

Analytical Investigation of Soliton Solutions to Three Quantum Zakharov-Kuznetsov Equations

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Abstract In the present paper, the two-dimensional quantum Zakharov-Kuznetsov (QZK) equation, three-dimensional quantum Zakharov-Kuznetsov equation and the three-dimensional modified quantum Zakharov-Kuznetsov equation are analytically investigated for exact solutions using the modified extended tanh-expansion based method. A variety of new and important soliton solutions are obtained including the dark soliton solution, singular soliton solution, combined dark-singular soliton solution and many other trigonometric function solutions. The used method is implemented on the Mathematica software for the computations as well as the graphical illustrations.

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1 Introduction

Partial differential equations are of great importance in many physical phenomena describing particularly science and engineering models. However, one can easily find that the quantum Zakharov-Kuznetsov (QZK) equations^[1–8] play vital roles in plasma physics that has to do with magnetized plasma and nonlinear ion-acoustic waves among others.^[9]

The (2+1)-dimensional quantum Zakharov-Kuznetsov (QZK) equation is given by

$$u_t + puu_x + q(u_{xxx} + u_{yyy}) + r(u_{xyy} + u_{xxy}) = 0, \quad (1)$$

the (3 + 1)-dimensional quantum Zakharov-Kuznetsov (QZK) equation reads

$$u_t + puu_x + qu_{zzz} + ru_{xxz} + su_{yyz} = 0, \quad (2)$$

while the (3+1)-dimensional modified quantum Zakharov-Kuznetsov (MQZK) equation is

$$u_t + pu^2u_x + qu_{zzz} + ru_{xxz} + su_{yyz} = 0, \quad (3)$$

where p , q , r , and s are real-valued constants. Furthermore, since solution of a given problem helps in understating the physics and mathematics behind; different methods have been proposed to solve varieties of evolution equations. These methods include the Kudryashov method,^[10–13] extended tanh based method,^[14–18] rational expansion method,^[19] Sine-Gordon and ansatz methods,^[20–23] Jacobi elliptic function method^[24] symmetry analysis method^[25] and Adomian decomposition

method^[26–27] and other methods (see, Refs. [28–38]) others.

However, in the present article, we intend to analytically treat the above equations given in Eqs. (1)–(3) by constructing certain solitary wave solutions using the modified extended tanh-expansion method with the Riccati differential equation;^[14–18] via the help of *Mathematica* software. The paper is organized as follows: Section 2 gives details about the methodology. Section 3 is for application of the method to the considered equations. In Sec. 4 we give some graphical representations of the obtained solutions; and Sec. 5 is for conclusion.

2 Analysis of the Method

We present the modified extended tanh expansion method by considering the nonlinear differential equation of the form:

$$G(u, u_t, u_x, u_{tt}, u_{xx}, u_t u_x, \dots) = 0, \quad (4)$$

where $u = u(x, t)$. Also, we use the wave transformation

$$u(x, t) = U(\xi), \quad \xi = ax - dt, \quad (5)$$

where a and d are nonzero constants. Substitution of transformation Eq. (5) into Eq. (4), we get a reduced ordinary differential equation of the form

$$P(U, U', U'', \dots) = 0, \quad (6)$$

where, ' is a derivative with respect to ξ . Further, the

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solution is assumed to be of the finite series of the form:

$$U(\xi) = a_0 + \sum_{n=1}^{n=N} a_n \Phi^n(\xi) + \sum_{n=1}^{n=N} \frac{b_n}{\Phi^n(\xi)},$$

$$a_N \neq 0, b_N \neq 0, \quad (7)$$

where $a_0, a_n, b_n, n = 1, 2, \dots, N$ are nonzero constants and N is a positive integer determined by balancing the highest order derivative with the highest nonlinear term in the equation, and $\Phi(\xi)$ satisfies the Riccati differential equation:

$$\Phi'(\xi) = w + \Phi^2(\xi), \quad (8)$$

where w is a constant. Further, the differential equation (8) admits the following solutions:

(i) if $w = 0$, then

$$\Phi(\xi) = -\frac{1}{\xi},$$

(ii) if $w > 0$, then

$$\Phi(\xi) = \sqrt{w} \tan(\sqrt{w}\xi),$$

$$\Phi(\xi) = -\sqrt{w} \cot(\sqrt{w}\xi).$$

(iii) if $w < 0$, then

$$\Phi(\xi) = -\sqrt{-w} \tanh(\sqrt{-w}\xi),$$

$$\Phi(\xi) = -\sqrt{-w} \coth(\sqrt{-w}\xi).$$

Substituting Eq. (8) and its necessary derivatives into Eq. (6) gives a polynomial in $\Phi(\xi)$ with $P(\Phi(\xi)) = 0$. Collecting coefficients of the obtained polynomials and subsequently setting each one to zero, we will get a set of over-determined algebraic equations for a_0, a_n, b_n ($n = 1, 2, \dots, N$), and f with the aid of symbolic computation using *Mathematica*. Finally, solving the algebraic equations and the above possible solutions of differential equation into Eq. (7), we obtain the solution of Eq. (4).

3 Application

3.1 The (2+1)-Dimensional Quantum Zakharov-Kuznetsov Equation

We consider the (2 + 1)-dimensional quantum Zakharov-Kuznetsov (QZK) equation of the form

$$u_t + pu_x u^2 + q(u_{xxx} + u_{yyy}) + r(u_{xyy} + u_{xxy}) = 0, \quad (9)$$

where $u = u(x, y, t)$, p, q , and r are real-valued constants. Employing the wave transformation,

$$u(x, y, t) = U(\xi), \quad \xi = ax + by - dt,$$

Eq. (9) becomes an ordinary differential equation:

$$-dU' + ap(U^2)' + (qa^3 + qb^3 + rab^2 + ra^2b)U'''' = 0, \quad (10)$$

integrating the above equation once with respect to ξ , get

$$-dU + apU^2 + (qa^3 + qb^3 + rab^2 + ra^2b)U'' = 0, \quad (11)$$

and we assume the constant of integration zero. Now, balancing U^2 and U'' by homogeneous balancing, we get $N = 2$. Thus, Eq. (9) has a solution of the form to

$$U(\xi) = a_0 + a_1 \Phi(\xi) + a_2 \Phi^2(\xi) + \frac{b_1}{\Phi(\xi)} + \frac{b_2}{\Phi^2(\xi)}, \quad (12)$$

where from Eq. (8)

$$\Phi'(\xi) = w + \Phi^2(\xi), \quad \Phi''(\xi) = 2\Phi(\xi)(w + \Phi^2(\xi)). \quad (13)$$

Then, putting the values of Eq. (12) and its necessary derivatives together with Eq. (13) into Eq. (11); collecting all the coefficients of same powers of $\Phi(\xi)$ and thereafter setting them to zero, we get the algebraic equations below:

$$6a^3qw^2b_2 + 6b^3qw^2b_2 + 6a^2brw^2b_2$$

$$+ 6ab^2rw^2b_2 + apb_2^2 = 0,$$

$$2a^3qw^2b_1 + 2b^3qw^2b_1 + 2a^2brw^2b_1$$

$$+ 2ab^2rw^2b_1 + 2apb_1b_2 = 0,$$

$$apb_1^2 - db_2 + 8a^3qwb_2 + 8b^3qwb_2 + 8a^2brwb_2$$

$$+ 8ab^2rwb_2 + 2apa_0b_2 = 0,$$

$$-db_1 + 2a^3qwb_1 + 2b^3qwb_1 + 2a^2brwb_1 + 2ab^2rwb_1$$

$$+ 2apa_0b_1 + 2apa_1b_2 = 0,$$

$$-da_0 + apa_0^2 + 2a^3qw^2a_2 + 2b^3qw^2a_2 + 2a^2brw^2a_2$$

$$+ 2ab^2rw^2a_2 + 2apa_1b_1 + 2a^3qb_2 + 2b^3qb_2$$

$$+ 2a^2brb_2 + 2ab^2rb_2 + 2apa_2b_2 = 0,$$

$$-da_1 + 2a^3qwa_1 + 2b^3qwa_1 + 2a^2brwa_1 + 2ab^2rwa_1$$

$$+ 2apa_0a_1 + 2apa_2b_1 = 0,$$

$$apa_1^2 - da_2 + 8a^3qwa_2 + 8b^3qwa_2 + 8a^2brwa_2$$

$$+ 8ab^2rwa_2 + 2apa_0a_2 = 0,$$

$$2a^3qa_1 + 2b^3qa_1 + 2a^2bra_1 + 2ab^2ra_1 + 2apa_1a_2 = 0,$$

$$6a^3qa_2 + 6b^3qa_2 + 6a^2bra_2 + 6ab^2ra_2 + apa_2^2 = 0,$$

Solving the above system via *Mathematica* software, we get the following:

Case 1

$$a_0 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap},$$

$$a_1 = a_2 = b_1 = 0,$$

$$b_2 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w^2}{ap},$$

$$d = -4(a+b)(a^2q - abq + b^2q + abr)w.$$

Case 2

$$a_0 = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap},$$

$$a_1 = a_2 = b_1 = 0,$$

$$b_2 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w^2}{ap},$$

$$d = 4(a+b)(a^2q - abq + b^2q + abr)w.$$

Case 3

$$a_0 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6(a^3q + b^3q + a^2br + ab^2r)}{ap}, \quad b_1 = b_2 = 0,$$

$$d = -4(a+b)(a^2q - abq + b^2q + abr)w.$$

Case 4

$$a_0 = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6(a^3q + b^3q + a^2br + ab^2r)}{ap}, \quad b_1 = b_2 = 0,$$

$$d = 4(a+b)(a^2q - abq + b^2q + abr)w.$$

Case 5

$$a_0 = -\frac{12(a+b)(a^2q - abq + b^2q + abr)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6(a^3q + b^3q + a^2br + ab^2r)}{ap}, \quad b_1 = 0,$$

$$b_2 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w^2}{ap},$$

$$d = -16(a+b)(a^2q - abq + b^2q + abr)w.$$

Case 6

$$a_0 = \frac{4(a^3qw + b^3qw + a^2brw + ab^2rw)}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6(a^3q + b^3q + a^2br + ab^2r)}{ap}, \quad b_1 = 0,$$

$$b_2 = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w^2}{ap},$$

$$d = 16(a+b)(a^2q - abq + b^2q + abr)w.$$

Thus, we get the following solutions for $w > 0$:

Solution 1

$$u_1(x, y, t) = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \tan^2(\sqrt{w}\xi)},$$

$$u_2(x, y, t) = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \cot^2(\sqrt{w}\xi)},$$

$$\xi = ax + by - 4(a+b)(a^2q - abq + b^2q + abr)wt. \quad (14)$$

Solution 2

$$u_3(x, y, t) = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \tan^2(\sqrt{w}\xi)},$$

$$u_4(x, y, t) = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \cot^2(\sqrt{w}\xi)},$$

$$\xi = ax + by - 4(a+b)(a^2q - abq + b^2q + abr)wt. \quad (15)$$

Solution 3

$$u_5(x, y, t) = -\frac{12(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a^3q + b^3q + a^2br + ab^2r)w}{ap} \tan^2(\sqrt{w}\xi)$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap} \cot^2(\sqrt{w}\xi),$$

$$\xi = ax + by - 16(a+b)(a^2q - abq + b^2q + abr)wt. \quad (16)$$

Solution 4

$$u_6(x, y, t) = \frac{4(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$-\frac{6(a^3q + b^3q + a^2br + ab^2r)w}{ap} \tan^2(\sqrt{w}\xi)$$

$$-\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap} \cot^2(\sqrt{w}\xi),$$

$$\xi = ax + by + 16(a+b)(a^2q - abq + b^2q + abr)wt. \quad (17)$$

Similarly, for $w < 0$ we get the following solutions:

Solution 5

$$u_7(x, y, t) = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$+\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \tanh^2(\sqrt{-w}\xi)},$$

$$u_8(x, y, t) = -\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$+\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \coth^2(\sqrt{-w}\xi)},$$

$$\xi = ax + by - 4(a+b)(a^2q - abq + b^2q + abr)wt. \quad (18)$$

Solution 6

$$u_9(x, y, t) = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$+\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \tanh^2(\sqrt{-w}\xi)},$$

$$u_{10}(x, y, t) = -\frac{2(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$+\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap \coth^2(\sqrt{-w}\xi)},$$

$$\xi = ax + by - 4(a+b)(a^2q - abq + b^2q + abr)wt. \quad (19)$$

Solution 7

$$u_{11}(x, y, t) = -\frac{12(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$+\frac{6(a^3q + b^3q + a^2br + ab^2r)w}{ap} \tanh^2(\sqrt{-w}\xi)$$

$$+\frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap} \coth^2(\sqrt{-w}\xi),$$

$$\xi = ax + by - 16(a+b)(a^2q - abq + b^2q + abr)wt. \quad (20)$$

Solution 8

$$u_{12}(x, y, t) = \frac{4(a+b)(a^2q - abq + b^2q + abr)w}{ap}$$

$$\begin{aligned}
& + \frac{6(a^3q + b^3q + a^2br + ab^2r)w}{ap} \tanh^2(\sqrt{-w}\xi) \\
& + \frac{6(a+b)(a^2q - abq + b^2q + abr)w}{ap} \coth^2(\sqrt{-w}\xi), \\
\xi = ax + by + 16(a+b)(a^2q - abq + b^2q + abr)wt. \quad (21)
\end{aligned}$$

3.2 The (3+1)-Dimensional Quantum Zakharov-Kuznetsov Equation

We consider the (3 + 1)-dimensional quantum Zakharov-Kuznetsov (QZK) equation of the form

$$u_t + pu_x u^2 + qu_{zzz} + ru_{xxz} + su_{yyz} = 0, \quad (22)$$

where p , q , r , and s are real-valued constants. Employing the wave transformation,

$$u(x, y, z, t) = U(\xi), \quad \xi = ax + by + cz - dt,$$

Eq. (22) becomes an ordinary differential equation:

$$-dU' + ap(U^2)' + (c^3q + a^2cr + b^2cs)U''' = 0, \quad (23)$$

integrating the above equation once with respect to ξ , get

$$-dU + apU^2 + (c^3q + a^2cr + b^2cs)U'' = 0, \quad (24)$$

and we assume the constant of integration zero. Balancing U^2 and U'' by homogeneous balancing, we get $N = 2$. Thus, Eq. (22) has a solution of the form to

$$U(\xi) = a_0 + a_1\Phi(\xi) + a_2\Phi^2(\xi) + \frac{b_1}{\Phi(\xi)} + \frac{b_2}{\Phi^2(\xi)}, \quad (25)$$

where from Eq. (8)

$$\Phi'(\xi) = w + \Phi^2(\xi), \quad \Phi''(\xi) = 2\Phi(\xi)(w + \Phi^2(\xi)). \quad (26)$$

Then, putting Eq. (25) and its necessary derivatives together with Eq. (26) into Eq. (24); collecting all the coefficients of same powers of $\Phi(\xi)$ and thereafter setting them to zero, we get the algebraic equations below:

$$\begin{aligned}
& 6c^3qw^2b_2 + 6a^2crw^2b_2 + 6b^2csw^2b_2 + apb_2^2, \\
& 2c^3qw^2b_1 + 2a^2crw^2b_1 + 2b^2csw^2b_1 + 2apb_1b_2, \\
& apb_1^2 - db_2 + 8c^3qwb_2 + 8a^2crwb_2 + 8b^2cswb_2 + 2apa_0b_2, \\
& -db_1 + 2c^3qwb_1 + 2a^2crwb_1 + 2b^2cswb_1 \\
& + 2apa_0b_1 + 2apa_1b_2, \\
& -da_0 + apa_0^2 + 2c^3qw^2a_2 + 2a^2crw^2a_2 + 2b^2csw^2a_2 \\
& + 2apa_1b_1 + 2c^3qb_2 + 2a^2crb_2 + 2b^2csb_2 + 2apa_2b_2, \\
& -da_1 + 2c^3qwa_1 + 2a^2crwa_1 + 2b^2cswa_1 \\
& + 2apa_0a_1 + 2apa_2b_1, \\
& apa_1^2 - da_2 + 8c^3qwa_2 + 8a^2crwa_2 + 8b^2cswa_2 + 2apa_0a_2, \\
& 2c^3qa_1 + 2a^2cra_1 + 2b^2csa_1 + 2apa_1a_2, \\
& 6c^3qa_2 + 6a^2cra_2 + 6b^2csa_2 + apa_2^2.
\end{aligned}$$

Solving the above system via *Mathematica* software, we get the following:

Case 7

$$\begin{aligned}
a_0 &= -\frac{6c(c^2q + a^2r + b^2s)w}{ap}, \\
a_1 &= a_2 = b_1 = 0,
\end{aligned}$$

$$b_2 = -\frac{6c(c^2q + a^2r + b^2s)w^2}{ap},$$

$$d = -4c(c^2q + a^2r + b^2s)w.$$

Case 8

$$a_0 = -\frac{2c(c^2q + a^2r + b^2s)w}{ap},$$

$$a_1 = a_2 = b_1 = 0,$$

$$b_2 = -\frac{6c(c^2q + a^2r + b^2s)w^2}{ap},$$

$$d = 4c(c^2q + a^2r + b^2s)w.$$

Case 9

$$a_0 = -\frac{6c(c^2q + a^2r + b^2s)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6c(c^2q + a^2r + b^2s)}{ap}, \quad b_1 = b_2 = 0,$$

$$d = -4c(c^2q + a^2r + b^2s)w.$$

Case 10

$$a_0 = -\frac{2c(c^2q + a^2r + b^2s)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6c(c^2q + a^2r + b^2s)}{ap}, \quad b_1 = b_2 = 0,$$

$$d = 4c(c^2q + a^2r + b^2s)w.$$

Case 11

$$a_0 = -\frac{12c(c^2q + a^2r + b^2s)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6c(c^2q + a^2r + b^2s)}{ap}, \quad b_1 = 0,$$

$$b_2 = -\frac{6c(c^2q + a^2r + b^2s)w^2}{ap},$$

$$d = -16c(c^2q + a^2r + b^2s)w.$$

Case 12

$$a_0 = \frac{4c(c^2q + a^2r + b^2s)w}{ap}, \quad a_1 = 0,$$

$$a_2 = -\frac{6c(c^2q + a^2r + b^2s)}{ap}, \quad b_1 = 0,$$

$$b_2 = -\frac{6c(c^2q + a^2r + b^2s)w^2}{ap},$$

$$d = 16c(c^2q + a^2r + b^2s)w.$$

Therefore, we get the following solutions for $w > 0$

Solution 9

$$\begin{aligned}
u_1(x, y, z, t) &= -\frac{6c(c^2q + a^2r + b^2s)w}{ap} \\
&\quad - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \cot^2(\sqrt{w}\xi),
\end{aligned}$$

$$\begin{aligned}
u_2(x, y, z, t) &= -\frac{6c(c^2q + a^2r + b^2s)w}{ap} \\
&\quad - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tan^2(\sqrt{w}\xi),
\end{aligned}$$

$$\xi = ax + by + cz - 4c(c^2q + a^2r + b^2s)wt. \quad (27)$$

Solution 10

$$u_3(x, y, z, t) = -\frac{2c(c^2q + a^2r + b^2s)w}{ap} - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \cot^2(\sqrt{w}\xi),$$

$$u_4(x, y, z, t) = -\frac{2c(c^2q + a^2r + b^2s)w}{ap} - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tan^2(\sqrt{w}\xi),$$

$$\xi = ax + by + cz + 4c(c^2q + a^2r + b^2s)wt. \quad (28)$$

Solution 11

$$u_5(x, y, z, t) = -\frac{12c(c^2q + a^2r + b^2s)w}{ap} - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tan^2(\sqrt{w}\xi) - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \cot^2(\sqrt{w}\xi),$$

$$\xi = ax + by + cz - 16c(c^2q + a^2r + b^2s)wt. \quad (29)$$

Solution 12

$$u_6(x, y, z, t) = \frac{4c(c^2q + a^2r + b^2s)w}{ap} - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tan^2(\sqrt{w}\xi) - \frac{6c(c^2q + a^2r + b^2s)w}{ap} \cot^2(\sqrt{w}\xi),$$

$$\xi = ax + by + cz + 16c(c^2q + a^2r + b^2s)wt. \quad (30)$$

Also, we get the following solutions for $w < 0$.

Solution 13

$$u_7(x, y, z, t) = -\frac{6c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \coth^2(\sqrt{-w}\xi),$$

$$u_8(x, y, z, t) = -\frac{6c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tanh^2(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz - 4c(c^2q + a^2r + b^2s)wt. \quad (31)$$

Solution 14

$$u_9(x, y, z, t) = -\frac{2c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \coth^2(\sqrt{-w}\xi),$$

$$u_{10}(x, y, z, t) = -\frac{2c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tanh^2(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz + 4c(c^2q + a^2r + b^2s)wt. \quad (32)$$

Solution 15

$$u_{11}(x, y, z, t) = -\frac{12c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tanh^2(\sqrt{-w}\xi) + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \coth^2(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz - 16c(c^2q + a^2r + b^2s)wt. \quad (33)$$

Solution 16

$$u_{12}(x, y, z, t) = \frac{4c(c^2q + a^2r + b^2s)w}{ap} + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \tanh^2(\sqrt{-w}\xi) + \frac{6c(c^2q + a^2r + b^2s)w}{ap} \coth^2(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz + 16c(c^2q + a^2r + b^2s)wt. \quad (34)$$

3.3 The (3+1)-Dimensional Modified Quantum Zakharov-Kuznetsov Equation

We consider the (3+1)-dimensional modified quantum Zakharov-Kuznetsov (QZK) equation of the form

$$u_t + pu_x u^3 + qu_{zzz} + ru_{xxz} + su_{yyz} = 0, \quad (35)$$

where $p, q, r,$ and s are real-valued constants. Employing the wave transformation

$$u(x, y, z, t) = U(\xi), \quad \xi = ax + by + cz - dt,$$

Eq. (35) becomes an ordinary differential equation:

$$-dU' + ap(U^3)' + (c^3q + a^2cr + b^2cs)U''' = 0, \quad (36)$$

integrating the above equation once with respect to ξ , get

$$-dU + apU^3 + (c^3q + a^2cr + b^2cs)U'' = 0, \quad (37)$$

and we assume the constant of integration zero. Balancing U^3 and U'' in the above equation by homogeneous balancing, we get $N = 1$. Thus, Eq. (37) has a solution of the form to

$$U(\xi) = a_0 + a_1\Phi(\xi) + \frac{b_1}{\Phi(\xi)}. \quad (38)$$

Then, putting Eq. (38) and its necessary derivatives together with Eq. (26) into Eq. (37); collecting all the coefficients of same powers of $\Phi(\xi)$ and thereafter setting them to zero, we get the algebraic equations below:

$$2c^3qw^2b_1 + 2a^2crw^2b_1 + 2b^2csw^2b_1 + apb_1^3 = 0,$$

$$3apa_0b_1^2 = 0, \quad 3apa_0a_1^2 = 0,$$

$$-db_1 + 2c^3qwb_1 + 2a^2crwb_1 + 2b^2cswb_1 + 3apa_0^2b_1 + 3apa_1b_1^2 = 0,$$

$$-da_0 + apa_0^3 + 6apa_0a_1b_1 = 0,$$

$$2c^3qa_1 + 2a^2cra_1 + 2b^2cra_1 + apa_1^3 = 0,$$

$$-da_1 + 2c^3qwa_1 + 2a^2crwa_1 + 2b^2cswa_1 + 3apa_0^2a_1 + 3apa_1^2b_1 = 0.$$

Solving the above system via *Mathematica* software, we get the following:

Case 13

$$a_0 = 0, \quad a_1 = \mp \frac{\sqrt{2}\sqrt{-c(c^2q + a^2r + b^2s)}}{\sqrt{ap}},$$

$$b_1 = 0, \quad d = 2c(c^2q + a^2r + b^2s)w.$$

Case 14

$$a_0 = a_1 = 0, \quad b_1 = \mp i \frac{\sqrt{2c}\sqrt{c^2q + a^2r + b^2s}}{\sqrt{ap}}w,$$

$$d = 2c(c^2q + a^2r + b^2s)w.$$

Therefore, we get the following solutions for $w > 0$.

Solution 17

$$u_{1,2}(x, y, z, t) = \mp \frac{\sqrt{2}\sqrt{-c(c^2q + a^2r + b^2s)}}{\sqrt{ap}}$$

$$\times \sqrt{w} \tan(\sqrt{w}\xi),$$

$$u_{3,4}(x, y, z, t) = \pm \frac{\sqrt{2}\sqrt{-c(c^2q + a^2r + b^2s)}}{\sqrt{ap}}$$

$$\times \sqrt{w} \cot(\sqrt{w}\xi),$$

$$\xi = ax + by + cz + 2c(c^2q + a^2r + b^2s)wt. \quad (39)$$

Solution 18

$$u_{5,6}(x, y, z, t) = \mp i \frac{\sqrt{2c}\sqrt{c^2q + a^2r + b^2s}}{\sqrt{ap}}$$

$$\times \sqrt{w} \cot(\sqrt{w}\xi),$$

$$u_{7,8}(x, y, z, t) = \pm i \frac{\sqrt{2c}\sqrt{c^2q + a^2r + b^2s}}{\sqrt{ap}}$$

$$\times \sqrt{w} \tan(\sqrt{w}\xi),$$

$$\xi = ax + by + cz + 2c(c^2q + a^2r + b^2s)wt. \quad (40)$$

Also, we get the following solutions for $w < 0$.

Solution 19

$$u_{9,10}(x, y, z, t) = \pm \frac{\sqrt{2}\sqrt{-c(c^2q + a^2r + b^2s)}}{\sqrt{ap}}$$

$$\times \sqrt{-w} \tanh(\sqrt{-w}\xi),$$

$$u_{11,12}(x, y, z, t) = \pm \frac{\sqrt{2}\sqrt{-c(c^2q + a^2r + b^2s)}}{\sqrt{ap}}$$

$$\times \sqrt{-w} \coth(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz + 2c(c^2q + a^2r + b^2s)wt. \quad (41)$$

Solution 20

$$u_{12,14}(x, y, z, t) = \pm i \frac{\sqrt{2c}\sqrt{c^2q + a^2r + b^2s}}{\sqrt{ap}}$$

$$\times \sqrt{-w} \coth(\sqrt{-w}\xi),$$

$$u_{15,16}(x, y, z, t) = \pm i \frac{\sqrt{2c}\sqrt{c^2q + a^2r + b^2s}}{\sqrt{ap}}$$

$$\times \sqrt{-w} \tanh(\sqrt{-w}\xi),$$

$$\xi = ax + by + cz + 2c(c^2q + a^2r + b^2s)wt. \quad (42)$$

4 Graphical Representations of the Solutions

In this section, we give some 2-dimensional and 3-dimensional graphical representations of some solutions at different time levels. Figures 1–2 give the solitary plots for the (2+1) QZK equation in Eq. (9); Figures 3–4 give the solitary plots for the (3+1) QZK equation in Eq. (22) while Figs. 5–6 give the solitary plots for the (3+1) modified QZK equation in Eq. (35).

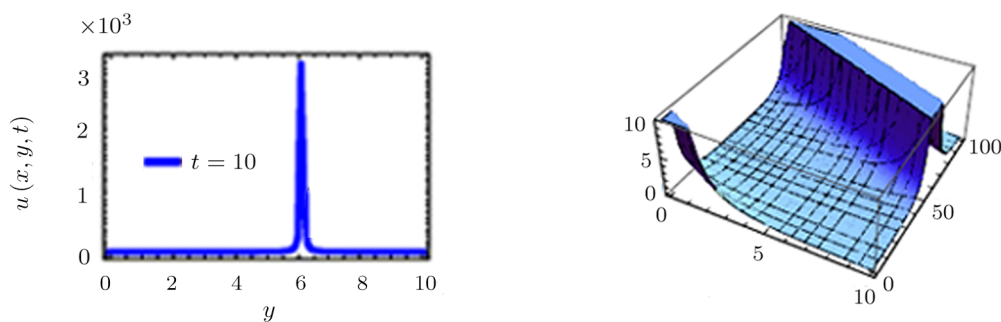


Fig. 1 Exact solutions for the (2+1) QZK equation in Eq. (9) at $t = 10$.

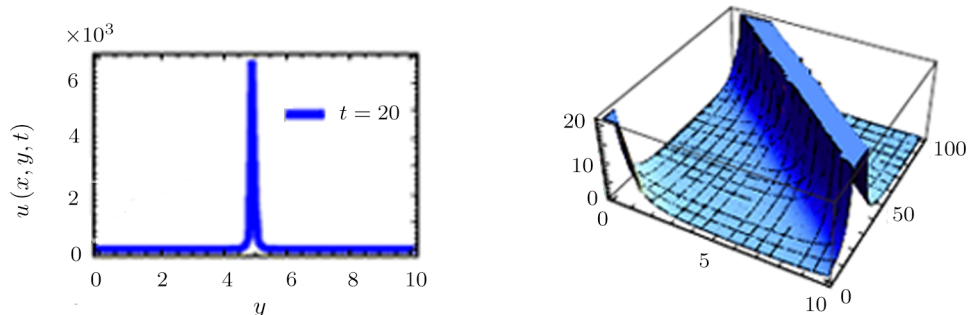


Fig. 2 Exact solutions for the (2+1) QZK equation in Eq. (9) at $t = 20$.

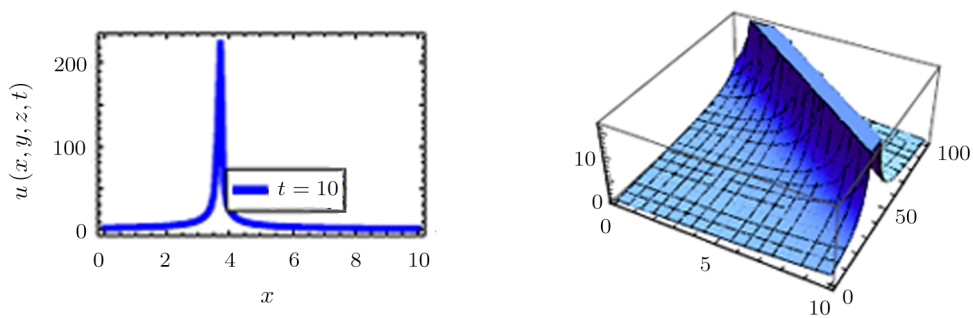


Fig. 3 Exact solutions for the (3+1) QZK equation in Eq. (22) at $t = 10$.

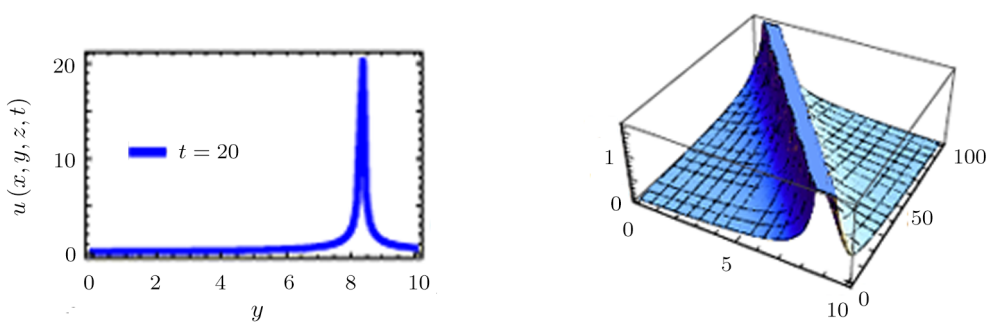


Fig. 4 Exact solutions for the (3+1) QZK equation in Eq. (22) at $t = 20$.

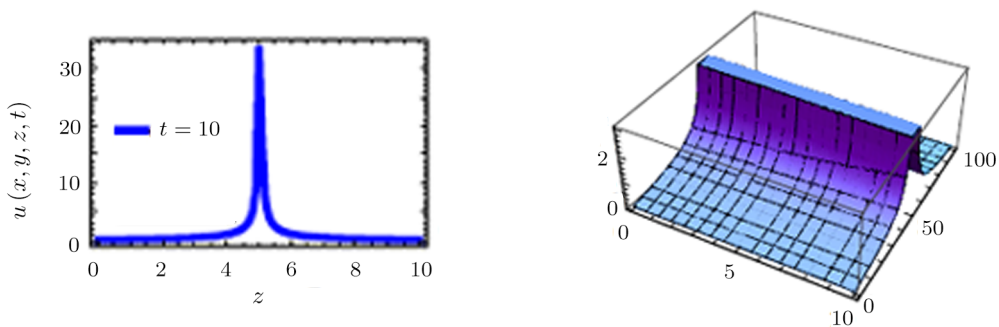


Fig. 5 Exact solutions for the (3+1) modified QZK equation in Eq. (35) at $t = 10$.

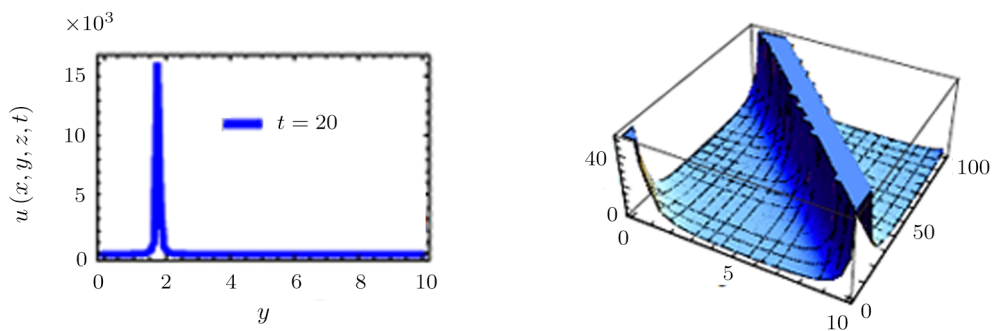


Fig. 6 Exact solutions for the (3+1) modified QZK equation in Eq. (35) at $t = 20$.

5 Conclusion

In conclusion, we have engaged the modified extended tanh-expansion method to construct various exact soliton solutions for the two-dimensional quantum Zakharov-Kuznetsov equation, three-dimensional quantum Zakharov-Kuznetsov equations, and three-dimensional modified quantum Zakharov-Kuznetsov equation. Various trigonometric and hyperbolic func-

tion solutions including the dark soliton solution, singular soliton solution, combined dark-singular soliton solution, and many others were obtained by the method and validated using the *Mathematica* software. These new soliton solutions obtained will help greatly in studying the physical aspects of the considered problems. Thus, the used method is promising and can be applied to several problems in this regards.

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