} + t_1 t_2\,, \quad \text{(E2)}$$

$$\gamma = (a-\epsilon)t_1 t_2\,. \quad \text{(E3)}$$

By Eqs. (16) and (18), we can express $\tilde{t}_{12}$ and t_1t_2 in terms of a, σ and ϵ, hence also α, β and γ. Thus,

$$\alpha - \beta + \gamma = -R/(a-\epsilon)\,, \quad a - \epsilon > 0\,, \quad \text{(E4)}$$

where

$$R = \epsilon^3 - A\epsilon^2 + B\epsilon - 1\,, \quad \text{(E5)}$$

$$A = 1/2(3a - 1 - \sigma)\,, \quad \text{(E6)}$$

$$B = 1/2(a^2 - 1 - (a-1)\sigma)\,. \quad \text{(E7)}$$

Equation (E4) may be tested for one exactly known solution: If $\epsilon = 1/2a$, hence $t_k = 1/2a$, it corresponds to $x_k = 1/2$. If $x_3 = 1/2$ say, it represents one of the superstable 3-cycle fixed points.[17] If $\epsilon = 1/2\ a$ in R Eq. (E5), we obtain:

$$R = -(\sigma^3 - \sigma^2 + 7\sigma + 1)\,. \quad \text{(E8)}$$

The right hand side of Eq. (E8) vanishes as the condition for the superstability of 3-cycle.[19] See Appendix I.

Appendix F: Reflection Symmetry

At $a = \tilde{a}$,

$$\tilde{q}(\tilde{t}) = \tilde{t}^3 - \tilde{\alpha}t^2 + \tilde{\beta}\tilde{t} - \tilde{\gamma}\,. \quad \text{(F1)}$$

If $\tilde{t}$ is replaced by $\tilde{t} = \tilde{\gamma} - \boldsymbol{t}$ in Eq. (E1), it may be expressed as:

$$\tilde{q}(\tilde{\gamma} - \boldsymbol{t}) = -\boldsymbol{t}^3 + P\boldsymbol{t}^2 - Q\boldsymbol{t} + R\,, \quad \text{(F2)}$$

where

$$P = 3\tilde{\gamma} - \tilde{\alpha}\,, \quad \text{(F3)}$$

$$Q = 3(2\tilde{\alpha} + 7) - 2\tilde{\alpha}\tilde{\gamma} + 4\tilde{\gamma} - 7\,, \quad \text{(F4)}$$

$$R = \tilde{\gamma}(\tilde{\gamma}^2 - \tilde{\alpha}\tilde{\gamma} + \tilde{\beta} - 1)\,. \quad \text{(F5)}$$

From Eqs. (16)–(18) at $a = \tilde{a}$ for which $\sigma = 0$, we can establish several simple identities between $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. If a product (e.g. $\tilde{\alpha}\tilde{\gamma}$) is involved, it may be reduced to single terms by $\tilde{a}^2 = 2\tilde{a} + 7$. In this way P, Q, and R are made simple as shown below: First, $P = 7$ by the identity, $\tilde{\alpha} = 3\tilde{\gamma} - 7$. Second, $Q = 14$ by the identity $\tilde{\alpha}\tilde{\gamma} - 3\tilde{\alpha} - 2\tilde{\gamma} = 0$. By this identity and the trigonal relation,

$$R = \tilde{\gamma}(\tilde{\gamma}^2 - \tilde{\alpha} - \tilde{\beta} - 1)\,. \quad \text{(F6)}$$

By the identity $\tilde{\gamma}^2 = 2\tilde{\alpha} + 7$ and the trigonal relation again, and by the identity $\tilde{\gamma}^2 = 6\tilde{\gamma} - 7$. $R = \tilde{\gamma}(6 - \tilde{\gamma}) = 7$. Thus, Eq. (E2) is:

$$\tilde{q}(\tilde{\gamma} - \boldsymbol{t}) = -\boldsymbol{t}^3 + 7\boldsymbol{t}^2 - 14\boldsymbol{t} + 7\,. \quad \text{(F7)}$$

If $a = 4$ and $\sigma = 1$, $\alpha = 7$, $\beta = 14$, and $\gamma = 7$. Thus,

$$\tilde{q}(\tilde{\gamma} - \boldsymbol{t}) = -q(\boldsymbol{t})\,, \qquad \text{QED}\,. \quad \text{(F8)}$$

Appendix G: Local Dynamics of Harmonic Chain

To see whether a spectral property shown in Secs. 12 and 13 exists in other systems, we consider a harmonic chain of N equal masses in periodic boundary conditions (N even number, mass m and coupling constant κ) defined by

$$H = \sum_{-N/2}^{N/2-1} p_i^2/2m + 1/2\kappa(q_i - q_{i+1})^2\,, \quad \text{(G1)}$$

where sites $-N/2$ and $N/2 - 1$ are nns. The time evolution of the momentum at site 0 is given formally by $p_0(t) = \exp\mathcal{L}tp_0$, $\mathcal{L}$ the Liouville operator. By the recurrence relations method[31] in units $m = \kappa = 1$,

$$p_0(t) = a_0(t)p_0 + a_1(t)((q_{-1} + q_1)/2 - q_0) \\ + a_2(t)(p_{-1} + p_1) + \cdots\,, \quad \text{(G2)}$$

where $a_0 = \langle p_0(t)p_0\rangle/\langle p_0p_0\rangle$, the autocorrelation function of p_0. If it is given, the others $a_1, a_2, \ldots, a_N$ are obtained from it by a recurrence relation: $a_1 = -1/2\dot{a}_0$, $a_2 = -\dot{a}_1 + a_0$, etc. It is known[32–33] that

$$a_0(t) = \sum_{s=1}^{N/2-1} \cos\omega_s t + 0(1/N)\,, \quad \text{(G3)}$$

$$\omega_s = 2|\sin(\pi s/N)|\,, \quad s = -N/2, \ldots, 0, \ldots, N/2 - 1\,. \quad \text{(G4)}$$

If $N \to \infty$, Eq. (G3) is also known:[33–34] $a_0 = \mathrm{J}_0(2t)$, J_0 the Bessel function of order 0. Thus, by the recurrence relation, $a_1 = \mathrm{J}_1(2t)$, $a_2 = \mathrm{J}_2(2t), \ldots$, $a_N = \mathrm{J}_N(2t)$. If $t \to \infty$, $a_0 = \mathrm{J}_0(2t) \to 0$. But $\langle p_0(t)p_0(t)\rangle/\langle p_0p_0\rangle = 1$. The "length" of $p_0(t)$, hence its measure, is an invariant of t.

Let $\tilde{a}_0(z) = \boldsymbol{L}a_0(t)$, $\boldsymbol{L}$ the Laplace transform operator. If $a_0(t) = \mathrm{J}_0(2t)$,

$$\tilde{a}_0(z) = \frac{1}{\sqrt{4 + z^2}}\,. \quad \text{(G5)}$$

Thus, if $\tilde{a}_0(z = \mathrm{i}\omega) = \pi\rho(\omega)$,

$$\rho(\omega) = \begin{cases} \dfrac{1}{\pi\sqrt{4-\omega^2}}\,, & -2 < \omega < 2\,. \\ 0\,, & \text{if otherwise}\,. \end{cases} \tag{G6}$$

Equation (G4) is the dispersion relation if $N \to \infty$. If $N \to \infty$, the spectral density $\rho(\omega)$ can be calculated by $\mathrm{d}(s/N)/\mathrm{d}\omega$.

If $s/N = y - 1/2$ in Eq. (G4), $\omega = 2|1 - 2\sin^2(\pi y/2)|$, $0 < y < 1$. If $N \to \infty$, all the values of y lie continuously in the interval from 0 to 1. We can now identify $y/2$ as the cyclic values of an aleph cycle. Thus the two spectra are simply related by

$$\omega = 2|1 - 2x|\,, \quad 0 < x < 1\,. \tag{G7}$$

We can now conclude that the two different systems, one linear but with infinitely many degrees of freedom and the other not linear with one degree of freedom, are 1 to 1 in spectral properties and they have the same spectral measure. This isomorphic relationship implies that chaos also exists in a harmonic chain.

Chaos in a harmonic chain? Consider $p_0(t)$ or its autocorrelation function (G2) and (G3), respectively for $N \to \infty$. It is a superposition of all frequencies which are much like those of an aleph cycle. It is chaotic. Chaos is in the time evolutions of local variables like p_0 or q_0 if $N \to \infty$.

According to the ergometric theory,[19] p_0 is ergodic if $\tilde{a}_0$ satisfies the condition that $W \equiv \tilde{a}_0(z = 0) \neq 0, \infty$. By Eq. (G5), p_0 is ergodic, which means that it is also chaotic. The trajectories of an aleph cycle are chaotic and ergodic if starting from almost everywhere. We have thus proved that an aleph cycle of the logistic map at "fully developed chaos" and a local variable of an infinite harmonic chain are dynamically equivalent.[29]

Appendix H: Conjugacy

If $a = 4$ in Eq. (1), $x' = 4x(1 - x)$, $x = (0, 1)$. If a transformation $x = \sin^2(\pi t/2)$, $t = (0, 1)$, is applied, one obtains what is a special case ($\lambda = 1$) of the tent map $t' = g(t) = 2\lambda t$ if $t = (0, 1/2)$ and $2\lambda(1 - t)$ if $t = (1/2, 1)$, $0 < \lambda \leq 1$. Observe that the slope at $t = 1/2$ is not determinable. It changes abruptly as t goes from below 1/2 to above 1/2. (The abrupt change in slope can be removed if the interval $(1/2, 1)$ is reflected to $(1, 1/2)$, which results in the Bernoulli shift map, a discontinuous map[18]) We regard the tent map not continuous in this sense. It would mean that the tent map has no force of the theorem of Sharkovskii and Li–Yorke behind. It means that there is no SL for the tent map. If such a limit is taken nonetheless, it is purely arbitrary.

If $g^n(t^*) = t^*$ at $\lambda = 1$, $n = 1, 2, 3, \ldots, t^*$ is also given by the right hand side of Eq. (37). Without SL, $t^* = Q$ only. For example, if $t = 1/2^k$, $k = 1, 2, 3, \ldots$, none of them are fixed points of g^n. Thus, $\mu(t^*) = 0$ whereas $\mu(t) = 1$, and t^* is contained in t in (0,1). Not every point in t is a fixed point of a cycle. We contend that conjugacy of the tent map and the logistic map is limited by the fact that the former is a discontinuous map while the latter a continuous map. The discontinuousness can make the tent map behave aberrantly. When $\lambda = 1/2$, $t^* = t = (0, 1/2)$, but $t^* \neq t$ for (1/2,0).

To our knowledge, there is no Sharkovskii ordering in the tent map that is found in the logistic map. This is perhaps the most significant difference between the two maps.

Appendix I: On the Slope of 3-Cycle at the Fixed Points

The slope of 3-cycle at a fixed point can be obtained from $(f^3 - x)/(f - x) = q.q'$. See Eq. (5). Let $m(x_1) = \mathrm{d}f^3(x_1)/\mathrm{d}x$. Since $m(x_1) = m(x_2) = m(x_3)$, we will denote it simply by m. Then,

$$\begin{aligned} m &= \mathrm{d}/\mathrm{d}x\{x + (f(x) - x)q.q'\}_{x_1} \\ &= 1 - t_{12}\,\mathrm{d}q(t_1)/\mathrm{d}t q'(t_1)\,, \end{aligned} \tag{I1}$$

where $t_{12} = t_1 - t_2$ and, with $t = ax$, $af(x_1) = t_2$. Now $\mathrm{d}q(t_1)/\mathrm{d}t = -t_{12}t_{31}$, recalling that $q(t_1) = 0$, and $q'(t_1) = \sigma(t_1 - a)(t_1 - 1)$, proved in Note 3 below, where $\sigma = (a^2 - 2a - 7)^{1/2}$. Also $(t_1 - a)(t_1 - 1) = t_{23}/t_{12}$, proved in Note 4. Thus,

$$m = 1 + \sigma t_{12}t_{23}t_{31}\,. \tag{I2}$$

Observe that when $\sigma = 0$ ($a = \tilde{a}$), $m = 1$. Further proved in Note 5 are: $t_{12}t_{23}t_{31} = \alpha^2 - 3\beta = r^2$, where $r = (\sigma^2 - \sigma + 7)^{1/2}$, see Eq. (27). Thus we have arrived at the general formula for the slope of 3-cycle:

$$m = 1 + \sigma r^2\,. \tag{I3}$$

Equation (I3), a cubic equation in σ, is directly solvable for any values of a. It is thus to be preferred over the earlier results of Gordon[13] and Lee,[15] both of which are sextic equations in a, valid only for special values. Recall that $\sigma > 0$ and $\sigma < 0$ go with $\{x\}$ and $\{x'\}$, respectively. When $\sigma = 0$, $x_i = x'_i$, $i = 1, 2, 3$.

If $\sigma > 0$, $1 < m \leq 8$. The fixed points x_1, x_2, x_3 are never stable. If $\sigma < 0$, $-8 \leq m < 1$ for the fixed points x'_1, x'_2, x'_3. It is thus possible for them to have $m = -1$ (instability) at some value of σ. For these fixed points one can define the 3-cycle window $-1 \leq m \leq 1$, centered on $m = 0$ (superstability).

If $m = -1$ in Eq. (I3), we obtain in terms of $s = -\sigma > 0$:

$$s^3 + s^2 + 7s - 2 = 0\,. \tag{I4}$$

By $s = t - 1/3$, it can be put in the reduced form:

$$t^3 + 20/3t - 115/27 = 0\,. \tag{I5}$$

There is one real positive root required:

$$\begin{aligned} t = 1/3[&\{5/2(13 + \sqrt{1809})\}^{1/3} \\ &+ \{5/2(13 - \sqrt{1809})\}^{1/3}] \end{aligned}$$

$$= 0.605\ 576\ 993 \cdots \qquad \text{(I6)}$$

Thus,

$$s = t - 1/3 = 0.272\ 243\ 650 \cdots , \qquad \text{(I7)}$$

where t is given by Eq. (I6), and

$$a = 1 + \sqrt{8 + s^2} = 3.841\ 499\ 008 \cdots , \qquad \text{(I8)}$$

in agreement with Gordon.[13]

If $m = 0$ in Eq. (I3), we obtain in terms of $s = -\sigma > 0$:

$$s^3 + s^2 + 7s - 1 = 0 . \qquad \text{(I9)}$$

By $s = t - 1/3$ it is put in the reduced form:

$$t^3 + 20/3t - 88/27 = 0 . \qquad \text{(I10)}$$

There is one real positive root required:

$$t = 1/3[\{4(11 + \sqrt{621})\}^{1/3} + \{4(11 - \sqrt{621})\}^{1/3}]$$
$$= 0.473\ 013\ 915 \cdots \qquad \text{(I11)}$$

Thus,

$$s = t - 1/3 = 0.139\ 680\ 582 \cdots , \qquad \text{(I12)}$$

where t is given by Eq. (I11), and

$$a = 1 + \sqrt{8 + s^2} = 3.831\ 874\ 055 \cdots , \qquad \text{(I13)}$$

in agreemet with our earlier work.[15]

Note 3 $q'(t_1) = \sigma(t_1 - a)(t_1 - 1)$,

$$\text{lhs} = t_1^3 - \alpha' t_1^2 + \beta' t_1 - \gamma' .$$

By $q(t_1) = 0$, t_1^3 is replaced:

$$\text{lhs} = (\alpha-\alpha')t_1^2-(\beta-\beta')t_1+(\gamma-\gamma') = \sigma(t_1-a)(t_1-1).$$

QED

To obtain the above, Eqs. (16)–(17) have been used.

Note 4 $(t_1 - a)(t_1 - 1) = t_{23}/t_{12}$

lhs $= t_1(t_1 - a) - t_1 + a = a - t_1 - t_2$, where $t_2 = t_1(a - t_1)$ and also $t_3 = t_2(a - t_2)$. Hence, $t_2 - t_3 = (t_1 - t_2)(a - t_1 - t_2)$. QED

Note 5 $t_{12}t_{23}t_{31} = \alpha^2 - 3\beta$

lhs $= t_1(t_2^2 - t_3^2) + t_2(t_3^2 - t_1^2) + t_3(t_1^2 - t_2^2)$.

By the 3-cycle condition, $t_2 = t_1(a-t_1)$, $t_3 = t_2(a-t_2)$, $t_1 = t_3(a - t_3)$, we obtain: $t_2^2 - t_3^2 = at_{23} - t_{31}$, $t_3^2 - t_1^2 = at_{31} - t_{12}$, $t_1^2 - t_2^2 = at_{12} - t_{23}$.

Hence, lhs $= -t_1t_{31} - t_2t_{12} - t_3t_{23} = t_1^2 + t_2^2 + t_3^2 - t_3t_1 - t_1t_2 - t_2t_3 = \alpha^2 - 3\beta$. By Eqs. (16) and (17), or by Eq. (A4) $\alpha^2 - 3\beta = 1/4(3a^2 - 6a + 7 + \sigma^2 - 4\sigma) = r^2$, by the defining relations $a^2 - 2a = \sigma^2 + 7$ and $r^2 = \sigma^2 - \sigma + 7$.

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