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CRE Solvability, Nonlocal Symmetry and Exact Interaction Solutions of the Fifth-Order Modified Korteweg-de Vries Equation*

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Abstract *This paper is concerned with the fifth-order modified Korteweg-de Vries (fmKdV) equation. It is proved that the fmKdV equation is consistent Riccati expansion (CRE) solvable. Three special form of soliton-cnoidal wave interaction solutions are discussed analytically and shown graphically. Furthermore, based on the consistent tanh expansion (CTE) method, the nonlocal symmetry related to the consistent tanh expansion (CTE) is investigated, we also give the relationship between this kind of nonlocal symmetry and the residual symmetry which can be obtained with the truncated Painlevé method. We further study the spectral function symmetry and derive the Lax pair of the fmKdV equation. The residual symmetry can be localized to the Lie point symmetry of an enlarged system and the corresponding finite transformation group is computed.*

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Key words: fifth-order modified Korteweg-de Vries equation, soliton-cnoidal wave interaction solution, nonlocal symmetry, consistent Riccati expansion

1 Introduction

Over the last few decades, constructing exact solutions for nonlinear evolution equations (NLEEs) have become an attractive topic in nonlinear science. Up to now, many effective approaches have been established to obtain exact solutions of NLEEs. Some of them are respectively the inverse scattering transformation,[1] the Darboux and Bäcklund transformations,[2−3] Hirota's bilinear method,[4] Painlevé analysis,[5] symmetry reductions,[6] the homogeneous balance method,[7] the tanh method[8] and the separated variable method,[9] and so on. However, these methods are insufficient to find interaction solutions among different nonlinear excitations.

Recently, it is found that the residue of truncated Painlevé expansion with respect to the singular manifold is just the nonlocal symmetry, which is called residual symmetry.[10−11] According to the novel results of the symmetry reduction with nonlocal symmetries, Lou[12] further proposed the consistent Riccati expansion (CRE) method. The CRE method can be used to identify CRE solvable systems and it is a more direct but much simpler method to find interaction solutions between a soliton and other nonlinear waves, such as the soliton-cnoidal waves, soliton-perodic waves, soliton-error function waves, and soliton-rational waves.[12−22]

In this paper, we would like to consider the following fifth-order modified Korteweg-de Vries (fmKdV) equation

$$u_t = u_{xxxxx} - 10u^2u_{xxx} - 40uu_xu_{xx} - 10u_x^3 + 30u^4u_x\,, \quad (1)$$

which possesses a close connection with the known fifth-order KdV equation

$$v_t = v_{xxxxx} + 10vv_{xxx} + 20v_xv_{xx} + 30v^2v_x\,, \quad (2)$$

by the Miura transformation

$$v = u_x - u^2\,, \quad (3)$$

that will convert Eq. (2) to Eq. (1). It is well known that the known fifth-order KdV equation has wide application in Physics, so the study of Eq. (1) is being of potential application in Physics besides the academic interest. The fmKdV equation (1) is a higher-order equation of the mKdV hierarchy, the Lax pair and bi-Hamiltonian structure were studied in Ref. [23]. In Ref. [24], a semidiscrete version for the fmKdV equation (1) was constructed from the three known semidiscrete mKdV fluxes. Kwak[25] proved the local well-posedness of the fmKdV equation (1) for low regularity Sobolev initial data via the energy method.

The paper is organised as follows. In Sec. 2, the CRE method is applied to prove the fmKdV equation is CRE solvable. In Sec. 3, starting from the last consistent differential equation, three special form of interaction solutions between the soliton and the cnoidal periodic wave of

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this equation are presented both analytically and graphically. In Sec. 4, the nonlocal symmetry related to the CTE and the nonlocal residual symmetry of the fmKdV equation are obtained. Also, the relationship between them is given. Then the corresponding finite transformation group is obtained by the localization of residual symmetry to the Lie point symmetry. The last section is a summary and discussion.

2 CRE Solvability and CTE Solvability

2.1 *CRE Solvability*

In this section we apply the CRE method in Ref. [21] to Eq. (1). According to the leading order analysis, the solution u is selected as the following ansatz ($R \equiv R(w)$)

$$u = u_1 R + u_0 \,, \tag{4}$$

where u_0, u_1, and w are functions of (x,t), and R is a solution of the Riccati equation

$$R_w = a_0 + a_1 R + a_2 R^2 \,, \tag{5}$$

which admits a solution $\tanh(w)$.

Substituting Eq. (4) with Eq. (5) into Eq. (1) and vanishing all the coefficients of R^i for all i, we obtain seven overdetermined differential equations with only three undetermined functions. It is fortunate that these overdetermined equations are consistent. As a result, we have

$$u_1 = -a_2 w_x \,, \quad u_0 = -\frac{1}{2}a_1 w_x - \frac{1}{2}\frac{w_{xx}}{w_x} \,, \tag{6}$$

and the function w satisfies a generalization of the Schwarzian form of Eq. (1)

$$\begin{aligned} &\mathcal{C} = \mathcal{S}_{xx} + \frac{3}{2}S^2 - \frac{5}{2}\delta(\mathcal{S}w_x^2 + w_{xx}^2) + \frac{3}{8}\delta^2 w_x^4 \,, \\ &\delta = a_1^2 - 4a_0 a_2 \,, \end{aligned} \tag{7}$$

where the notations C and S are defined as

$$\mathcal{C} = \frac{w_t}{w_x} \,, \quad \mathcal{S} = \frac{w_{xxx}}{w_x} - \frac{3}{2}\frac{w_{xx}^2}{w_x^2} \,. \tag{8}$$

From the definition in Ref. [12], we deduce that the fmKdV equation is CRE solvable.

2.2 *CTE Solvability*

We consider a special solution of the Riccati equation (5) as follows

$$R = \tanh(w) \,, \tag{9}$$

the truncated expansion expression (4) is converted to

$$u = u_0 + u_1 \tanh(w) \,, \tag{10}$$

where u_0, u_1, and w are determined by Eqs. (6) and (7) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $\delta = 4$. Hence, we obtain

$$u_1 = w_x \,, \quad u_0 = -\frac{1}{2}\frac{w_{xx}}{w_x} \,, \tag{11}$$

and w needs to satisfy

$$\mathcal{C} = \mathcal{S}_{xx} + \frac{3}{2}S^2 - 10(\mathcal{S}w_x^2 + w_{xx}^2) + 6w_x^4 \,. \tag{12}$$

From above, it shows that the fmKdV equation is consistent tanh expansion (CTE) solvable. It is obvious that a CRE solvable system must be consistent tanh expansion (CTE) solvable, and vice versa.

In summary, we can establish the following nonauto-BT theorem for Eq. (1).

Theorem 1 If w is a solution of Eq. (12), then,

$$u = w_x \tanh(w) - \frac{1}{2}\frac{w_{xx}}{w_x} \tag{13}$$

is a solution of the fmKdV equation (1).

3 Exact Solutions from Theorem 1

By means of Theorem 1, we can derive some exact solutions of the fmKdV equation (1), in particularly the interaction solutions between one soliton and other kinds of complicated waves. Next, some special types of solutions are given.

3.1 *Soliton Solution*

A quite trivial straight line solution for w has the form

$$w = k_1 x + \omega_1 t + d \,, \tag{14}$$

where k_1 and d are the free constants, and ω_1 is determined by the dispersion relation

$$\omega_1 = 6k_1^5 \,. \tag{15}$$

Substituting Eq. (14) into the CTE result (13) leads to one kink soliton solution

$$u = k_1 \tanh(k_1 x + 6k_1^5 t) \,. \tag{16}$$

3.2 *Soliton-Cnoidal Wave Interaction Solutions*

To find out the soliton-cnoidal wave interaction solutions, we assume w in the form

$$w = k_1 x + \omega_1 t + W \,, \quad W \equiv W(\xi) \,, \quad \xi = k_2 x + \omega_2 t \,. \tag{17}$$

Substituting Eq. (17) into Eq. (12), we can find that W_1 satisfies

$$\begin{aligned} &W_{1\xi}^2 = C_0 + C_1 W_1 + C_2 W_1^2 + C_3 W_1^3 + C_4 W_1^4 \,, \\ &W_1 \equiv W_\xi \,, \end{aligned} \tag{18}$$

with

$$\begin{aligned} C_4 &= 4 \,, \\ C_0 &= \frac{k_1[k_2(C_1 k_2^2 - C_2 k_1 k_2 + C_3 k_1^2) - 4k_1^3]}{k_2^4} \,, \\ \omega_1 &= \frac{1}{2}C_1 k_2^4 (C_2 k_2 - 4C_3 k_1) \\ &\quad - \frac{1}{8}C_2 k_1 k_2^3 (5C_2 k_2 - 26C_3 k_1) \\ &\quad + \frac{1}{8}C_3 k_1^3 k_2 (304k_1 - 21C_3 k_2) \end{aligned}$$

$$+10k_1^2k_2^2(2C_1k_2-3C_2k_1)-104k_1^5\,,$$
$$\omega_2=2k_1k_2^3(4C_1k_2+C_2k_1)+22k_1^3k_2(4k_1-C_3k_2)-\frac{1}{8}k_2^5(4C_1C_3-3C_2^2)-\frac{5}{8}C_3k_1k_2^3(2C_2k_2-3C_3k_1)\,, \tag{19}$$

while all the other constants remain free. Then the explicit solution of the fmKdV equation writes as

$$u=(k_1+k_2W_1)\tanh(k_1x+\omega_1t+W)-\frac{1}{2}\frac{k_2^2W_{1\xi}}{k_1+k_2W_1}\,. \tag{20}$$

It is known that the solutions of Eq. (18) can be expressed in terms of Jacobi elliptic functions. Thus, the solution (20) reveals the interactions between one soliton and cnoidal periodic waves. In the following we will list three nontrivial cases to obtain this kind of solution.

Case 1 The first simple solution of Eq. (18) is given by

$$W_1=\mu_0+\mu_1\mathrm{sn}(m\xi,n)\,. \tag{21}$$

Substituting Eqs. (19) and (21) into Eq. (18) yields

$$C_1=2\mu_0[m^2(n^2+1)-8\mu_0^2]\,,$$
$$C_2=-m^2(n^2+1)+24\mu_0^2\,,\quad C_3=-16\mu_0\,,$$
$$k_1=\frac{1}{2}k_2(m-2\mu_0)\,,\quad \mu_1=\frac{1}{2}mn\,. \tag{22}$$

Then the exact soliton-cnoidal wave interaction solution can be derived as

$$u=-\frac{1}{2}k_2m(nS+1)\tanh\Big[-\frac{1}{2}k_2mx+\frac{1}{16}k_2^5m^5(5n^4-50n^2-3)t-\frac{1}{2}\ln(D-nC)-\lambda\Big]-\frac{1}{2}\frac{k_2mnCD}{nS+1}\,, \tag{23}$$

where $\{k_2,m,n,\lambda\}$ are arbitrary constants, $\xi=(1/8)k_2[8x+k_2^4m^4(3n^4+2n^2+43)t]$. Hereafter, S, C and D are the usual Jacobian elliptic functions sn, cn and dn with modulus n, respectively.

Figure 1 plots one kink soliton in the cnoidal periodic wave background expressed by Eq. (23), and the parameters are fixed at

$$k_1=-0.5\,,\quad k_2=\mu_0=m=1\,,\quad \mu_1=0.25\,,\quad n=0.5\,,\quad \lambda=0\,,\quad \omega_1=-4.511\,718\,8\,,\quad \omega_2=5.460\,937\,5\,. \tag{24}$$

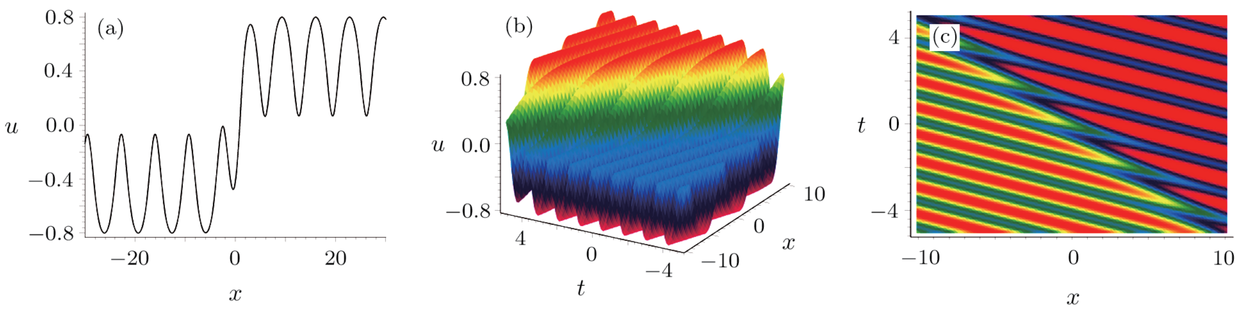

Fig. 1 The first special form of soliton-cnoidal wave interaction solution of u expressed by Eq. (23) with the parameters being fixed at Eq. (24). (a) The soliton-cnoidal wave structure at $t=0$; (b) The evolution of the soliton-cnoidal wave structure; (c) The density plot for the soliton-cnoidal wave structure.

Case 2 As the second example, we consider the solution of Eq. (18) as

$$W=cE_\pi(\mathrm{sn}(\xi,n),\mu,n)\,, \tag{25}$$

which leads to the soliton-cnoidal wave interaction solution of Eq. (1):

$$u=-\frac{k_1+ck_2-\mu k_1S^2}{\mu S^2-1}\tanh[k_1x+\omega_1t+cE_\pi(\mathrm{sn}(\xi,n),\mu,n)]+\frac{c\mu k_2^2SCD}{(\mu S^2-1)(k_1+ck_2-\mu k_1S^2)}\,, \tag{26}$$

where $\{k_2,\mu,n\}$ are three independent constants, $E_\pi(\zeta,\mu,n)$ is the third type of incomplete elliptic integral, and

$$c=\frac{\sqrt{\mu(\mu-1)(\mu-n^2)}}{\mu}\,,\quad k_1=\frac{\sqrt{\mu(\mu-1)(\mu-n^2)}k_2}{\mu(\mu-1)}\,,$$
$$\omega_1=-\frac{2k_2^5[(\mu^2+4\mu-8)n^4-2(2\mu^2-\mu-4)n^2-3]\sqrt{\mu(\mu-1)(\mu-n^2)}}{\mu(\mu-1)^3}\,,$$
$$\omega_2=\frac{2k_2^5[(3\mu^2+4\mu+8)n^4-2(4\mu^2+7\mu+4)n^2+8\mu^2+4\mu+3]}{(\mu-1)^2}\,. \tag{27}$$

Figure 2 shows the structure of the soliton-cnoidal wave interaction solution (26) with the parameters chosen as $k_2=\mu=0.5$ and $n=1.3$.

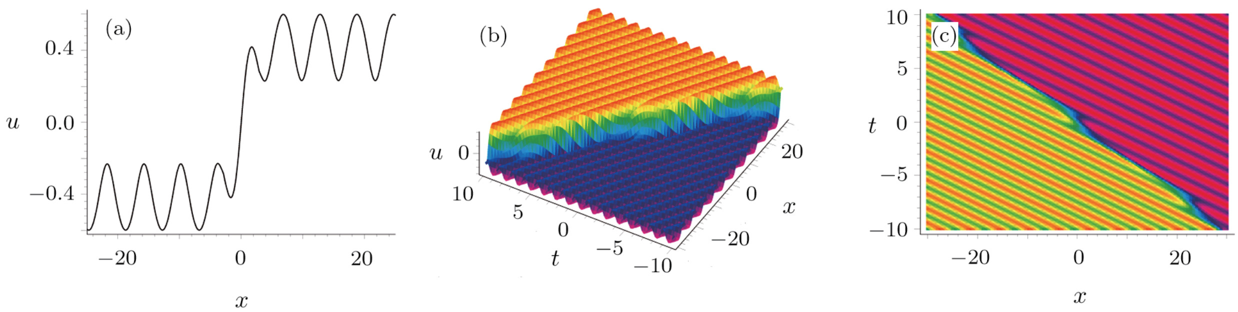

Fig. 2 The second special form of soliton-cnoidal wave interaction solution of u generated by Eq. (26) with the parameters chosen as $k_2 = \mu = 0.5$ and $n = 1.3$. (a) The soliton-cnoidal wave structure at $t = 0$; (b) The dynamical evolution of the soliton-cnoidal wave; (c) The density plot for time evolution.

Case 3 The third special solution of Eq. (18) is taken as the form

$$W = A\text{arctanh}[\text{sn}(\xi, n)]\,, \tag{28}$$

in this case, the interaction solution for Eq. (1) is obtained as:

$$u = \frac{k_2^2 + 4k_1^2(1 - S^2) - k_2^2 n^2 S^2 + 4k_1 k_2 CD}{2[2k_1(1 - S^2) + k_2 CD]} \tanh\Big[k_1 x + \omega_1 t + \frac{1}{2}\text{arctanh}(S)\Big] + \frac{k_2^2(n^2 - 1)S}{2\big[2k_1(1 - S^2) + k_2 CD\big]}\,, \tag{29}$$

where k_1 and n are two arbitrary constants, and

$$A = \frac{1}{2}\,, \quad k_2 = \frac{2k_1}{n}\,, \quad \omega_1 = \frac{2k_1^5(3n^4 + 50n^2 - 5)}{n^4}\,, \quad \omega_2 = \frac{4k_1^5(43n^4 + 2n^2 + 3)}{n^5}\,. \tag{30}$$

Figure 3 displays the third special form of soliton-cnoidal wave interaction solution for the field u given by Eq. (29) with the parameters determined as $k_1 = 0.6$ and $n = 1.5$.

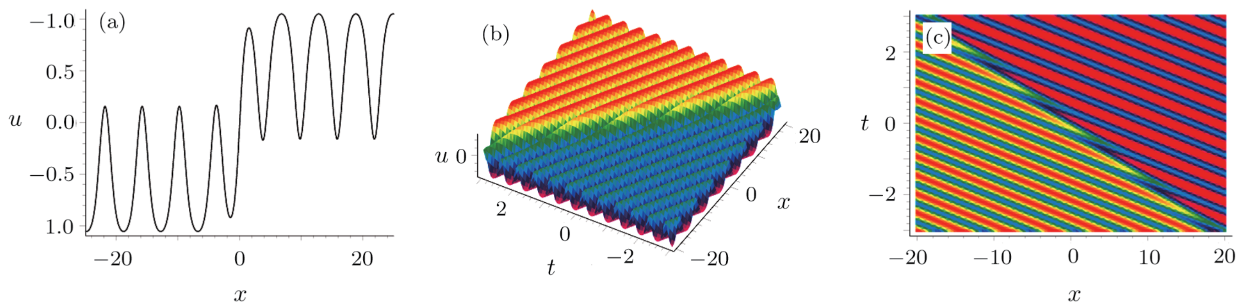

Fig. 3 The third special form of soliton-cnoidal wave interaction solution of u given by Eq. (29) with the parameters determined as $k_1 = 0.6$ and $n = 1.5$. (a) One-dimensional image at $t = 0$; (b) The corresponding three-dimensional view; (c) The density plot for soliton-cnoidal wave u.

In the ocean, there are some typical nonlinear waves such as soliton-cnoidal periodic wave. The interaction solutions may be useful for describing many more interesting physical phenomena, such as the Fermionic quantum plasma.[26]

4 Nonlocal Symmetry and Its Localization

4.1 *Nonlocal Symmetry*

Symmetries, including nonlocal symmetries, play an important role in nonlinear mathematical physics. In this subsection, the nonlocal symmetries of the fmKdV equation (1) will be studied. To seek the nonlocal symmetry related to the CTE, a nonauto-BT theorem for the fmKdV equation (1) is given as

Theorem 2 If w is a solution to Eq. (12), then the fmKdV equation (1) has a solution

$$u = -w_x - \frac{1}{2}\frac{w_{xx}}{w_x}\,. \tag{31}$$

Proof By direct calculation for substituting Eq. (31) into the fmKdV equation (1) by using the w equation (12). □

It is known that a symmetry σ^u of the fmKdV equation (1) is defined as a solution of its linearized equation

$$\sigma_t^u - \sigma_{xxxxx}^u + 10u^2\sigma_{xxx}^u + 40uu_x\sigma_{xx}^u + 10(4uu_{xx} +3u_x^2 - 3u^4)\sigma_x^u + 20(uu_{xxx} + 2u_xu_{xx} - 6u^3u_x)\sigma^u = 0\,. \quad (32)$$

That means Eq. (1) is form invariant under the infinitesimal transformation

$$u \to u + \epsilon\sigma^u\,, \quad (33)$$

with ϵ being an infinitesimal parameter.

Proposition 1 The fmKdV equation (1) possesses a nonlocal symmetry

$$\sigma^u = w_x \mathrm{e}^{2w}\,, \quad (34)$$

where w satisfies Eq. (12).

Proof By direct calculation for substituting (34) into Eq. (32) by using the nonauto-BT (31) in Theorem 2 and the w Eq. (12). □

Now, we make the following transformation

$$\phi = \frac{1}{1 - \tanh(w)}\,. \quad (35)$$

Substituting Eq. (35) into Eq. (34) leads to

$$\sigma^u = \phi_x\,, \quad (36)$$

which is the residual symmetry of Eq. (1).

Here we can derive the residual symmetry (36) from the truncated painlevé expansion. For the fmKdV equation (1), we truncate the Laurent series as

$$u = \frac{u_0}{\phi} + u_1\,, \quad (37)$$

where $\phi = \phi(x,t)$ is the singular manifold, and functions u_0 and u_1 are determined from the requirement for solution u to satisfy Eq. (1).

Substituting Eq. (37) into Eq. (1) and comparing the coefficients of each powers of $1/\phi$, we can simply find

$$u_0 = \phi_x\,, \quad u_1 = -\frac{1}{2}\frac{\phi_{xx}}{\phi_x}\,, \quad (38)$$

and the Schwarzian form of Eq. (1)

$$\mathcal{C} = \mathcal{S}_{xx} + \frac{3}{2}S^2\,, \quad (39)$$

with the Schwarzian derivative

$$C = \frac{\phi_t}{\phi_x}\,, \quad S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}\,. \quad (40)$$

The Schwarzian form (39) is invariant under the Möbious transformation

$$\phi \to \frac{a + b\phi}{c + d\phi}\,, \quad (ad \neq bc)\,, \quad (41)$$

which means the function ϕ possesses the Lie point symmetry in the form of

$$\sigma^\phi = b_0 + b_1\phi + b_2\phi^2\,, \quad (42)$$

with arbitrary constants b_0, b_1, and b_2. From the above standard truncated Painlevé expansion, we have the following nonauto-BT theorem.

Theorem 3 If the field ϕ is a solution of the Schwarzian equation (39), then

$$u = -\frac{1}{2}\frac{\phi_{xx}}{\phi_x} \quad (43)$$

is a solution of the fmKdV equation (1).

Proof By direct verification for substituting Eq. (43) into the fmKdV equation (1) with the help of the Schwarzian equation (39).

Based on the definition of residual symmetry,[10] Eq. (36) is the residual symmetry of Eq. (1). The residual symmetry (36) can be also obtained by using Schwarzian form (39) and nonauto-BT (43) in Theorem 3.[20,27–29]

It should be noted that the solution ϕ of the Schwarzian equation (39) is just the spectral function related to u, therefore, Eq. (36) is also the spectral function symmetry of Eq. (1). It is straightforward to derive the Lax pair of Eq. (1) as follows:

$$\phi_{xx} + 2u\phi_x = 0\,,$$
$$\phi_t + 2(u_{xxx} + 2uu_{xx} - u_x^2 - 6u^2u_x - 3u^4)\phi_x = 0\,, \quad (44)$$

which is simpler than the result in Ref. [23].

4.2 *Localization of Residual Symmetry*

According to the Lie's first theorem, the initial value problem related with the nonlocal residual symmetry (36) will be expressed as

$$\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{\phi_x}(\varepsilon)\,, \quad \hat{u}(0) = u\,. \quad (45)$$

It is difficult to solve the initial value problem (45) due to the intrusion of the function $\hat{\phi}(\varepsilon)$ and its differentiation.[11] To eliminate the space derivative of the field ϕ, the potential field f is defined as

$$f = \phi_x\,. \quad (46)$$

Now the nonlocal residual symmetry of Eq. (1) is localized to a Lie point symmetry

$$\sigma^u = f\,, \quad \sigma^f = -2\phi f\,, \quad \sigma^\phi = -\phi^2\,, \quad (47)$$

for the related prolonged system

$$u_t = u_{xxxxx} - 10u^2u_{xxx} - 40uu_xu_{xx} - 10u_x^3 + 30u^4u_x\,,$$
$$u = -\frac{1}{2}\frac{\phi_{xx}}{\phi_x}\,, \quad f = \phi_x\,, \quad (48)$$

with the Lie point symmetry vector

$$V = f\frac{\partial}{\partial u} - 2\phi f\frac{\partial}{\partial f} - \phi^2\frac{\partial}{\partial \phi}\,. \quad (49)$$

The initial value problem (45) is correspondingly transformed

$$\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{f}(\varepsilon)\,, \quad \hat{u}(0) = u\,,$$
$$\frac{\mathrm{d}\hat{f}(\varepsilon)}{\mathrm{d}\varepsilon} = -2\hat{\phi}(\varepsilon)\hat{f}(\varepsilon)\,, \quad \hat{f}(0) = f\,,$$

$$\frac{\mathrm{d}\hat{\phi}(\varepsilon)}{\mathrm{d}\varepsilon} = -\hat{\phi}^2(\varepsilon)\,, \quad \hat{\phi}(0) = \phi\,. \tag{50}$$

The solution of the above initial value problem (50) leads to the following BT theorem for the prolonged system (48).

Theorem 4 If $\{u, f, \phi\}$ is a solution of the prolonged system (48), so is $\{\hat{u}, \hat{f}, \hat{\phi}\}$ with

$$\hat{u}(\varepsilon)=u+\frac{f\varepsilon}{1+\phi\varepsilon},\ \ \hat{f}(\varepsilon)=\frac{f}{(1+\phi\varepsilon)^2},\ \ \hat{\phi}(\varepsilon)=\frac{\phi}{1+\phi\varepsilon}. \tag{51}$$

It is worth noticing that the nonlocal residual symmetry (36) is just the infinitesimal form of the symmetry group transformation (51). Furthermore, if we set

$$1+\phi\varepsilon = \phi\,, \quad f\varepsilon = \phi_x\,, \tag{52}$$

then the first equation of Eq. (51) is nothing but the truncated Painlevé expansion (37) with Eq. (38).

5 Summary and Discussion

In summary, the fmKdV equation is proved to be CRE integrable and abundant interaction solution between the soliton and the cnoidal periodic waves including arbitrary constants are obtained. Meanwhile, for the fmKdV equation, the nonlocal symmetry related to the CTE is derived. Under the transformation $\phi = 1/(1-\tanh(w))$, this kind of nonlocal symmetry is changed as the residual symmetry which can be obtained obtained with the truncated Painlevé method. We find that the residual symmetry is just the spectral function symmetry and derive the Lax pair of the fmKdV equation. To solve the initial value problem related by the residual symmetry, the residual symmetry is readily localized to Lie point symmetry by introducing multiple new dependent variables, the corresponding finite transformation group is found by solving the initial value problem of the Lie's first principle.

In addition, the CRE method is a powerful method for dealing with exact interaction solutions to NLEEs. The relationship between the CRE and the consistent sine-cosine expansion is an interesting problem, and we hope to investigate it further in the future.

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References

[1] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19** (1967) 1095.

[2] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).

[3] C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, Cambridge (2002).

[4] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.

[5] J. Weiss, M. Taboe, and G. Carnevale, J. Math. Phys. **24** (1983) 522.

[6] S. Y. Lou, Phys. Lett. A **151** (1990) 133.

[7] M. L. Wang, Y. B. Zhou, and Z. B. Li, Phys. Lett. A **216** (1996) 67.

[8] W. Malfliet and W. Hereman, Phys. Scr. **54** (1996) 563.

[9] X. Y. Tang, S. Y. Lou, and Y. Zhang, Phys. Rev. E **66** (2002) 046601.

[10] S. Y. Lou, Residual Symmetries and Bäcklund, Transformations, arXiv:1308.1140v1.

[11] X. N. Gao, S. Y. Lou, and X. Y. Tang, J. High Energy Phys. **05** (2013) 029.

[12] S. Y. Lou, Stud. Appl. Math. **134** (2015) 372.

[13] S. Y. Lou, X. P. Cheng, and X. Y. Tang, Chin. Phys. Lett. **31** (2014) 070201.

[14] C. L. Chen and S. Y. Lou, Chin. Phys. Lett. **30** (2013) 110202.

[15] W. G. Cheng and B. Li, Z. Naturforsch. A. **71** (2016) 351.

[16] W. G. Cheng and B. Li, Adv. Math. Phys. **2016** (2016) 4874392.

[17] B. Ren, J. Yu, and X. Z. Liu, Commun. Theor. Phys. **65** (2016) 341.

[18] W. G. Cheng, B. Li, and Y. Chen, Commun. Theor. Phys. **63** (2015) 549.

[19] D. Yang, S. Y. Lou, and W. F. Yu, Commun. Theor. Phys. **60** (2013) 387.

[20] B. Ren, Phys. Scr. **90** (2015) 065206.

[21] Y. H. Wang, Appl. Math. Lett. **38** (2014) 100.

[22] X. R. Hu and Y. Q. Li, Appl. Math. Lett. **51** (2016) 20.

[23] A. Choudhuri, B. Talukdar, and U. Das, Z. Naturforsch. A. **64** (2009) 171.

[24] T. Zhou, Z. N. Zhu, and P. He, Sci. China Math. **56** (2013) 123.

[25] C. Kwak, J. Differential Equations **260** (2016) 7683.

[26] A. J. Keane, A. Mushtaq, and M. S. Wheatland, Phys. Rev. E **83** (2011) 066407.

[27] W. G. Cheng, B. Li, and Y. Chen, Commun. Nonlinear Sci. Numer. Simul. **29** (2015) 198.

[28] B. Ren, X. P. Cheng, and J. Lin, Nonlinear Dyn. **86** (2016) 1855.

[29] B. Ren, Commun. Nonlinear Sci. Numer. Simulat. **42** (2017) 456.