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Infinitely Many Symmetries of Konopelchenko–Dubrovsky Equation*

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Abstract *A set of generalized symmetries with arbitrary functions of t for the Konopelchenko–Dubrovsky (KD) equation in 2+1 space dimensions is given by using a direct method called formal function series method presented by Lou. These symmetries constitute an infinite-dimensional generalized* w_∞ *algebra.*

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1 Introduction

How to get the infinitely many symmetries for an integrable nonlinear system is becoming a very important subject. In (1+1)-dimensional case, one of the effective methods is to use strong symmetry (recursion operator).[1−6] We can get some sets of infinitely many symmetries by using strong symmetry to some seed symmetries. However, it is usually quite difficult to get symmetries for (2+1)-dimensional nonlinear models. In Ref. [7] ∼ [11], Lou had presented an effective method, formal series symmetry approach, to get symmetries of (2+1)-dimensional integrable models. For example, a set of generalized symmetries of the Kadomtsev–Petviashvili (KP) equation

$$u_{tx} = (6uu_x - u_{xxx})_x - 3u_{yy} \tag{1}$$

can be expressed as the following formula:

$$\sigma_n(f) = \frac{1}{2n!3^{n+1}} \sum_{k=0}^{n+1} f^{(n+1-k)} (K_2' - \partial_t)^k y^n \,, \quad (n = 0, 1, 2, \ldots)\,, \tag{2}$$

where f is an arbitrary function of t, $f^{(k)} = \partial^k f/\partial t^k$, and the linearized operator K_2' reads

$$K_2' = -\partial_x^3 + 6\partial_x u - 3\partial_x^{-1}\partial_y^2 \,. \tag{3}$$

The generalized Lie algebra constituted by these symmetries is as follows:

$$[\sigma_n(f_1) - \sigma_m(f_2)] = \frac{1}{3}\sigma_{n+m-2}((m+1)\dot{f}_1 f_2 - (n+1)\dot{f}_2 f_1)\,, \tag{4}$$

where f_1 and f_2 are arbitrary functions of t, $\dot{f} = \partial f/\partial t$, and the Lie product [,] is defined by

$$[A, B] = \frac{\partial}{\partial \varepsilon}[A(u + \varepsilon B) - B(u + \varepsilon A)]_{\varepsilon=0} = A'B - B'A\,. \tag{5}$$

The Konopelchenko–Dubrovsky (KD) equation

$$u_t - u_{xxx} - 6\beta uu_x + \frac{3\alpha^2 u^2 u_x}{2} - 3w_y + 3\alpha u_x w = 0\,, \quad w_x = u_y \tag{6}$$

is one of the important (2+1)-dimensional integrable models. Obviously, it is a generalization of other two well-known (2+1)-dimensional integrable models, the KP equation ($\alpha = 0$) and the modified KP (mKP) equation ($\beta = 0$), and many of its interesting properties have been given by many authors. However, in our knowledge, it is not known whether the generalized w_∞-type symmetries exist, similar to its special cases, the KP and mKP models. Section 2 of this paper is devoted to obtaining the infinitely many symmetries of the KD equation by means of the formal series approach. Section 3 is a short summary and discussion.

2 Symmetries with Arbitrary Functions of t and Related w_∞ Algebra

Substituting the transformation

$$u \to u + \varepsilon p, \quad w \to w + \varepsilon q\,, \tag{7}$$

where ε is an infinitesimal parameter, into Eq. (6) yields its linearized form

$$p_t - p_{xxx} - 6\beta u_x p - 6\beta u p_x + 3\alpha^2 u u_x p + \frac{3}{2}\alpha^2 u^2 p_x - 3q_y + 3\alpha w p_x + 3\alpha u_x q = 0\,, \tag{8}$$

$$q_x = p_y\,, \tag{9}$$

and we say that

$$\sigma = \begin{pmatrix} p \\ q \end{pmatrix} \tag{10}$$

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is a symmetry of Eq. (6). In other words, a solution of the linearized system (8) and (9) is a symmetry of the KD equation. As for the other (2+1)-dimensional models, like KP and Toda systems,[7,10] we can look for the symmetries of Eq. (6) having the form

$$p(f)=\sum_{k=0}^{n} f^{(n-k)}p[k]\,,$$

$$q(f)=\sum_{k=0}^{n+1} f^{(n+1-k)}q[k]\,, \tag{11}$$

where f is an arbitrary function of t, $f^{(k)}=\partial^k f/\partial t^k$, and $p[k]$ and $q[k]$ are functions of x,y,u,w and their derivatives but not time-dependent explicitly. Substituting Eq. (11) into Eqs. (9) and (10) yields

$$\begin{aligned}&\sum_{k=0}^{n} f^{(n+1-k)}p[k]+\sum_{k=0}^{n} f^{(n-k)}p_t[k]-\sum_{k=0}^{n} f^{(n-k)}p_{xxx}[k]-6\beta u_x\sum_{k=0}^{n} f^{(n-k)}p[k]\\&\quad-6\beta u\sum_{k=0}^{n} f^{(n-k)}p_x[k]+3\alpha^2uu_x\sum_{k=0}^{n} f^{(n-k)}p[k]+\frac{3}{2}\alpha^2u^2\sum_{k=0}^{n} f^{(n-k)}p_x[k]\\&\quad-3\sum_{k=0}^{n+1} f^{(n+1-k)}q_y[k]+3\alpha w\sum_{k=0}^{n} f^{(n-k)}p_x[k]+3\alpha u_x\sum_{k=0}^{n+1} f^{(n+1-k)}q[k]=0\,,\\&\sum_{k=0}^{n+1} f^{(n+1-k)}q_x[k]=\sum_{k=0}^{n} f^{(n-k)}p_y[k]\,.\end{aligned}\tag{12}$$

Equation (12) should be true at any order of time-derivative of f because of its arbitrariness. That means the equation

$$\begin{aligned}&p[k]+p_t[k-1]-p_{xxx}[k-1]-6\beta u_xp[k-1]-6\beta up_x[k-1]+3\alpha^2uu_xp[k-1]\\&\quad+\tfrac{3}{2}\alpha^2u^2p_x[k-1]-3q_y[k]+3\alpha wp_x[k-1]+3\alpha u_xq[k]=0\,,\\&q_x[k]=p_y[k-1]\end{aligned}\tag{13}$$

should be satisfied. The generalized solutions of Eq. (13) read

$$\begin{aligned}&p[0]-3q_y[0]+3\alpha u_xq[0]=0, &&\Rightarrow\quad q[0]=g(y),\\&q_x[0]=0\,, &&\quad\;\; p[0]=3g'(y)-3\alpha u_xg(y),\qquad k=0\,;\end{aligned}\tag{14}$$

$$\begin{aligned}&p[k]=(-\partial_t+\partial_x^3+6\beta\partial_xu-\tfrac{3}{2}\alpha^2\partial_xu^2+3\partial_x^{-1}\partial_y^2-3\alpha w\partial_x-3\alpha u_x\partial_x^{-1}\partial_y)p[k-1]\,,\\&q[k]=\partial_x^{-1}\partial_yp[k-1],\qquad k\ge1\,;\end{aligned}\tag{15}$$

$$\begin{aligned}&p_t[n]-p_{xxx}[n]-6\beta u_xp[n]-6\beta up_x[n]+3\alpha^2uu_xp[n]+\tfrac{3}{2}\alpha^2u^2p_x[n]-3q_y[n+1]\\&\quad+3\alpha wp_x[n]+3\alpha u_xq[n+1]=0\,,\\&q_x[n+1]=p_y[n],\qquad k=n+1\,.\end{aligned}\tag{16}$$

So, from Eqs. (14) $\sim$ (16), we can get $p[k]$, $(k=0\ldots n)$ and $q[k],(k=0\ldots n+1)$. The only thing left is to determine the function $g(y)$. In fact, we know from Eqs. (10) and (15) that $p[n]$ and $q[n+1]$ are time-independent symmetries of Eq. (6). Therefore, we can determine $g(y)$ if we substitute $p[n]$ and $q[n+1]$ (they are expressions about the unknown function $g(y)$).

Here we first select $g(y)$ as $-1/6\alpha$, $-y/18\alpha$, $-y^2/108\alpha$, $-y^3/972\alpha$ when $n=0,1,2,3$. Accordingly, we can have the following four sets of symmetries of Eq. (6):

$$\sigma_0(f)=\begin{pmatrix} f\dfrac{u_x}{2}\\ -\dfrac{\dot f}{6\alpha}+f\dfrac{w_x}{2}\end{pmatrix},\tag{17}$$

$$\sigma_1(f)=\begin{pmatrix} \dot f\Big(-\dfrac{1}{6\alpha}+\dfrac{u_xy}{6}\Big)+fw_x\\ -\dfrac{\ddot f}{18\alpha}+\dot f\Big(\dfrac{w_xy}{6}+\dfrac{u}{6}-\dfrac{\beta}{3\alpha^2}\Big)+fw_y\end{pmatrix},\tag{18}$$

$$\sigma_2(f)=\begin{pmatrix} \ddot f\Big(-\dfrac{y}{18\alpha}+\dfrac{u_xy^2}{36}\Big)+\dot f\Big(\dfrac{yu_y}{3}+\dfrac{u}{6}+\dfrac{xu_x}{6}-\dfrac{\beta}{3\alpha^2}\Big)+f\dfrac{u_t}{2}\\ \dfrac{\partial^3 f}{\partial t^3}\Big(\dfrac{-y^2}{108\alpha}\Big)+\ddot f\Big(\dfrac{-x}{18\alpha}+\dfrac{yu}{9}+\dfrac{y^2u_y}{18}-\dfrac{\beta y}{9\alpha^2}\Big)+\dot f\Big(\dfrac{w}{3}+\dfrac{yw_y}{3}+\dfrac{xw_x}{6}-\dfrac{2\beta^2}{3\alpha^3}\Big)+f\dfrac{w_t}{2}\end{pmatrix},\tag{19}$$

$$\sigma_3(f)=\begin{pmatrix}
\dfrac{\partial^3 f}{\partial t^3}\Big(-\dfrac{y^2}{108\alpha}+\dfrac{y^3u_x}{324}\Big)+\ddot f\Big(-\dfrac{x}{18\alpha}+\dfrac{yu}{18}+\dfrac{y^2u_y}{18}+\dfrac{xyu_x}{18}-\dfrac{\beta y}{9\alpha^2}\Big)+\dot f\Big(\dfrac{w}{2}+\dfrac{yw_y}{2}\\
+\dfrac{xw_x}{3}+\beta yuu_x+\dfrac{yu_{xxx}}{6}+\dfrac{\alpha u^2}{12}-\dfrac{\beta u}{3\alpha}-\dfrac{\alpha wyu_x}{2}-\dfrac{2\beta yw_x}{3\alpha}+\dfrac{2\beta yu_y}{3\alpha}-\dfrac{\alpha^2yu^2u_x}{4}\\
-\dfrac{2\beta^2}{3\alpha^3}\Big)+f\Big(-2\alpha^2u^2w_x-2\alpha^2u_x\displaystyle\int w_xudx-4\alpha ww_x+2\alpha\int w_x^2dx\\
-2\alpha u_x\int w_ydx-2\alpha\int w_yu_xdx+8\beta uw_x+4\beta u_xw+2\int w_{yy}dx+2w_{xxx}\Big)\\[2ex]
\dfrac{\partial^4 f}{\partial t^4}\Big(\dfrac{-y^3}{972\alpha}\Big)+\dfrac{\partial^3 f}{\partial t^3}\Big(\dfrac{-xy}{54\alpha}+\dfrac{y^2u}{108}+\dfrac{y^3u_y}{324}-\dfrac{\beta y^2}{54\alpha^2}\Big)+\ddot f\Big(\dfrac{yw}{9}+\dfrac{y^2w_y}{18}+\dfrac{xyw_x}{18}+\dfrac{xu}{18}-\dfrac{\beta x}{9\alpha^2}\\
-\dfrac{2\beta^2y}{9\alpha^3}\Big)+\dot f\Big(\dfrac{2\int w_ydx}{3}+\dfrac{y\int w_{yy}dx}{2}-\dfrac{\beta w}{3\alpha}+\beta yuw_x+\dfrac{\beta u^2}{2}-\dfrac{\alpha^2yu^2w_x}{4}-\dfrac{\alpha^2u^3}{12}\\
+\dfrac{xw_y}{3}+\dfrac{u_{xx}}{6}+\dfrac{\alpha wu}{6}-\dfrac{2\alpha\int wu_xdx}{3}-\dfrac{\alpha y\int w_yu_xdx}{2}-\dfrac{4\beta^3}{3\alpha^4}+\dfrac{yw_{xxx}}{6}-\dfrac{\alpha y\int w_{xx}wdx}{2}\Big)\\
+f\Big(-2\alpha^2u^2w_y+2\alpha^2u\int u_xw_ydx-2\alpha^2\int uw_xdxw_x+6\alpha\iint w_xw_{xy}dxdx\\
+2\alpha\iint uw_{xyy}dxdx-4\alpha ww_y-2\alpha\int w_ydxw_x-4\beta\int w_yu_xdx+8\beta uw_y\\
-2\alpha^2uww_x+2\iint w_{yyy}dxdx+2w_{xxy}-4\beta\int ww_{xx}dx-2\alpha u\int w_{yy}dx\\
+8\beta ww_x+2\alpha^2u\int ww_{xx}dx\Big)
\end{pmatrix}. \tag{20}$$

And the time-independent symmetries are

$$K_0=\begin{pmatrix}\dfrac{u_x}{2}\\ \dfrac{w_x}{2}\end{pmatrix}, \tag{21}$$

$$K_1=\begin{pmatrix}w_x\\ w_y\end{pmatrix}, \tag{22}$$

$$K_2=\frac{1}{2}\begin{pmatrix}u_t\\ w_t\end{pmatrix}=\begin{pmatrix}\dfrac{u_{xxx}}{2}+3\beta uu_x-\dfrac{3\alpha^2u^2u_x}{4}+\dfrac{3w_y}{2}-\dfrac{3\alpha u_xw}{2}\\ \dfrac{3\int w_{yy}dx}{2}+3\beta uw_x-\dfrac{3}{4}\alpha ww_x+\dfrac{3\alpha\int w_x^2dx}{2}-\dfrac{3\alpha\int u_xw_ydx}{2}+\dfrac{w_{xxx}}{2}\end{pmatrix}, \tag{23}$$

$$K_3=\begin{pmatrix}-2\alpha^2u^2w_x-2\alpha^2u_x\int w_xudx-4\alpha ww_x+2\alpha\int w_x^2dx-2\alpha u_x\int w_ydx-\\
2\alpha\int w_yu_xdx+8\beta uw_x+4\beta u_xw+2\int w_{yy}dx+2w_{xxx}\\
-2\alpha^2u^2w_y+2\alpha^2u\int u_xw_ydx-2\alpha^2\int uw_xdxw_x+6\alpha\iint w_xw_{xy}dxdx+\\
2\alpha\iint uw_{xyy}dxdx-4\alpha ww_y-2\alpha\int w_ydxw_x-4\beta\int w_yu_xdx+8\beta uw_y-\\
2\alpha^2uww_x+2\iint w_{yyy}dxdx+2w_{xxy}-4\beta\int ww_{xx}dx-2\alpha u\int w_{yy}dx\\
+8\beta ww_x+2\alpha^2u\int ww_{xx}dx\end{pmatrix}. \tag{24}$$

In $\sigma_3(f)$, if we take $f=t$, we can get $\sigma_3(t)=t\begin{pmatrix}p[3]\\ q[4]\end{pmatrix}+M$. And we can find that

$$M=\begin{pmatrix}\dfrac{w}{2}+\dfrac{yw_y}{2}+\dfrac{xw_x}{3}+\beta yuu_x+\dfrac{yu_{xxx}}{6}+\dfrac{\alpha u^2}{12}-\dfrac{\beta u}{3\alpha}-\dfrac{\alpha wyu_x}{2}-\dfrac{2\beta yw_x}{3\alpha}+\dfrac{2\beta yu_y}{3\alpha}-\dfrac{\alpha^2yu^2u_x}{4}-\dfrac{2\beta^2}{3\alpha^3}\\
\dfrac{2\int w_ydx}{3}+\dfrac{y\int w_{yy}dx}{2}-\dfrac{\beta w}{3\alpha}+\beta yuw_x+\dfrac{\beta u^2}{2}-\dfrac{\alpha^2yu^2w_x}{4}-\dfrac{\alpha^2u^3}{12}+\dfrac{xw_y}{3}+\dfrac{u_{xx}}{6}\\
+\dfrac{\alpha wu}{6}-\dfrac{2\alpha\int wu_xdx}{3}-\dfrac{\alpha y\int w_yu_xdx}{2}-\dfrac{4\beta^3}{3\alpha^4}+\dfrac{yw_{xxx}}{6}-\dfrac{\alpha y\int w_{xx}wdx}{2}\end{pmatrix} \tag{25}$$

is a master symmetry of the KD equation. According to the approach presented in Refs. [1] and [12], we can use the master symmetry M and the seed symmetry K_0 to yield

$$K_1=-6[M,K_0]\,,\quad K_2=-3[M,K_1]\,,\quad K_3=-2[M,K_2]\,,\quad\cdots\quad K_n=-\frac{6}{n!}[M,K_{n-1}],\quad n=1,2,\ldots$$

Replacing M with $\sigma_3(f_1)$, we can get

$$\sigma_n(f)=-\frac{6}{n!}[\sigma_3(f_1),K_{n-1}]\,,\qquad \dot f_1=f,\quad n=1,2,3,\ldots \tag{26}$$

In Eq. (26), we have defined $\partial_x^{-1}C=Cx+h(y,t)$.

In terms of the method presented in Ref. [7], we can determine the function $g(y)$ in a comparatively simple way. If we set the dimension of x, β as $[L]$, then according to Eq. (6), we can know each dimension of y, t, u, w. They are $[L]^2, [L]^3, [L]^{-3}, [L]^{-4}$. From Eqs. (16) $\sim$ (19) and (25) we can find $q_0(f)/f, q_1(f)/f, q_2(f)/f, q_3(f)/f, \ldots, q_n(f)/f$ have the dimensions $[L]^{-3}, [L]^{-4}, [L]^{-5}, [L]^{-6}, \ldots, [L]^{-n-3}$. So it can be known that $g(y)f^{(n+1)}/f$ has dimension $[L]^{-n-3}$.

That is to say, the dimension of $g(y)$ is $[L]^{2n}$. Therefore we can know the form of $g(y)$

$$g(y) = \frac{-y^n}{2n!3^{n+1}\alpha}, \quad n = 0, 1, 2, 3, \ldots \tag{27}$$

Finally, the generalized symmetries of KD equation can be written as the following formula:

$$\sigma_n(f) = \begin{pmatrix} \sum_{k=0}^{n} f^{(n-k)}(-\partial_t + \partial_x^3 + 6\beta u_x + 6\beta u\partial_x - 3\alpha^2 uu_x - \frac{3}{2}\alpha^2 u^2\partial_x \\ +3\partial_x^{-1}\partial_y^2 - 3\alpha w\partial_x - 3\alpha u_x\partial_x^{-1}\partial_y)^k \frac{ny^{n-1} - \alpha^2 u_x y^n}{2n!3^n\alpha} \\ \sum_{k=0}^{n+1} f^{(n+1-k)}\Big[(\partial_x^{-1}\partial_y)(-\partial_t + \partial_x^3 + 6\beta u_x + 6\beta u\partial_x - 3\alpha^2 uu_x \\ -\frac{3}{2}\alpha^2 u^2\partial_x + 3\partial_x^{-1}\partial_y^2 - 3\alpha w\partial_x - 3\alpha u_x\partial_x^{-1}\partial_y)^k\Big] \frac{ny^{n-1} - \alpha^2 u_x y^n}{2n!3^n\alpha} \end{pmatrix}, \quad n = 0, 1, 2, 3, \ldots \tag{28}$$

In order to remove the otiose parts, we change the form of $\delta_n(f)$ as follows:

$$\sigma_n(f) = \sigma_n'(f) + \frac{2\beta}{\alpha}\sigma_{n-1}(f)\,, \quad n = 0, 1, 2, 3, \ldots$$

Finishing all detailed calculations, we discuss the algebra of the generalized symmetries. By calculating in detail, we can find all generalized symmetries constitute a closed infinite-dimensional Lie algebra:

$$[\sigma_n(f_1), \sigma_m(f_2)] = \frac{-1}{6}\sigma_{m+n-2}[(m+1)\dot{f}_1 f_2 - (n+1)f_1\dot{f}_2]\,, \quad m, n = 0, 1, 2, 3, \ldots\,,$$

$$\sigma_{-1}(f) = 0, \quad \sigma_{-2}(f) = 0\,. \tag{29}$$

We can find that it is isomorphic to that of the KP equation.[7]

3 Summary

In this paper, we have obtained infinitely many symmetries with arbitrary functions of t for Konopelchenko–Dubrovsky equation by using formal series symmetry approach presented in Refs. [7] ~ [11]. These symmetries are shown by the formula (24). And we have also found this equation's master symmetry, which is shown by expression (25). Using the master symmetry and the seed K_0, we can obtain all the time-independent symmetries K_n, as the standard symmetry method introduced in Refs. [1] and [12]. Replacing M with $\sigma_3(f_1)$, we get expressions (23). And by using it all the generalized symmetries can be obtained. As the same method used in KP equation in Ref. [7], we have fixed the form of function $g(y)$ in a comparatively simple way instead of substituting $p[n]$ and $q[n+1]$ into Eq. (13) to determine it by analysis of the dimension. After obtaining the generalized symmetries, we have also calculated the symmetry algebra of the model and find that it has the generalized w_∞-type symmetry algebra structure, which is similar to its special cases, the KP and mKP models. The uniform commutation relation of the algebra is shown by Eq. (29). Because the symmetry plays such an important role in mathematics, physics, and many other mathematically-based sciences, its study is attracting more and more attention, especially for the (2+1)-dimensional integrable systems. And some other new effective methods are presented in Refs. [13] ~ [15].

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