Two-Body Scattering in $(1+1)$ Dimensions by a Semi-relativistic Formalism and a Hulthén Interaction Potential

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(Received November 3, 2014; revised manuscript received January 7, 2015)

Abstract *Scattering solutions of two-body Spinless Salpeter Equation (SSE) are investigated in the center of mass frame with a repulsive, symmetric Hulthén potential in one spatial dimension. Transmission and reflection coefficients are calculated and discussed.*

PACS numbers: 03.65.Ca, 03.65.Pm, 03.65.Nk
Key words: Spinless Salpeter Equation, transmission and reflection coefficients, Hulthén potential

1 Introduction

The Bethe–Salpeter equation (BSE), obtained from a relativistic field theory, provides us with a reliable basis to study many fermions.[1−6] The equation, however, appears as an integral equation which complicated in many cases. If we neglect the spin degrees of freedom and apply some approximations to the original equation, we are left with the so-called Spinless Salpeter Equation (SSE).[7−8] The latter, possesses two important features; it is an acceptable relativistic generalization of the Schrödinger equation, and, has a simpler structure in configuration space in comparison with the BSE. Until now, one has succeeded in transforming the integral equation into a differential equation. To be more precise, the reduced equation includes the kinetic term under a square root. References[9−15] have suggested novel ideas to obtain bounds on the energy eigenvalues or approximations to the wave function. On the other hand, the Hulthén potential[16] is one of the most important short range model potentials in the physics. This potential has been widely used in a number of areas such as nuclear and particle physics,[17] atomic physics,[18−19] molecular physics,[20−22] and chemical physics.[23] Unfortunately, quantum mechanical equations with the Hulthén potential can be solved analytically only for states with zero angular momentum,[16,24] i.e. the angular momentum quantum number l=0, so several techniques were used to obtain approximate solutions for non-relativistic Schrödinger equation with the Hulthén potential for $l=0$. Bound-state eigenvalues were solved numerically[25−26] and quasi-analytically using variational,[25,27] perturbation,[28] shifted $1/N$ expansion,[29−30] NU,[31−32] SUSYQM,[33−36] and AIM[37] methods. In the relativistic case Hulthén potentials have also been discussed in Refs. [38–46] and references therein.

The paper is organized as follows: Section 2 introduces the SSE in one spatial dimension. Next, we find our exact solutions in terms of Hypergeometric functions together with algebraic expressions for the transmission and reflection coefficients in Sec. 3. Here some numerical results are illustrated. Conclusions are in Sec. 4.

2 The Two-Body Hamiltonian

The spinless Salpeter equation for two particles with arbitrary masses m_1, m_2 interacting by a potential $V(x)$ in (1+1) dimensions and in the center of mass system appears as[1,7−9]

$$\Big[\sqrt{-\Delta+m_1^2}+\sqrt{-\Delta+m_2^2}+V(x)-M\Big]\Phi=0$$
$$\text{where } \Delta=\frac{\mathrm{d}^2}{\mathrm{d}x^2} \text{ and } M=E-m_1-m_2\,, \tag{1}$$

and $E-V$ Eq. (1) can be heuristically approximated by a local equation for small values of is then written as[13−14]

$$\Big[-\frac{\Delta}{2\mu}+V-\frac{\mu^2}{2\eta^3}(E-V)^2\Big]U(x)=EU(x) \tag{2}$$

where μ,η are defined by $\mu=m_1m_2/(m_1+m_2)$ and $\eta=\mu[m_1m_2/(m_1m_2-3\mu^2)]^{1/3}$, and U is the transformed wave function.

We consider the problem in (1+1) dimensions for the symmetric Hulthén potential[47−48] given by

$$V(x)=\frac{V_0}{\exp(\alpha|x|)-q}\,. \tag{3}$$

This potential is finite at the origin if $q<1$. The parameter α^{-1} defines the size of the potential barrier.

3 Reflection and Transmission Coefficients

As we are searching for the scattering states of the equation, we will first study the wave functions for $x<0$.

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From substitution of Eq. (3) into Eq. (2), we find

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\Big(E-\frac{V_0}{\exp(-ax)-q}\Big)\Big(2\mu+\Big(\frac{\mu}{\eta}\Big)^3\times\Big(E-\frac{V_0}{\exp(-ax)-q}\Big)\Big)\Big]U(x)=0\,. \tag{4}$$

In the single-particle limit $(\mu/\eta)^3=1$, and $\mu=m$, where m is the light mass. For equal masses, m, we have $(\mu/\eta)^3=1/4$ and $\mu=m/2$. In numerical application, Fig. 1 and Table 1, we consider the equal masses case and use $m=1$.

Applying the new variable $\mathrm{e}^{-ax}=q/y_L$ in Eq. (4), we find

$$\Big[y_L(1-y_L)\frac{\mathrm{d}^2}{\mathrm{d}y_L^2}+(1-y_L)\frac{\mathrm{d}}{\mathrm{d}y_L}+\frac{1}{y_L(1-y_L)}\times(y_L^2w_1+y_Lw_2+w_3)\Big]U_L(y_L)=0\,, \tag{5}$$

where

$$w_1=\frac{\mu^3E^2}{\eta^3\alpha^2}+\frac{\mu^3V_0^2}{\eta^3\alpha^2q^2}+\frac{2\mu^3EV_0}{\eta^3\alpha^2q}+\frac{2\mu V_0}{q\alpha^2}+\frac{2\mu E}{\alpha^2}\,,$$
$$w_2=-\frac{2\mu^3E^2}{\eta^3\alpha^2}-\frac{2\mu^3EV_0}{\eta^3\alpha^2q}-\frac{2\mu V_0}{\alpha^2q}-\frac{4\mu E}{\alpha^2}\,,$$
$$w_3=\frac{\mu^3E^2}{\eta^3\alpha^2}+\frac{2\mu E}{\alpha^2}\,.$$

A trial wave function is used to write the equation into the Hypergeometric form

$$U_L(y_L)=y_L^{\eta_L}(1-y_L)^{\lambda_L}\phi_L(y_L)\,. \tag{6}$$

This substitution enables us to write the wave functions as a sum of incoming, transmitted and reflected parts. Equation (5) now takes the form

$$y_L(1-y_L)\frac{\mathrm{d}^2\phi_L(y)}{\mathrm{d}y_L^2}+[1+2\eta_L-(2\eta_L+2\lambda_L+1)y_L]\frac{\mathrm{d}\phi_L(y)}{\mathrm{d}y_L}-(\eta_L^2+\lambda_L^2+2\eta_L\lambda_L+w_1)\phi_L(y_L)=0\,, \tag{7}$$

Eq. (7) is identified as the hyperpergeometric equation[48−54] on the left side ($x<0$) with solution

$$U_L(y_L)=A_1y_L^{\eta_L}(1-y_L)^{\lambda_L}{}_2F_1(a,b;d;y_L)+A_2y_L^{-\eta_L}(1-y_L)^{\lambda_L}{}_2F_1(a+1-d,b+1-d;2-d;y_L)\,, \tag{8}$$

where

$$d=2\eta_L+1,\quad a=\eta_L+\lambda_L-\mathrm{i}\sqrt{w_1},\quad b=\eta_L+\lambda_L+\mathrm{i}\sqrt{w_1},\quad \Omega=w_1+w_2+w_3,\quad \lambda_L=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\Omega}\,.$$

For $x>0$ by applying the new variable $\mathrm{e}^{ax}=\tilde{q}/y_R$ in Eq. (2), we have

$$\Big[y_R(1-y_R)\frac{\mathrm{d}^2}{\mathrm{d}y_R^2}+(1-y_R)\frac{\mathrm{d}}{\mathrm{d}y_R}+\frac{1}{y_R(1-y_R)}(y_R^2w_4+y_Rw_5+w_6)\Big]U_R(y_R)=0\,, \tag{9}$$

where

$$w_4=\frac{\mu^3E^2}{\eta^3\alpha^2}+\frac{\mu^3V_0^2}{\eta^3\alpha^2\tilde{q}^2}+\frac{2\mu^3EV_0}{\eta^3\alpha^2\tilde{q}}+\frac{2\mu V_0}{\tilde{q}\alpha^2}+\frac{2\mu E}{\alpha^2},\quad w_5=-\frac{2\mu^3E^2}{\eta^3\alpha^2}-\frac{2\mu^3EV_0}{\eta^3\alpha^2\tilde{q}}-\frac{2\mu V_0}{\alpha^2\tilde{q}}-\frac{4\mu E}{\alpha^2},\quad w_6=\frac{\mu^3E^2}{\eta^3\alpha^2}+\frac{2\mu E}{\alpha^2}\,.$$

Next putting $U_R(y_R)=y_R^{\eta_R}(1-y_R)^{\lambda_R}\phi_R(y_R)$, we may write the wave function for ($x>0$)

$$U_R(y_R)=A_3y_R^{\eta_R}(1-y_R)^{\lambda_R}{}_2F_1(a',b';d';y_R)+A_4y_R^{-\eta_R}(1-y_R)^{\lambda_R}{}_2F_1(a'+1-d',b'+1-d';2-d';y_R)\,, \tag{10}$$

where

$$d'=2\eta_R+1,\quad a'=\eta_R+\lambda_R-\mathrm{i}\sqrt{w_4},\quad b'=\eta_R+\lambda_R+\mathrm{i}\sqrt{w_4},\quad \Omega'=w_4+w_5+w_6,\quad \lambda_R=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\Omega'}\,.$$

We are interested in the case where the particles are initially incident on the potential from the left. Since no wave is reflected from the region $x>0$ to the left, the constant A_3 must vanish and we have

$$U_R(y_R)=A_4y_R^{-\eta_R}(1-y_R)^{\lambda_R}{}_2F_1(a'+1-d',b'+1-d';2-d';y_R)\,. \tag{11}$$

To proceed further, let us consider the asymptotic behaviors of the wave functions in both sides, i.e.

$$x\to-\infty,\quad y_L\to0,\quad (1-y_L)^{\lambda_L}\to1,\quad y_L^{\pm\eta_L}\to q^{\pm\eta_L}\,\mathrm{e}^{\pm\alpha x\eta_L},$$
$$x\to\infty,\quad y_R\to0,\quad (1-y_R)^{\lambda_R}\to1,\quad y_R^{\mp\eta_R}\to\tilde{q}^{\mp\eta_R}\,\mathrm{e}^{\pm\alpha x\eta_R}.$$

As a result, we write the total wave function as

$$U(x)\to\begin{cases}A_1q^{\eta_L}\,\mathrm{e}^{ax\eta_L}+A_2q^{-\eta_L}\,\mathrm{e}^{-ax\eta_L}, & x\to-\infty\,,\\ A_4\tilde{q}^{-\eta_R}\,\mathrm{e}^{ax\eta_R}, & x\to+\infty\,.\end{cases} \tag{12}$$

In order to obtain the explicit expressions for A_2 and A_4 introduced in Eqs. (8) and (11), we have to use the continuity conditions of wave functions and their derivatives, i.e. $U_L(x=0)=U_R(x=0)$ and $U'_L(x=0)=U'_R(x=0)$, which respectively yield

$$A_1\delta_1^{\eta_L}{}_1\delta_2^{\lambda_L}\xi_1+A_2\delta_1^{-\eta_L}\delta_2^{\lambda_L}\xi_2=A_4\delta_3^{-\eta_R}\delta_3^{\lambda_R}\xi_3\,, \tag{13a}$$

and

$$A_1\Big[\eta_L\delta_1^{\eta_L-1}\delta_2^{\lambda_L}\xi_1-\lambda_L\delta_1^{\eta_L}\delta_2^{\lambda_L-1}\xi_1+\delta_1^{\eta_L}\delta_2^{\lambda_L}\frac{(a)(b)}{d}\xi_4\Big]$$

$$+A_2\Big[-\eta_L\delta_1^{-\eta_L-1}\delta_2^{\lambda_L}\xi_2-\lambda_L\delta_1^{-\eta_L}\delta_2^{\lambda_L-1}\xi_2+\delta_1^{-\eta_L}\delta_2^{\lambda_L}\frac{(a+1-d)(b+1-d)}{2-d}\xi_5\Big]$$
$$=A_4\Big[-\eta_R\delta_3^{-\eta_R-1}\delta_4^{\lambda_R}\xi_3-\lambda_R\delta_3^{-\eta_R}\delta_4^{\lambda_R-1}\xi_3+\delta_3^{-\eta_R}\delta_4^{\lambda_R}\frac{(a'+1-d')(b'+1-d')}{2-d'}\xi_6\Big]\,, \tag{13b}$$

with

$$\delta_1=q,\quad \delta_2=1-q,\quad \delta_3=\tilde{q},\quad \delta_4=1-\tilde{q}\,. \tag{14a}$$

and

$$\begin{aligned}\xi_1&={}_2F_1(a,b;d;\delta_1)\,,\\ \xi_2&={}_2F_1(a+1-d,b+1-d;2-d;\delta_1)\,,\\ \xi_3&={}_2F_1(a'+1-d',b'+1-d';2-d';\delta_3)\,,\\ \xi_4&={}_2F_1(a+1,b+1;d+1;\delta_1)\,,\\ \xi_5&={}_2F_1(a+2-d,b+2-d;3-d;\delta_1)\,,\\ \xi_6&={}_2F_1(a'+2-d',b'+2-d';3-d';\delta_3)\,.\end{aligned} \tag{14b}$$

The wave function in Eq. (8) can be written as $U_L=U_{\rm inc}+U_{\rm ref}$ for $x\to-\infty$ where $U_{\rm inc}$ is the incident wave and $U_{\rm ref}$ is the reflected wave. Similarly, as $x\to\infty$ the wavefunction in Eq. (10) is $U_R=U_{\rm trans}$ where $U_{\rm trans}$ is the transmitted wave and in non-relativistic quantum mechanics, the transmission coefficient and reflection coefficient are used to describe the behavior of waves incident on a barrier. The transmission coefficient represents the probability flux of the transmitted wave relative to that of the incident wave. It is often used to describe the probability of a particle tunneling through a barrier. R and T, the transmission and reflection coefficients are defined in terms of current density as

$$R=\Big|\frac{\text{reflected current density}}{\text{incident current density}}\Big|=\Big|\frac{J_{\rm ref}}{J_{\rm inc}}\Big|\,, \tag{15a}$$

$$T=\Big|\frac{J_{\rm trans}}{J_{\rm inc}}\Big|\,, \tag{15b}$$

where the indices ref, inc, and trans stand for reflected, incident, and transmitted, respectively. To calculate R and T, we need to find $J_{\rm ref}$, $J_{\rm inc}$, and $J_{\rm trans}$. We use Eq. (16) for the incident current density (or incident flux) as well as reflected and transmitted fluxes

$$\vec{J}=\frac{\hbar}{2mi}(U^*\vec{\nabla}U-U\vec{\nabla}U^*)\,. \tag{16}$$

The current density for the left and right hand sides respectively are $\vec{J}_L=\vec{J}_{\rm inc}+\vec{J}_{\rm ref}$, and $\vec{J}_R=\vec{J}_{\rm trans}$ so the incoming, reflected and transmitted fluxes are

$$\vec{J}_{\rm inc}=\frac{\hbar}{2mi}(U^*_{\rm inc}\vec{\nabla}U_{\rm inc}-U_{\rm inc}\vec{\nabla}U^*_{\rm inc})\,,$$
$$\vec{J}_{\rm ref}=\frac{\hbar}{2mi}(U^*_{\rm ref}\vec{\nabla}U_{\rm ref}-U_{\rm ref}\vec{\nabla}U^*_{\rm ref})\,,$$
$$\vec{J}_{\rm trans}=\frac{\hbar}{2mi}(U^*_{\rm trans}\vec{\nabla}U_{\rm trans}-U_{\rm trans}\vec{\nabla}U^*_{\rm trans})\,. \tag{17}$$

By using Eqs. (13)–(17) in summary, the reflection and transmission coefficients are

$$R=\frac{J_{\rm ref}}{J_{\rm inc}}=\frac{U^*_{\rm ref}\vec{\nabla}U_{\rm ref}-U_{\rm ref}\vec{\nabla}U^*_{\rm ref}}{U^*_{\rm inc}\vec{\nabla}U_{\rm inc}-U_{\rm inc}\vec{\nabla}U^*_{\rm inc}}=\Big|\frac{A_2}{A_1}\Big|^2\,,$$
$$T=\frac{J_{\rm trans}}{J_{\rm inc}}=\frac{U^*_{\rm trans}\vec{\nabla}U_{\rm trans}-U_{\rm trans}\vec{\nabla}U^*_{\rm trans}}{U^*_{\rm inc}\vec{\nabla}U_{\rm inc}-U_{\rm inc}\vec{\nabla}U^*_{\rm inc}}=\Big|\frac{A_4}{A_1}\Big|^2. \tag{18}$$

The reflection coefficient R, and the transmission coefficient T, are obtained as

$$\Big|\frac{A_2}{A_1}\Big|^2=\Big|\frac{g_4g_3-g_1g_6}{g_2g_6-g_5g_3}\Big|^2,\qquad \Big|\frac{A_4}{A_1}\Big|^2=\Big|\frac{g_4g_2-g_1g_5}{g_6g_2-g_3g_5}\Big|^2, \tag{19}$$

where

$$g_1=\delta_1^{\eta_L}\delta_2^{\lambda_L}\xi_1,\quad g_2=\delta_1^{-\eta_L}\delta_2^{\lambda_L}\xi_2,\quad g_3=\delta_3^{-\eta_R}\delta_4^{\lambda_R}\xi_3\,,$$
$$g_4=\eta_L\delta_1^{\eta_L-1}\delta_2^{\lambda_L}\xi_1-\lambda_L\delta_1^{\eta_L}\delta_2^{\lambda_L-1}\xi_1+\frac{ab}{d}\delta_1^{\eta_L}\delta_2^{\lambda_L}\xi_4\,,$$
$$g_5=-\eta_L\delta_1^{-\eta_L-1}\delta_2^{\lambda_L}\xi_2-\lambda_L\delta_1^{-\eta_L}\delta_2^{\lambda_L-1}\xi_2+\frac{(a+1-d)(b-d+1)}{(2-d)}\delta_1^{-\eta_L}\delta_2^{\lambda_L}\xi_5\,,$$
$$g_6=-\eta_R\delta_3^{-\eta_R}\delta_4^{\lambda_R}\xi_3-\lambda_R\delta_3^{-\eta_R}\delta_4^{\lambda_R-1}\xi_3+\frac{(a'+1-d')(b'-d'+1)}{(2-d')}\delta_3^{-\eta_R}\delta_4^{\lambda_R}\xi_6\,.$$

If we put $(\mu^2/2\eta^3)(E-V)^2\ll(E-V)$ or $(\mu^2/2\eta^3)(E-V)\ll 1$ in Eq. (2) transmission and reflection coefficients by Schrödinger equation for the Hulthén potential will be obtain.[54]

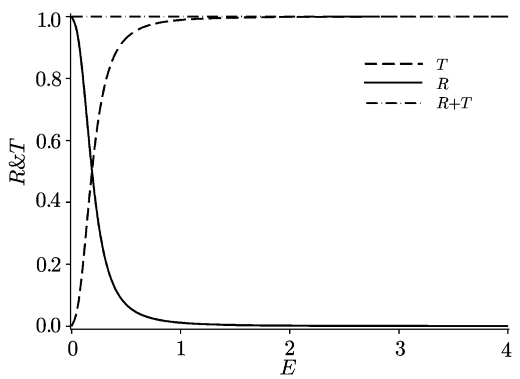

Fig. 1 Reflection and Transmission coefficients versus energy for $\mu=0.5$, $\eta=0.79$, $\alpha=0.5$, $V_0=0.2$, $q=\tilde{q}=0.5$.

Table 1 Transmission coefficient as a function of selected energies for $\mu=0.5$, $\eta=0.79$, $\alpha=0.5$, $V_0=0.2$, $q=\tilde{q}=0.5$.

E	$T(E)$
0.1	0.181 68
0.2	0.540 86
0.3	0.771 33
0.4	0.879 11

As a typical example in Fig. 1 we have plotted R, T versus E, as we expect, the relation $R+T=1$ is satisfied. Table 1 shows some relevant numerical values of T versus E.

4 Conclusion

In the present work, we obtain the exact solutions of (1+1)-dimensional two-body Spinless Salpeter equation for the Hulthén potential in terms of hypergeometric functions. Using the basic properties of hypergeometric functions, we obtain formulas for calculating both transmission and reflection coefficients.

Acknowledgments

We wish to give our sincere gratitude to the referee for his technical comments.

References

[1] E.E. Salpeter and H.A. Bethe, Phys. Rev. **84** (1951) 1232.
[2] G.C. Wick, Phys. Rev. **96** (1954) 1124.
[3] P. Maris and C.D. Roberts, Int. J. Mod. Phys. E **12** (2003) 297.
[4] L. Chang and C.D. Roberts, Phys. Rev. Lett. **103** (2009) 081601.
[5] P. Maris and C.D. Roberts, Phys. Rev. C **56** (1997) 3369.
[6] N. Nakanishi, Prog. Theor. Phys. Suppl. **43** (1969) 1.
[7] W. Lucha and F.F. Schöberl, Int. J. Mod. Phys. A **14** (1999) 2309.
[8] W. Lucha and F.F. Schöberl, Fiz. B **8** (1999) 193.
[9] W. Lucha and F.F. Schöberl, Int. J. Mod. Phys. A **17** (2002) 2233.
[10] W. Lucha and F.F. Schöberl, Phys. Rev. A **54** (1996) 3790.
[11] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. E **17** (2008) 1107.
[12] W. Lucha and F.F. Schöberl, Phys. Rev. D **50** (1994) 5443.
[13] S. Hassanabadi and A.A. Rajabi, Mod. Phys. Lett. A **27** (2012) 1250057.
[14] S. Zarrinkamar, A.A. Rajabi, H. Hassanabadi, and H. Rahimov, Phys. Scr. **84** (2011) 065008.
[15] S. Zarrinkamar, A.A. Rajabi, and H. Hassanabadi, Few-Body Sys. **52** (2011) 165.
[16] L. Hulthén, Ark. Mat. Astron. Fys. A **28** (1942) 5.
[17] B. Durand and L. Durand, Phys. Rev. D **23** (1981) 1092.
[18] R. Dutt, K. Chowdhury, and Y.P. Varshni, J. Phys. A: Math. Gen. **18** (1985) 1379.
[19] T. Xu, Z.Q. Cao, Y.C. Ou, Q.S. Shen, and G.L. Zhu, Chin. Phys. **15** (2006) 1172.
[20] I.S. Bitensky, V.K. Ferleger, and I.A. Wojciechowski, Nucl. Instrum. Methods, Phys. Res. Sect. B **125** (1997) 201.
[21] C.S. Jia, J.Y. Wang, S. He, and L.T. Sun, J. Phys. A: Math. Gen. **33** (2000) 6993.
[22] J.X. Sun and L.Y. Zhang, Acta Phys. Sin. **45** (1996) 1953.
[23] J.A. Olson and D.A. Micha, J. Chem. Phys. **68** (1978) 4252.
[24] S. Flügge, *Practical Quantum Mechanics*, Vol. 1, Springer, Berlin (1994).
[25] Y.P. Varshni, Phys. Rev. A **41** (1990) 4682.
[26] M.A. Nunez, Phys. Rev. A **47** (1993) 3620.
[27] S.H. Patil, J. Phys. A: Math. Gen. **34** (2001) 3153.
[28] P. Matthys and H. De Meyer, Phys. Rev. A **38** (1988) 1168.
[29] A.Z. Tang and F.T. Chan, Phys. Rev. A **35** (1987) 911.
[30] B. Roy and R. Roychoudhury, J. Phys. A: Math. Gen. **20** (1987) 3051.
[31] S.M. Ikhdair and R. Sever, J. Math. Chem. **42** (2007) 461.
[32] M. Aktas and R. Sever, J. Mol. Struct. **710** (2004) 219.
[33] E.D. Filho and R.M. Ricotta, Mod. Phys. Lett. A **10** (1995) 1613.
[34] B. Gönül, O. Özer, Y. Cancelik, and M. Kocak, Phys. Lett. A **275** (2000) 238.
[35] S.W. Qian, B.W. Huang, and Z.Y. Gu, New J. Phys. **4** (2002) 13.
[36] B. Gönül, Chin. Phys. Lett. **21** (2003) 1685.
[37] O. Bayrak, G. Kocak, and I. Boztosun, J. Phys. A: Math. Gen. **39** (2006) 11521.
[38] F. Dominguez-Adame, Phys. Lett. A **136** (1989) 175.
[39] L. Chetouani, L. Guechi, A. Lecheheb, T.F. Hammann, and A. Messouber, Physica A **234** (1996) 529.
[40] B. Talukdar, A. Yunus, and M.R. Amin, Phys. Lett. A **141** (1989) 326.
[41] S.Z. Hu and R.K. Su, Acta Phys. Sinica **40** (1991) 1201 (in Chinese).
[42] J.Y. Guo, J. Meng, and F.X. Xu, Chin. Phys. Lett. **20** (2003) 602.
[43] A.D. Alhaidari, J. Phys. A **37** (2004) 5805.
[44] C.Y. Chen, D.Sh. Sun, and F.L. Lu, Phys. Lett. A **370** (2007) 219.
[45] D. Agboola, Phys. Scr. **80** (2009) 065304.
[46] N. Saad, Phys. Scr. **76** (2007) 6.
[47] K. Sogut and A. Havare, J. Phys. A **43** (2010) 225204.
[48] G.F. Wei, W.L. Chen, H.Y. Wang, and Y.Y. Li, Chin. Phys. B **18** (2009) 3633.
[49] O. Aydodu, A. Arda, and R. Sever, J. Math. Phys. **53** (2012) 102111.
[50] CHEN Chang-Yuan, SUN Dong-Sheng, LIU Cheng-Lin, and LU Fa-Lin, Commun. Theor. Phys. **55** (2011) 399.
[51] Z. Molaee, H. Hassanabadi, and S. Zarrinkamar, Commun. Theor. Phys. **60** (2013) 25.
[52] C.Y. Chen, F.L. Lu, and D.S. Sun, Acta Phys. Sin. **56** (2007) 6204 (in Chinese).
[53] H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar, and N. Salehi, Few-Body Systems **54** (2013) 2009.
[54] A. Arda, O. Aydogdu, and R. Sever, arXiv:1107.3204v1 [math-ph] 16 Jul. 2011.