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# Statefinder Parameters for Coupled Quintessence Scenario in a Power Law Case*

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***Abstract*** *We investigate a coupled quintessence scenario, which can provide a natural solution to the cosmic coincidence problem. We assume that the mass of dark matter particles depends on a power law function of the scalar field associated to dark energy and meanwhile the scalar field evolves in a power law potential. Since the dynamics of this system is dominated by an attractor solution, the mass of dark matter particles is forced to change with time as to ensure that the ratio between the energy densities of dark matter and dark energy becomes a constant at late times, and one thus solves the cosmic coincidence problem naturally. We then apply a statefinder diagnostic to this coupled quintessence scenario. It is shown that the evolving trajectory of this scenario in the s-r diagram is quite different from those of other dark energy models.*

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**Key words:** coupled quintessence scenario, power-law case, attractor solution, statefinder diagnostic

There are more and more evidences[1−3] supporting that the present universe is dominated by dark sectors. Combined analysis of cosmological observations, especially the Wilkinson Microwave Anisotropy Probe (WMAP) satellite experiment,[2] shows that dark energy (DE) occupies about 73% of the energy of our universe, and dark matter (DM) about 23%. The usual baryon matter, which can be described by our known particle theory, occupies only about 4% of the total energy of the universe. The accelerated expansion of the present universe is attributed to the fact that DE is an exotic component with negative pressure, such as the cosmological constant[4] or a scalar field with a proper potential (i.e. the so-called quintessence).[5] The cosmological constant $\Lambda$ (or vacuum energy) has the equation of state $w = -1$. The cosmological model that consists of a mixture of vacuum energy and cold dark matter (CDM) is called LCDM (or $\Lambda$ CDM), while the so-called QCDM cosmology is based upon a mixture of CDM and quintessence field. The energy density and the negative pressure are provided by the quintessence scalar field $\phi$ slowly evolving down its potential $V(\phi)$. The equation of state of the quintessence $-1 < w < 0$ is guaranteed by the slow evolution. However, as is well known, there are two difficulties arising from all of these scenarios, namely, the "fine-tuning" problem, and the "cosmic coincidence" problem. The cosmic coincidence problem[6] states: Since the energy densities of DE and DM scale so differently during the expansion of the universe, why are they nearly equal today? To get this coincidence, it appears that their ratio must be set to a specific, infinitesimal value in the very early universe.

A possible solution to this cosmic coincidence problem may be provided by introducing a coupling between quintessence DE and CDM. This coupling is often described by the variable-mass particle (VAMP) scenario.[7] The VAMP scenario assumes that the CDM particles interact with the scalar DE field resulting in a time-dependent mass, i.e. the mass of the CDM particles evolves according to some function of the scalar field $\phi$. It has been shown that if we assume the quintessence scalar field $\phi$ evolves in an exponential potential and let the CDM particle mass also depend exponentially on $\phi$, the late time behavior of the cosmological equations gives accelerated expansion and, a constant ratio between DM energy density $\rho_\chi$ and DE energy density $\rho_\phi$.[8] This behavior relies on the existence of an attractor solution, which makes the effective equation of state of DE mimic the effective equation of state of DM at late times so that the late time cosmology is insensitive to the initial conditions for DE and DM. Therefore, the scenario of coupled quintessence with VAMPs in an exponential case solves the cosmic coincidence problem in this sense.

In this letter, however, we will give another solution to the problem of cosmic coincidence by using the coupled quintessence scenario in another case — the power law case.[9] We assume that the DM particle mass depends on a power law function of scalar field $\phi$ and the scalar field evolves in a power law potential and show that this case also has a stable attractor thus also solves the cosmic coincidence problem. Then we perform a statefinder diagnostic for this coupled quintessence model. The statefinder parameters introduced by Sahni *et al.*[10] are proven to be useful tools to characterize and differentiate between

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various DE models. We show in this letter that the evolving trajectory of this scenario in the $s$-$r$ diagram is quite different from those of other DE models.

Consider, now, the CDM particle $\chi$ with mass $M$ depending on a power law function of the DE field $\phi$,

$$M_\chi(\phi) = M_* \phi^{-\alpha} \,, \tag{1}$$

where $\phi$ is expressed in units of the reduced Planck mass $M_p$ ($M_p \equiv 1/\sqrt{8\pi G} = 2.436 \times 10^{18}$ GeV), and $\alpha$ is a positive constant. The scalar field has a power law potential,

$$V(\phi) = V_* \phi^\beta \,, \tag{2}$$

where $\beta$ is a positive constant. Since the CDM particle is stable, its number density $n_\chi$ must obey the equation,

$$\dot{n}_\chi + 3Hn_\chi = 0 \,, \tag{3}$$

where the dot denotes a derivative with respect to time, $H = \dot{a}/a$ represents the Hubble parameter, and $a(t)$ is the scale factor of the universe. The energy density of DM $\rho_\chi$ is also $\phi$-dependent, which is given by

$$\rho_\chi(\phi) = M_\chi(\phi) n_\chi \,, \tag{4}$$

and it follows that

$$\dot{\rho}_\chi + 3H\rho_\chi = -\alpha \frac{\dot{\phi}}{\phi} \rho_\chi \,. \tag{5}$$

Since the total energy of DE and DM is conserved, the equation of motion for DE can be obtained,

$$\dot{\rho}_\phi + 3H\rho_\phi(1 + w_\phi) = \alpha \frac{\dot{\phi}}{\phi} \rho_\chi \,, \tag{6}$$

where the usual parameter of equation of state for the homogeneous scalar field is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \,. \tag{7}$$

The equations of motion (5) and (6) can also be written in the form of effective equations of state for DM and DE,

$$\dot{\rho}_\chi + 3H\rho_\chi(1 + w_\chi^{(e)}) = 0 \,, \tag{8}$$

$$\dot{\rho}_\phi + 3H\rho_\phi(1 + w_\phi^{(e)}) = 0 \,, \tag{9}$$

where

$$w_\chi^{(e)} = \frac{\alpha}{3H} \frac{\dot{\phi}}{\phi} = \frac{\alpha}{3} \frac{\phi'}{\phi} \,, \tag{10}$$

$$w_\phi^{(e)} = w_\phi - \frac{\alpha}{3H} \frac{\dot{\phi}}{\phi} \frac{\rho_\chi}{\rho_\phi} = w_\phi - \frac{\alpha}{3} \frac{\phi'}{\phi} \frac{\rho_\chi}{\rho_\phi} \,, \tag{11}$$

are the effective equation of state parameters for DM and DE, respectively. Primes denote derivatives with respect to $u = \ln(a/a_0) = -\ln(1+z)$, where $z$ is the red-shift, and $a_0$ represents the current scale factor. From Eq. (6) or (9) one can get the equation of motion for the scalar field $\phi$,

$$\ddot{\phi} + 3H\dot{\phi} = \frac{\alpha}{\phi} \rho_\chi - \frac{\beta}{\phi} V \,. \tag{12}$$

The Friedmann equation for a spatially flat universe with DE, DM, baryons, and radiation reads

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_\chi(\phi) + \rho_{\rm b} + \rho_{\rm rad} \,, \tag{13}$$

where $\rho_{\rm b}$ and $\rho_{\rm rad}$ are the energy densities of baryons and radiation, respectively, and the Planck normalization $M_p = 1$ has been used. Using the Friedmann equation (13), the equation of motion of field $\phi$ can be written in terms of the derivatives of $\phi$ with respect to $u$,

$$\frac{1}{3} \frac{(\rho_\chi + \rho_{\rm b} + \rho_{\rm rad} + V)}{1 - \phi'^2/6} \phi'' + \frac{1}{2}\Big(\rho_\chi + \rho_{\rm b} + \frac{2}{3}\rho_{\rm rad} + 2V\Big)\phi' = \frac{\alpha}{\phi}\rho_\chi - \frac{\beta}{\phi} V \,, \tag{14}$$

Since we are interested in the late-time behavior, we can assume $\rho_{\rm b}, \rho_{\rm rad} \ll \rho_\chi, \rho_\phi$. In this limit we can get a stable attractor solution in the field space[9.11]

$$\phi = \phi_0 \, {\rm e}^{-[3/(\alpha+\beta)]u} \,, \tag{15}$$

such that

$$\Omega_\phi \simeq 1 - \Omega_\chi = \frac{\alpha}{\alpha + \beta} \,, \tag{16}$$

and

$$w_\phi^{(e)} = w_\chi^{(e)} = W = -\frac{\alpha}{\alpha + \beta} \,. \tag{17}$$

When the attractor is reached, the energy densities of DM and DE will evolve at a constant ratio depending only on $\alpha$ and $\beta$, thus solving the cosmic coincidence problem.

It is shown by Eq. (17) that $W$ is negative and may lead, if $W < -1/3$, to an accelerated expansion of the universe. To understand how it is possible to get both acceleration and constant ratio between DM and DE one may look at the scaling behavior of the energy densities on the attractor (15),

$$\rho_\chi \sim \phi^{-\alpha} \, {\rm e}^{-3u} \sim \rho_\phi \sim \phi^\beta \sim \, {\rm e}^{-3(1+W)u} \,. \tag{18}$$

The scaling behavior of DM deviates from the usual scaling way ${\rm e}^{-3u}$ due to the $\phi$-dependence of the DM mass. The interaction between DM and DE forces their effective equation of state parameters to become an equal negative constant $W$, and thus solving the coincidence problem and at the same time resulting in an accelerated expansion.

The time evolution of the density parameters for different components (including also $\rho_{\rm b}$ and $\rho_{\rm rad}$) and the effective equation of state parameters for DE and DM for a typical solution is plotted in Fig. 1. Notice that the attractor solution is going to be reached today in this example.

In what follows we will perform a statefinder diagnostic for this coupled quintessence scenario. Since more and more DE models are constructed for interpreting or describing the cosmic acceleration, there exists the problem of discriminating between the various contenders. In order to be able to differentiate between those competing cosmological scenarios involving DE, a sensitive and robust diagnostic for DE models is a must. For this purpose a diagnostic proposal that makes use of parameter pair $\{r, s\}$, the so-called "statefinder", was introduced by Sahni *et al.*[10] The statefinder probes the expansion dynamics of the universe through higher derivatives of the expansion factor $\dddot{a}$ and is a natural companion to the deceleration parameter, which depends upon $\ddot{a}$. The statefinder pair $\{r, s\}$ is defined as follows:

$$r \equiv \frac{\dddot{a}}{aH^3}, \qquad s \equiv \frac{r-1}{3(q-1/2)}\,. \tag{19}$$

The statefinder is a "geometrical" diagnostic in the sense that it depends upon the expansion factor and hence upon the metric describing space-time.

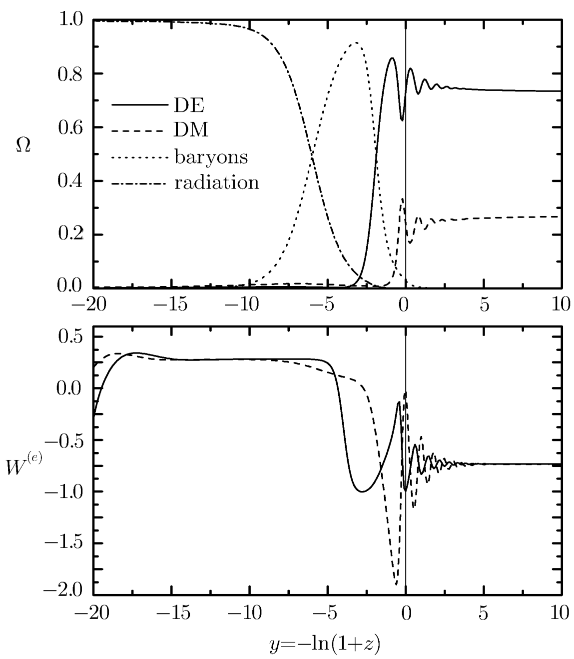


**Fig. 1** The evolution of the density parameters for different components and the effective equation of state parameters for DE and DM. The corresponding model parameters are: $\alpha = 11$, $\beta = 4$, $M_* = 230\rho_0/n_{\chi 0}$, and $V_* = 0.1\rho_0$.

Trajectories in the $s$-$r$ plane corresponding to different cosmological models exhibit qualitatively different behaviors. The spatially flat LCDM scenario corresponds to a fixed point in the diagram

$$\{s, r\}\Big|_{\rm LCDM} = \{0, 1\}\,. \tag{20}$$

Departure of a given DE model from this fixed point provides a good way of establishing the "distance" of this model from LCDM.[12] As demonstrated in Refs. [10] and [12] ∼ [14] the statefinder can successfully differentiate between a wide variety of DE models including the cosmological constant, quintessence, the Chaplygin gas, braneworld models, and interacting DE models. The interacting DE model analyzed in Ref. [14] cannot solve but only alleviate the cosmic coincidence problem. In this letter we will apply a diagnostic to the coupled quintessence scenario in power law case, which can provide a natural solution to the coincidence problem and we will show explicitly the difference between this scenario and other DE models.

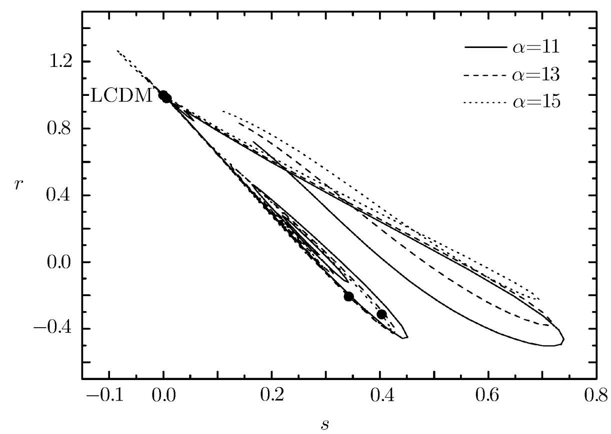


**Fig. 2** The $s$-$r$ diagram: evolution trajectories of $r(s)$ for the variable interval $u \in [-2, 2]$. Selected curves $r(s)$ for $\alpha = 11$, 13, and 15, respectively. Dots locate the current values of the statefinder pair $\{s, r\}$.

It can be followed that the statefinder parameters can be expressed in terms of the total energy density $\rho$ and the total pressure $p$ in the universe:

$$r = 1 + \frac{9(\rho+p)}{2\rho}\frac{\dot{p}}{\dot{\rho}}, \quad s = \frac{(\rho+p)}{p}\frac{\dot{p}}{\dot{\rho}}\,. \tag{21}$$

Since the total energy of the universe is conserved, we have $\dot{\rho} = -3H(\rho + p)$. Then making use of $\dot{\rho}_\phi = -3H(1 + w_\phi^{(e)})\rho_\phi$ and $\dot{\rho}_{\rm rad} = -4H\rho_{\rm rad}$, we can get

$$\frac{\dot{p}}{H} = p' = [w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\rho_\phi - \frac{4}{3}\rho_{\rm rad}\,. \tag{22}$$

Hence, the statefinder parameters for the coupled quintessence scenario can be obtained,

$$r = 1 - \frac{3}{2}[w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\Omega_\phi + 2\Omega_{\rm rad}\,, \tag{23}$$

$$s = \frac{-3[w'_\phi - 3w_\phi(1 + w_\phi^{(e)})]\Omega_\phi + 4\Omega_{\rm rad}}{9w_\phi\Omega_\phi + 3\Omega_{\rm rad}}\,. \tag{24}$$

The deceleration parameter is also given as

$$q = \frac{1}{2}(1 + 3w_\phi\Omega_\phi + \Omega_{\rm rad})\,. \tag{25}$$

In Fig. 2, we show the time evolution of the statefinder pair $\{r, s\}$. The plot is for variable interval $u \in [-2, 2]$, and the selected evolution trajectories of $r(s)$ corresponds to $\beta = 4$ and $\alpha = 11$, 13 and 15, respectively, and the other model parameters are the same as those used in Fig.1. The distance from this scenario to LCDM scenario can be seen explicitly in this diagram. We notice that the trajectories of this model will pass through LCDM fixed point. It is of interest to find that the trajectory of $r(s)$ will form swirl before reaching the attractor, which is quite different from other DE models (see Refs. [10] and [12] $\sim$ [14]). It is demonstrated again that the statefinder can successfully characterize and differentiate between various DE models. As complementarity for the diagnostic, we also plot the evolution trajectories of statefinder pair $\{r, q\}$ in Fig. 3.

In summary, we study in this letter the statefinder of the coupled quintessence scenario. We first study the solution to the cosmic coincidence problem provided by the DE scenario, in which the mass of DM particles is assumed to depend on a power law function of the scalar field with a power law potential associated to DE. Then we perform a statefinder diagnostic for this coupled quintessence scenario. It is shown that the evolving trajectory of this scenario in the $s$-$r$ plane is quite different from those of other DE models. We hope that the future high precision observations (e.g. SNAP) will be capable of determining these statefinder parameters and consequently shed light on the nature of DE.

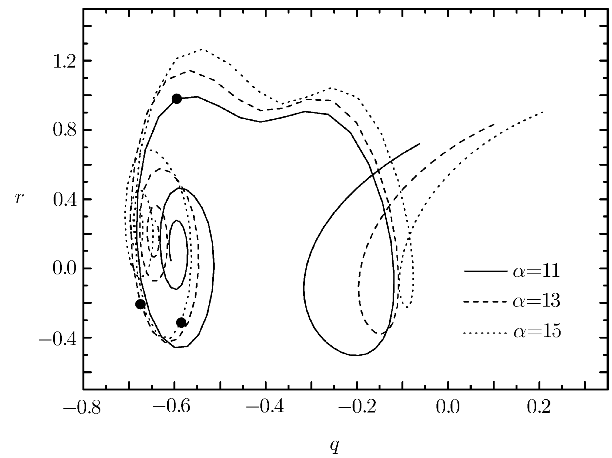


**Fig. 3** The $q$-$r$ diagram: evolution trajectories of $r(q)$ for the variable interval $u \in [-2, 2]$. Selected curves $r(q)$ for $\alpha = 11$, 13, and 15, respectively. Dots locate the current values of the statefinder pair $\{q, r\}$.

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