Commun. Theor. Phys. (Beijing, China) **43** (2005) pp. 519–522
Vol. 43, No. 3, March 15, 2005

Constructing Even- and Odd-Negative Binomial States*

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(Received December 15, 2003)

Abstract *By suitably choosing the normalization factors we introduce the even- and odd-negative binomial states. In some limit cases they approach the even- and odd-coherent states, respectively. We also derive a new eigenvector equation that the negative binomial state satisfies as a nonlinear coherent state.*

PACS numbers: 42.50.Dv
Key words: even- and odd-negative binomial states, normalization, coherent states

Recently, in the field of quantum optics, much attention has been paid to superpositions of quantum states (so-called quantum state engineering) because they exhibit nonclassical properties, such as sub-Poissonian statistics, antibunching, and quadrature squeezing.[1,2] These states are closely related to some probability distribution functions. For example, in the Fock space spanned by the number state $|n\rangle = a^{\dagger n}/\sqrt{n!}|0\rangle$, the binomial state (BS)[3]

$$|\eta, M\rangle = \sum_{n=0}^{\infty} \sqrt{\binom{M}{n} \eta^n (1-\eta)^{M-n}} |n\rangle, \quad 0 < \eta < 1 \tag{1}$$

links to the binomial distribution, while the negative binomial state (NBS)[4−6] corresponds to the negative binomial distribution, and the negative hypergeometric state[7] to the negative hypergeometric distribution, the Abel state[8] to the Abel-binomial distribution, etc. The NBS is constructed by a linear combination of numbers states with coefficients chosen in such a way that the photon number distribution is negative binomial, i.e.,

$$|\eta, s, \varphi\rangle = \sum_{n=0}^{\infty} \sqrt{b(n,\eta,s)}\, \mathrm{e}^{\mathrm{i}n\varphi} |n\rangle, \quad b(n,\eta,s) \equiv \binom{n+s}{n} \eta^{s+1} (1-\eta)^n, \quad 0 < \eta < 1\,. \tag{2}$$

The purpose of this work is to derive the correct form of even- and odd-negative binomial states. The motivation to do this is, as one can see shortly later, that the even- and odd-negative binomial states in some limiting cases will approach the even- and odd-coherent states, the latter are so-called Schrödinger-cat states — macroscopic quantum superposition states, which is the major topic of quantum optics.[2] Yurke and Stoler[9] have pointed out that some nonlinear systems (say, a Kerr-like medium) may convert a coherent state into a Schrödinger-cat states. In reference to the even-binomial state defined in Refs. [10] and [11],

$$|\eta, M\rangle_e = C_e \sum_{n=0}^{[M/2]} \sqrt{\binom{M}{2n} \eta^{2n} (1-\eta)^{M-2n}} |2n\rangle\,, \tag{3}$$

and the odd-binomial state

$$|\eta, M\rangle_e = C_o \sum_{n=0}^{[(M-1)/2]} \sqrt{\binom{M}{2n+1} \eta^{2n+1} (1-\eta)^{M-2n-1}} |2n+1\rangle\,, \tag{4}$$

one may directly introduce the even-negative binomial state

$$|\eta, s, \varphi\rangle_e = N_e \sum_{n=0}^{\infty} \sqrt{\binom{2n+s}{2n} \eta^{s+1} (1-\eta)^{2n}}\, \mathrm{e}^{\mathrm{i}2n\varphi} |2n\rangle\,, \tag{5}$$

and the odd-negative binomial state

$$|\eta, s, \varphi\rangle_o = N_o \sum_{n=0}^{\infty} \sqrt{\binom{2n+1+s}{2n+1} \eta^{s+1} (1-\eta)^{2n+1}}\, \mathrm{e}^{\mathrm{i}(2n+1)\varphi} |2n+1\rangle\,, \tag{6}$$

respectively. Then what are the normalization factors N_e and N_o? Here the subscripts o and e mean even and odd, respectively. The answer to this question is not trivial. If we do not know them, miscellaneous properties of $|\eta, s, \varphi\rangle_e$ and $|\eta, s, \varphi\rangle_o$ are hardly calculated exactly. One may naturally think of determining N_e by tackling the equation

$${}_e\langle \eta, s, \varphi | \eta, s, \varphi\rangle_e = |N_e|^2 \eta^{s+1} \sum_{n=0}^{\infty} \binom{2n+s}{2n} (1-\eta)^{2n} = 1\,. \tag{7}$$

But solving this equation seems mathematically difficult. However, we can derive it by considering the following expression

$$K \equiv (1+x)^{-s-1} + (1-x)^{-s-1}\,. \tag{8}$$

*The project supported by National Natural Science Foundation of China under Grant No. 10175057

Using the formula of negative binomial expansion

$$(1+x)^{-s-1}=\sum_{n=0}^{\infty}\binom{n+s}{n}(-1)^n x^n\,, \tag{9}$$

we have

$$K=\sum_{n=0}^{\infty}\binom{n+s}{n}[(-1)^n+1]x^n\,, \tag{10}$$

which implies that only the terms of even-power of x survive, so

$$K=2\sum_{n=0}^{\infty}\binom{2n+s}{2n}x^{2n}\,. \tag{11}$$

Let $1-\eta=x$, we have

$$\eta^{s+1}\sum_{n=0}^{\infty}\binom{2n+s}{2n}(1-\eta)^{2n}=\frac{1}{2}\eta^{s+1}[(2-\eta)^{-s-1}+\eta^{-s-1}]\,. \tag{12}$$

Hence the solution to Eq. (7) is

$$N_e=\sqrt{\frac{2}{1+(2/\eta-1)^{-s-1}}}\,, \tag{13}$$

which is s-dependent. On the other hand, by considering the following expression

$$K'\equiv(1+x)^{-s-1}-(1-x)^{-s-1}=\sum_{n=0}^{\infty}\binom{n+s}{n}[(-1)^n-1]x^n\,, \tag{14}$$

we see that only odd-power terms of x survive,

$$K'=-2\sum_{n=0}^{\infty}\binom{2n+1+s}{2n+1}x^{2n+1}\,, \tag{15}$$

which yields

$$\eta^{s+1}\sum_{n=0}^{\infty}\binom{2n+1+s}{2n+1}(1-\eta)^{2n+1}=\frac{1}{2}\eta^{s+1}[\eta^{-s-1}-(2-\eta)^{-s-1}]\,, \tag{16}$$

thus the normalization factor for odd–NBS is

$$N_o=\sqrt{\frac{2}{1-(2/\eta-1)^{-s-1}}}\,, \tag{17}$$

which is different from Eq. (13). One may compare Eqs. (13) and (17) with the normalized coefficients of even- and odd-binomial states,

$$C_e=\sqrt{\frac{2}{1+(1-2\eta)^M}},\qquad C_o=\sqrt{\frac{2}{1-(1-2\eta)^M}}\,, \tag{18}$$

which we have derived in Refs. [10] and [11], and see the difference and similarity.

We now examine some limiting cases of $|\eta,s,\varphi\rangle_e$. When $s\to\infty$, $1-\eta\to 0$, while $s(1-\eta)\equiv|\alpha|^2$ keeps finite (note that in this limit the negative binomial distribution goes to the Poisson distribution), from Eq. (2) we see

$$b(0,\eta,s)=\eta^{s+1}=(1+\eta-1)^{s+1}=\eta(1-|\alpha|^2/s)^s\to\mathrm{e}^{-|\alpha|^2}\,, \tag{19}$$

so

$$1+(2/\eta-1)^{-s-1}=1+\eta^{s+1}(1+1-\eta)^{-(s+1)}=1+\eta^{s+1}\Big(1+\frac{|\alpha|^2}{s}\Big)^{-(s+1)}\to 1+\mathrm{e}^{-2|\alpha|^2}\,, \tag{20}$$

and the ratio of two adjacent terms in Eq. (2) reduces to

$$\frac{b(2n,\eta,s)}{b(2n-2,\eta,s)}=\frac{b(2n,\eta,s)}{b(2n-1,\eta,s)}\frac{b(2n-1,\eta,s)}{b(2n-2,\eta,s)}=\frac{(2n+s)(2n-1+s)}{2n(2n-1)}(1-\eta)^2\to\frac{s^2(1-\eta)^2}{2n(2n-1)}=\frac{|\alpha|^4}{2n(2n-1)}\,. \tag{21}$$

It then follows that

$$b(2n,\eta,s)=\frac{|\alpha|^4}{2n(2n-1)}\frac{|\alpha|^4}{(2n-2)(2n-3)}\cdots\frac{|\alpha|^4}{2\cdot 1}b(0,\eta,s)\to\frac{|\alpha|^{4n}}{(2n)!}\mathrm{e}^{-|\alpha|^2}\,. \tag{22}$$

Thus the even-NBS approaches

$$|\eta,s,\varphi\rangle_e\to\sqrt{\frac{2}{1+\mathrm{e}^{-2|\alpha|^2}}}\,\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\mathrm{e}^{\mathrm{i}\varphi}|\alpha|)^{2n}}{\sqrt{(2n)!}}|2n\rangle=\frac{1}{\sqrt{\cosh(|\alpha|^2)}}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}|2n\rangle,\quad \alpha=\mathrm{e}^{\mathrm{i}\varphi}r\,, \tag{23}$$

which is just the even-coherent state. In a similar manner we can prove that in the same limit the odd-NBS goes to the odd-coherent state,

$$|\eta,s,\varphi\rangle_o\to\sqrt{\frac{2}{1-\,\mathrm{e}^{-2|\alpha|^2}}}\,\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\,\mathrm{e}^{\,\mathrm{i}\varphi}r)^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle=\frac{1}{\sqrt{\sinh(|\alpha|^2)}}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle\,. \tag{24}$$

Having derived the correct normalization factors for even- and odd-NBS we are capable of examining their properties. By using $a|n\rangle=\sqrt{n}|n-1\rangle$, we see

$$\begin{aligned}a|\eta,s,\varphi\rangle_e&=\sqrt{\frac{2}{1+(2/\eta-1)^{-s-1}}}\sqrt{\frac{(s+1)(1-\eta)}{\eta}}\sum_{n=1}^{\infty}\sqrt{\binom{2n-1+s+1}{2n-1}\eta^{s+1+1}(1-\eta)^{2n-1}}\,\mathrm{e}^{\,\mathrm{i}2n\varphi}|2n-1\rangle\\&=\sqrt{\frac{1-(2/\eta-1)^{-s-2}}{1+(2/\eta-1)^{-s-1}}}\sqrt{\frac{(s+1)(1-\eta)}{\eta}}\,\mathrm{e}^{\,\mathrm{i}\varphi}|\eta,s+1,\varphi\rangle_o\,,\end{aligned} \tag{25}$$

and

$$\begin{aligned}a|\eta,s,\varphi\rangle_o&=\sqrt{\frac{2}{1-(2/\eta-1)^{-s-1}}}\sqrt{\frac{(s+1)(1-\eta)}{\eta}}\sum_{n=0}^{\infty}\sqrt{\binom{2n+s+1}{2n}\eta^{s+1+1}(1-\eta)^{2n}}\,\mathrm{e}^{\,\mathrm{i}(2n+1)\varphi}|2n\rangle\\&=\sqrt{\frac{1+(2/\eta-1)^{-s-2}}{1-(2/\eta-1)^{-s-1}}}\sqrt{\frac{(s+1)(1-\eta)}{\eta}}\,\mathrm{e}^{\,\mathrm{i}\varphi}|\eta,s+1,\varphi\rangle_e\,.\end{aligned} \tag{26}$$

It then follows that

$$a^2|\eta,s,\varphi\rangle_e=\frac{1-\eta}{\eta}\sqrt{(s+1)(s+2)}\sqrt{\frac{1+\eta(2/\eta-1)^{-s-2}}{1-(2/\eta-1)^{-s-1}}}|\eta,s+2,\varphi\rangle_e\,. \tag{27}$$

We now set up an eigenvector equation for the even-negative binomial state. From Eq. (5) we see that the sum is from 0 to infinity, so the form of the eigenvector equation is

$$f(N)a^2|\eta,s,\varphi\rangle_e=\,\mathrm{e}^{\,\mathrm{i}2\varphi}(1-\eta)|\eta,s,\varphi\rangle_e\,, \tag{28}$$

where

$$f(N)=\frac{1}{\sqrt{(N+2+s)(N+1+s)}}\,. \tag{29}$$

On the other hand, from Eq. (6) we can derive that the eigenvector equation for odd-negative binomial state is

$$\frac{1}{\sqrt{(N+3+s)(N+2+s)}}a^2|\eta,s,\varphi\rangle_o=\,\mathrm{e}^{\,\mathrm{i}2\varphi}(1-\eta)|\eta,s,\varphi\rangle_o\,. \tag{30}$$

Hence the even- and odd-negative binomial states can also be viewed as nonlinear even- and odd-coherent states, respectively. Using these recursive relations one can discuss some non-classical properties of even- and odd-NBS without any difficulty. The results indicate that both even- and odd-NBS exhibit squeezing, antibunching, and photon sub-Poissonian statistics. For this paper's brevity we do not list the details here. The even- or odd-negative binomial photon distribution may be physically implemented by absorption mechanism from thermal photon beam.

We now turn to the question if the NBS can be viewed as some kind of nonlinear coherent states (NLCS). NLCS have been paid attention recently because they may appear as stationary states of the center-of-mass motion of a trapped and bichromatically laser-driven ion far from the Lamb–Dicke limit.[12–14] As reference [13] indicated, NLCS are defined as the right-hand eigenstates of $f(a^\dagger a)a$, where $f(a^\dagger a)$ is a nonlinear operator-valued function of the number operator $a^\dagger a$. We now derive a new eigenvector equation that the NBS satisfies as a nonlinear coherent state. For this purpose, we first recast the NBS into the SU(1,1) coherent state form, by selecting the following Bose operator realization of the SU(1,1) generators,

$$R_-=a\sqrt{2\lambda-1+a^\dagger a},\quad R_+=\sqrt{2\lambda-1+a^\dagger a}a^\dagger,\quad R_3=a^\dagger a+\lambda\,. \tag{31}$$

In fact, from $[a,a^\dagger]=1$, one can see

$$[R_-,\ R_+]=2R_3,\quad [R_3,\ R_+]=R_+,\quad [R_3,\ R_-]=-R_-\,. \tag{32}$$

Correspondingly, the Casimir operator is

$$C\equiv R_3^2-\frac{1}{2}(R_+R_-+R_-R_+)\,. \tag{33}$$

Due to

$$\binom{2\lambda-1+n}{n}^{1/2}|n\rangle=\frac{1}{n!}(\sqrt{2\lambda-1+a^\dagger a}a^\dagger)^n|0\rangle=\frac{1}{n!}R_+^n|0\rangle\,, \tag{34}$$

and letting $\eta = \mathrm{sech}^2 r$, $s = 2\lambda - 1$ we can construct the state[5]

$$\sum_{n=0}^{\infty}(\tanh^2 r)^{n/2}(\mathrm{sech}^2 r)^{\lambda}\begin{pmatrix}2\lambda-1+n\\ n\end{pmatrix}^{1/2}\mathrm{e}^{\mathrm{i}n\varphi}|n\rangle = (\mathrm{sech}^2 r)^{\lambda}\exp(R_+\,\mathrm{e}^{\mathrm{i}\varphi}\tanh r)|0\rangle\,. \tag{35}$$

Comparing Eq. (35) with Eq. (2) we can recast the NBS as an SU(1,1) coherent state,

$$|\eta = \mathrm{sech}^2 r,\ s = 2\lambda - 1,\ \varphi\rangle = (\mathrm{sech}^2 r)^{\lambda}\exp(R_+\,\mathrm{e}^{\mathrm{i}\varphi}\tanh r)|0\rangle \equiv |r,\lambda\rangle\,. \tag{36}$$

For other SU(1,1) coherent states, such as the single-mode Hermite polynomial state, the two-variable Hermite polynomial state, and the Laguerre polynomial state, we refer to Refs. [15] $\sim$ [17] respectively. Further, by operating $R_3 - \lambda$ on Eq. (36) and using Eq. (32) we have

$$(R_3-\lambda)|r,\lambda\rangle = (\mathrm{sech}^2 r)^{\lambda}[R_3-\lambda, \exp(R_+\,\mathrm{e}^{\mathrm{i}\varphi}\tanh r)]|0\rangle = R_+\tanh r|r,\lambda\rangle \tag{37}$$

or

$$(a^\dagger a - R_+\tanh r)|r,\lambda\rangle = 0\,. \tag{38}$$

Operating a on Eq. (38) we see

$$a(a^\dagger a - R_+\tanh r)|r,\lambda\rangle = [(a^\dagger a+1)a - \sqrt{2\lambda + a^\dagger a}(a^\dagger a+1)\tanh r]|r,\lambda\rangle = 0\,. \tag{39}$$

Thus $|r,\lambda\rangle$ satisfies a new eigenvector equation

$$\frac{1}{\sqrt{2\lambda + a^\dagger a}}a|r,\lambda\rangle = \tanh r|r,\lambda\rangle\,, \tag{40}$$

with the eigenvalue being $\tanh r$, which is a nonlinear coherent state-like equation. In particular, when $\lambda = 1/2$, $s = 0$, from Eqs. (36) and (31) we see

$$|r,\lambda=1/2\rangle = \mathrm{sech} r\exp(\sqrt{a^\dagger a}a^\dagger\,\mathrm{e}^{\mathrm{i}\varphi}\tanh r)|0\rangle \equiv |\alpha\rangle, \quad \alpha \equiv \mathrm{e}^{\mathrm{i}\varphi}\tanh r \tag{41}$$

and equation (40) becomes the eigenvector equation for the Susskind–Glogower phase operator

$$\frac{1}{\sqrt{1 + a^\dagger a}}a|r,\lambda\rangle = \tanh r|r,\lambda\rangle\,. \tag{42}$$

By introducing the following bra to Eq. (41)

$$\langle\alpha\| = \langle 0|\exp\Big(\alpha^*\frac{1}{\sqrt{1+a^\dagger a}}a\Big) = \sum_{n=0}^{\infty}\langle n|\frac{\alpha^{*n}}{n!}\,, \tag{43}$$

using the technique of integral within an ordered product of operators[18] we can prove the following completeness relation for NBS,

$$\int\frac{\mathrm{d}^2\alpha}{\pi}\,\mathrm{e}^{-|\alpha|^2}|\alpha\rangle\langle\alpha\| = 1\,. \tag{44}$$

In summary, we have determined the normalization constant for the even- and odd-NBS and analyzed their intrinsic relation to the even- and odd-coherent states. This deepens our knowledge about the negative binomial state. We also derive a new eigenvector equation that the negative binomial state satisfies as a nonlinear coherent state.

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