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Persistent Current in Two Connected Mesoscopic Rings with Reservoirs*

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Abstract *The persistent current and transmission of two connected Aharonov–Bohm rings coupled to external leads are solved analytically. The conditions on which we may observe persistent current, zero or perfect transmission as well as Fano resonance are studied.*

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1 Introduction

Persistent currents in normal metal systems are an intriguing phenomenon occurring in thermodynamic equilibrium. They are quantum-mechanical in nature and reflect the coherence of the wave function throughout the system.[1] After the original works of Byers, Bloch and Imry *et al.*,[2−4] great efforts have been devoted to the subject both on theoretical[5−13] and experimental sides.[14,15]

Persistent current is determined by

$$I = -\frac{1}{c}\frac{\partial J}{\partial \Phi}$$

where J is the thermodynamic potential (e.g. E at zero temperature and F at finite temperatures for canonical systems, $F - \mu N$ for grand canonical ones), Φ is the Aharonov–Bohm (AB) flux. There are two important facts about persistent current: (i) I is exactly a periodic function of Φ with a period $\Phi_0 = hc/e$. This obviously includes the case $I = 0$. (ii) The condition on which I is non-zero in general is that the wavefunction of the charge carriers should not be localized in a small part of the ring and that they should stay coherent along its circumference L. Denote the length over which phase memory is retained by L_ϕ and the localization length, if any, by ξ, and this means $L \leq L_\phi$, $L \leq \xi$.[1]

Theoretical treatments to date have mostly been concentrated on isolated rings. Persistent currents occur not only in the isolated rings but also in rings connected via leads to electron reservoirs, namely in open systems.[16−21] Mailly, Chapelier, and Benoit measured the persistent current in both the closed and open rings.[22] They developed a dedicated SQUID technique and detected a periodic signal with a period h/e, which is the signature of a persistent current with an amplitude 4 ± 2 nA. Buttiker gave a conceptually simple approach of a small metal loop connected to an electron reservoir.[23] Scattering processed in the loop is elastic. Inelastic processes occur only in the reservoir and are the source of dissipation. A.M. Jayannavar and P. Singha Deo considered a system of one metallic ring coupled to two electron reservoirs and showed that in the presence of a transport current, the persistent current can flow in the ring, even in the absence of a magnetic field.[17] This is purely a quantum effect and is related to the current magnification in the loop. In this paper, we extended one AB ring to two connected ones coupled to two electron reservoirs and gave the analytical solutions of the transmission as well as the persistent current. In Sec. 2 we present the theoretical treatment, and section 3 is devoted to results and conclusions.

2 Theoretical Treatment

In the present study, we consider two connected metallic loop coupled to two electron reservoirs via ideal wires as shown in Fig. 1. Following the method of quantum waveguide transport on network, it is straightforward to set up a scattering problem and to calculate the reflection and transmission as well as the persistent current.[24] We introduce wave functions to describe incoming wave with momentum k, incident upon the system from the left, $\Psi_i = \mathrm{e}^{\mathrm{i}kx} + a\,\mathrm{e}^{-\mathrm{i}kx}$ with a denoting the reflection coefficient, and transmitted wave as $\Psi_t = h\,\mathrm{e}^{\mathrm{i}kx}$. Similarly, the wavefunctions in the upper and the lower branch of the two rings can be written as

$$\Psi_2 = c_1\,\mathrm{e}^{\mathrm{i}kx} + c_2\,\mathrm{e}^{-\mathrm{i}kx}, \quad \Psi_3 = d_1\,\mathrm{e}^{\mathrm{i}kx} + d_2\,\mathrm{e}^{-\mathrm{i}kx},$$
$$\Psi_4 = c_3\,\mathrm{e}^{\mathrm{i}kx} + c_4\,\mathrm{e}^{-\mathrm{i}kx}, \quad \Psi_5 = d_3\,\mathrm{e}^{\mathrm{i}kx} + d_4\,\mathrm{e}^{-\mathrm{i}kx}.$$

The boundary conditions, the conservation of the current density and the continuity of the wavefunctions, are to be imposed at these nodes. Here we can choose a gauge for the vector potential in which the magnetic field appears

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only in the boundary conditions: the electrons propagating from the junction point $A(B)$ to the point $B(C)$ along the upper-half circle picks up a phase factor $\mathrm{e}^{\mathrm{i}\alpha_1}(\mathrm{e}^{\mathrm{i}\alpha_2})$, and along the lower-half loop the phase factor is given by $\mathrm{e}^{-\mathrm{i}\alpha_1}(\mathrm{e}^{-\mathrm{i}\alpha_2})$, where $\alpha_1(\alpha_2)$ is the magnetic flux enclosed by the ring I(II) in units of the flux quantum Φ_0. Applying the boundary condition as mentioned above, we then have a set of equations as follows:

$$
\begin{aligned}
&1 + a = c_1 + c_2\,\mathrm{e}^{-\mathrm{i}\alpha_1}\,,\\
&1 + a = d_1 + d_2\,\mathrm{e}^{-\mathrm{i}\alpha_1}\,,\\
&1 - a = c_1 - c_2\,\mathrm{e}^{-\mathrm{i}\alpha_1} + d_1 - d_2\,\mathrm{e}^{-\mathrm{i}\alpha_1}\,,\\
&c_3 + c_4\,\mathrm{e}^{-\mathrm{i}\alpha_2} = d_3 + d_4\,\mathrm{e}^{\mathrm{i}\alpha_2}\,,\\
&c_1\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_1} + c_2\,\mathrm{e}^{-\mathrm{i}kl} = d_1\,\mathrm{e}^{-\mathrm{i}kl}\,\mathrm{e}^{-\mathrm{i}\alpha_1} + d_2\,\mathrm{e}^{-\mathrm{i}kl}\,,\\
&c_3 + c_4\,\mathrm{e}^{-\mathrm{i}\alpha_2} = d_1\,\mathrm{e}^{-\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_1} + d_2\,\mathrm{e}^{-\mathrm{i}kl}\,,\\
&c_1\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_1} - c_2\,\mathrm{e}^{-\mathrm{i}kl} + d_1\,\mathrm{e}^{-\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_1} - d_2\,\mathrm{e}^{-\mathrm{i}kl}\\
&\qquad = c_3 - c_4\,\mathrm{e}^{-\mathrm{i}\alpha_2} + d_3 - d_4\,\mathrm{e}^{\mathrm{i}\alpha_2}\,,\\
&c_3\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_2} + c_4\,\mathrm{e}^{-\mathrm{i}kl} = h\,,\\
&d_3\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{-\mathrm{i}\alpha_2} + d_4\,\mathrm{e}^{-\mathrm{i}kl} = h\,,\\
&c_3\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{\mathrm{i}\alpha_2} - c_4\,\mathrm{e}^{-\mathrm{i}kl} + d_3\,\mathrm{e}^{\mathrm{i}kl}\,\mathrm{e}^{-\mathrm{i}\alpha_2} - d_4\,\mathrm{e}^{-\mathrm{i}kl} = h\,,
\end{aligned}
$$

where l is the half-circumference of the rings. Solving the equations algebraically, we can get the useful physical parameters: reflection $R = |a|^2$, transmission $T = |h|^2$, and currents

$$
\begin{aligned}
I_{c1} &= |c_1|^2 - |c_2|^2\,, \quad I_{d1} = |d_1|^2 - |d_2|^2\,,\\
I_{c2} &= |c_3|^2 - |c_4|^2\,, \quad I_{d2} = |d_3|^2 - |d_4|^2\,,
\end{aligned}
$$

as denoted in Fig. 1.

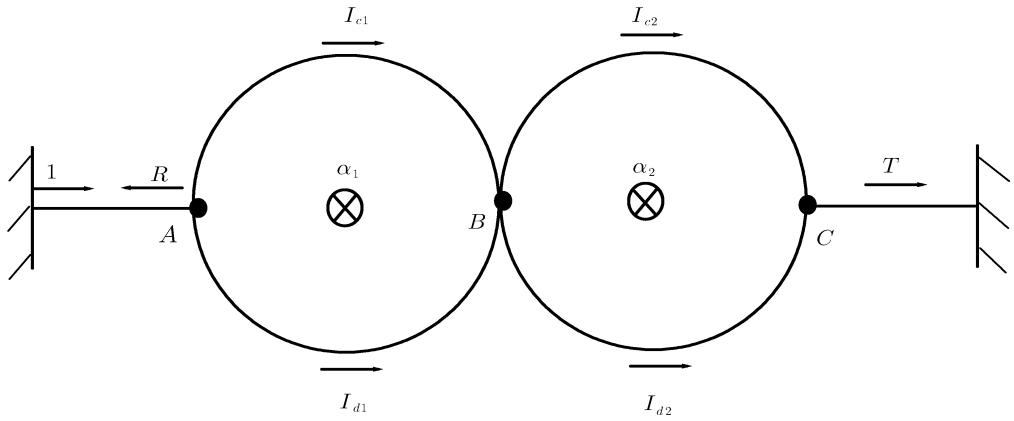


Fig. 1 Schematic picture of two AB rings connected to external leads and threaded by magnetic fluxs.

The expressions are

$$R = 1 - (4\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2)/\Omega\,, \tag{1}$$

$$
\begin{aligned}
I_{c1} = {} & (2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 - 6\sin kl\cos^3 kl\sin\alpha_1\cos\alpha_1 - 2\sin kl\cos kl\sin\alpha_1\cos\alpha_1\\
& + 4\sin kl\cos kl\sin\alpha_1\cos\alpha_1\cos^2\alpha_2)/\Omega\,,
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
I_{d1} = {} & (2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 + 6\sin kl\cos^3 kl\sin\alpha_1\cos\alpha_1 + 2\sin kl\cos kl\sin\alpha_1\cos\alpha_1\\
& - 4\sin kl\cos kl\sin\alpha_1\cos\alpha_1\cos^2\alpha_2)/\Omega\,,
\end{aligned} \tag{3}
$$

$$I_{c2} = (2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 - \sin 2kl\cos^2\alpha_1\sin 2\alpha_2)/\Omega\,, \tag{4}$$

$$I_{d2} = (2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 + \sin 2kl\cos^2\alpha_1\sin 2\alpha_2)/\Omega\,, \tag{5}$$

$$T = (4\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2)/\Omega\,, \tag{6}$$

$$\Omega = (3\cos^2 kl + 1)(3\cos^4 kl + \cos^2 kl - 4\cos^2 kl\cos^2\alpha_2 - 4\cos^2 kl\cos^2\alpha_1 + 2\cos^2\alpha_1\cos^2\alpha_2 + \cos^4\alpha_1 + \cos^4\alpha_2)\,. \tag{7}$$

Here, we define the persistent current circling the rings clockwise as positive and anti-clockwise as negative. Then the persistent current in our system can be written as

$$I_{p1} = \frac{|I_{d1}| - I_{d1} + I_{c1} - |I_{c1}|}{2}\,, \qquad I_{p2} = \frac{|I_{d2}| - I_{d2} + I_{c2} - |I_{c2}|}{2}\,. \tag{8}$$

3 Results and Discussions

From the above solutions, we can rewrite I_{c1} and I_{d1} as

$$I_{c1} = (A + B)/\Omega\,, \qquad I_{d1} = (A - B)/\Omega\,,$$

where

$$A = 2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2\,, \tag{9}$$

$$B = -6\sin kl\cos^3 kl\sin\alpha_1\cos\alpha_1 - 2\sin kl\cos kl\sin\alpha_1\cos\alpha_1 + 4\sin kl\cos kl\sin\alpha_1\cos\alpha_1\cos^2\alpha_2\,. \tag{10}$$

In order to have persistent current in the first ring, $I_{c1}I_{d1}$ must be negative, i.e.,

$$\frac{(A+B)(A-B)}{\Omega^2} < 0\,. \tag{11}$$

In other words $A^2 < B^2$ and $\Omega \neq 0$. $\Omega = 0$ defines the poles in the transmission amplitude, which corresponds to Fano resonance.[25] In our system, Fano resonance corresponds to strong persistent currents, which is a hybridized state between a bound state and a continuum state. It leads to a quasi-bound state showing resonance or antiresonance in the transmission.[26] Here we assume $\Omega \neq 0$.

(i) $A = 0$, $B \neq 0$, and then $A^2 < B^2$ is always satisfied, $I_{c1} = -I_{d1} = B/\Omega$, so the persistent current $I_{p1} = B/\Omega$. To satisfy this condition, we have $\cos\alpha_2 = 0$ and $\sin kl \neq 0$, $\cos\alpha_1 \neq 0$. In this case, $R = 1$, $T = 0$, and $I_{c2} = I_{d2} = 0$. It will show a total reflection and no transmission, then a persistent current can be found only in the first ring.

Figures 2(a) and 2(b) give the dependence of persistent current around the first ring I_{p1} on the flux α_1 and dimensionless wave vector kl, respectively, at $\alpha_2 = \pi/2$. $I_{p1} > (<)0$ means the direction of the current is clockwise (anti-clockwise) as mentioned above. Obviously both of the amplitude and the direction of persistent current are periodic functions of flux and wave vector. When α_1 increases from 0 to $\pi/2$, taking $kl = \pi/3$ as an example, I_{p1} will increase to its maximum 2.3 and then decrease to 0 at $\pi/2$. The direction of I_{p1} is anti-clockwise. When $\pi/2 \leq \alpha_1 \leq \pi$, I_{p1} shows the same behavior but its direction is clockwise. The period is π.

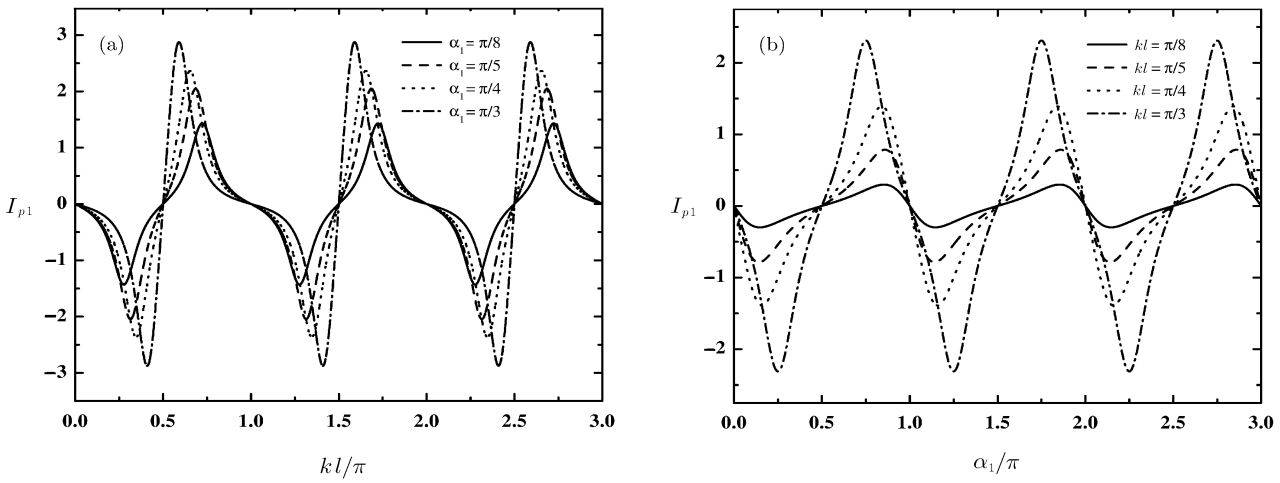


Fig. 2 Dependence of persistent current in the first ring I_{p1} on the dimensionless wave vector kl (a) and the flux α_1 (b) at $\alpha_2 = \pi/2$. No persistent current in the second ring in this case.

(ii) When $B = 0$, $A^2 < B^2$ will never be satisfied. So no persistent current will occurs in this case.

(iii) In general case, $A \neq 0$, $B \neq 0$. From $A^2 < B^2$ we have

$$-9\sin^2\alpha_1\cos^6 kl + 6\sin^2\alpha_1\cos 2\alpha_2\cos^4 kl + (4\cos^2\alpha_2\sin^2\alpha_2 + \sin^2\alpha_1 + 3\cos^2\alpha_1\cos^4\alpha_2$$
$$-4\cos^2\alpha_1\cos^2\alpha_2)\cos^2 kl + \cos^2\alpha_1\sin^4\alpha_2 < 0,\ \text{and}\ \sin kl, \cos\alpha_1, \cos\alpha_2 \neq 0\,. \tag{12}$$

If $\alpha_1 = \alpha_2 = \alpha$, the above inequation is

$$f_1(kl, \alpha) = -9\sin^2\alpha\cos^6 kl + 6\sin^2\alpha\cos 2\alpha\cos^4 kl + (4\cos^2\alpha\sin^2\alpha + \sin^2\alpha$$

$$+ 3\cos^6\alpha - 4\cos^4\alpha)\cos^2 kl + \cos^6\alpha < 0\,, \qquad \sin kl, \cos\alpha_1, \cos\alpha_2 \neq 0\,. \tag{13}$$

Persistent current circling the first ring can be found when $f_1(kl,\alpha) < 0$, as shown in Fig. 3. I_{p1} is a periodic function of kl and α with the period being π. When α changes from $\pi/8$ to $\pi/3$, the area of kl satisfying $f_1(kl) < 0$ is widen, which means that one can find persistent current in a wider kl area. Meanwhile, the amplitude of the persistent current I_{p1} increases a little.

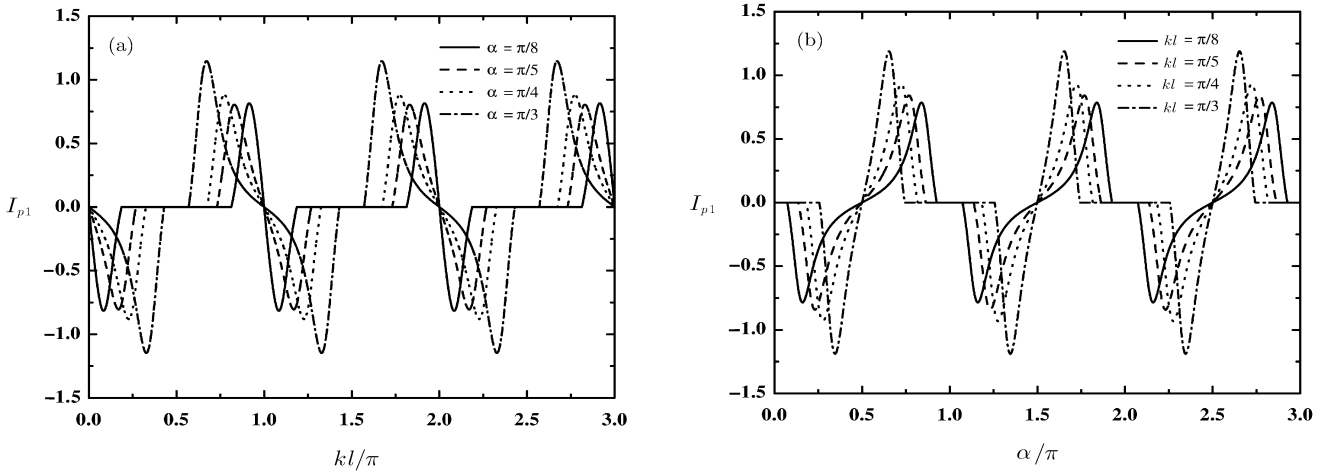


Fig. 3 Persistent current in the first ring under the condition of $f_1(kl,\alpha) < 0$ (formula (13)) at $\alpha_1 = \alpha_2 = \alpha$ for the given α (a) and kl (b).

In the same manner, we can rewrite I_{c2} and I_{d2} as

$$I_{c2} = (C + D)/\Omega\,, \qquad I_{d2} = (C - D)/\Omega\,,$$

where

$$C = A = 2\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2\,, \tag{14}$$

and

$$D = -\sin 2kl\cos^2\alpha_1\sin 2\alpha_2\,. \tag{15}$$

In order to have persistent current in the second ring, $I_{c2}I_{d2}$ should be negative, which means $C^2 < D^2$, if $\Omega \neq 0$.

(i) If $C = 0$, then $D = 0$, $I_{c2} = I_{d2} = 0$, there is no persistent current in the second ring.

(ii) If $D = 0$, $C^2 < D^2$ is never satisfied, no persistent current in this case either.

(iii) General case: $C \neq 0$, $D \neq 0$, $C^2 < D^2$, we have

$$f_2(kl,\alpha_2) = \sin^2 kl - 4\cos^2 kl\sin^2\alpha_2 < 0\,, \quad \text{and} \quad \sin kl, \cos\alpha_1, \cos\alpha_2 \neq 0\,. \tag{16}$$

Figure 4 gives the persistent current circling the second ring when $f_2(kl,\alpha) < 0$. I_{p2} is also the periodic function of kl and α. When α changes from $\pi/8$ to $\pi/3$, the maximum of I_{p2} changes from 0.6 to 0.27, decreasing by 55%.

Then, we would like to discuss the conditions for zero and perfect transmission from our analytical solutions. If $\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 = 0$ and $\Omega \neq 0$, we will get $T = 0$ and $R = 1$, which means zero transmission and perfect reflection. In the system, there is no persistent current when $\sin^2 kl\cos^2\alpha_1 = 0$, but a periodic persistent current can be found in the first ring when $\cos^2\alpha_2 = 0$ and $\sin^2 kl\cos^2\alpha_1 \neq 0$.

To obtain perfect transmission ($T = 1$), from Eqs. (6) and (7) we see that the following relation must be satisfied,

$$\begin{aligned} 4\sin^2 kl\cos^2\alpha_1\cos^2\alpha_2 = {} & (3\cos^2 kl + 1)(3\cos^4 kl + \cos^2 kl - 4\cos^2 kl\cos^2\alpha_2 \\ & - 4\cos^2 kl\cos^2\alpha_1 + 2\cos^2\alpha_1\cos^2\alpha_2 + \cos^4\alpha_1 + \cos^4\alpha_2)\,. \end{aligned} \tag{17}$$

If $\alpha_1 = \alpha_2 = \alpha$, equation (17) will become

$$4\sin^2 kl\cos^4\alpha = (3\cos^2 kl + 1)(3\cos^4 kl + \cos^2 kl - 8\cos^2 kl\cos^2\alpha + 4\cos^4\alpha)\,. \tag{18}$$

The solutions are

$$\cos^2 kl = 0\,, \qquad \cos^2 kl = -\frac{1}{3} + \frac{4}{3}\cos^2\alpha - \frac{2}{9}\cos^4\alpha \pm \frac{2}{9}\cos^2\alpha\sqrt{12\sin^2\alpha + \cos^4\alpha}\ .$$

The first solution is $\cos kl = 0$, i.e. $kl = (2n+1)\pi/2$. It was found that $I_{c1} = I_{d1} = I_{c2} = I_{d2} = 1/2$, except $\alpha = (2n+1)\pi/2$ (where $T = 0$, $R = 1$).

If $\Omega = 0$, but the numerator of T is non-zero (defining the poles of T), $T \to \infty$. It corresponds to Fano resonance states implying a response even if the input tends to zero. From Eq. (7) we now have

$$(3\cos^2 kl + 1)[-\cos^4 kl + \cos^2 kl + (\cos^2\alpha_1 + \cos^2\alpha_2 - 2\cos^2 kl)^2] = 0\,. \tag{19}$$

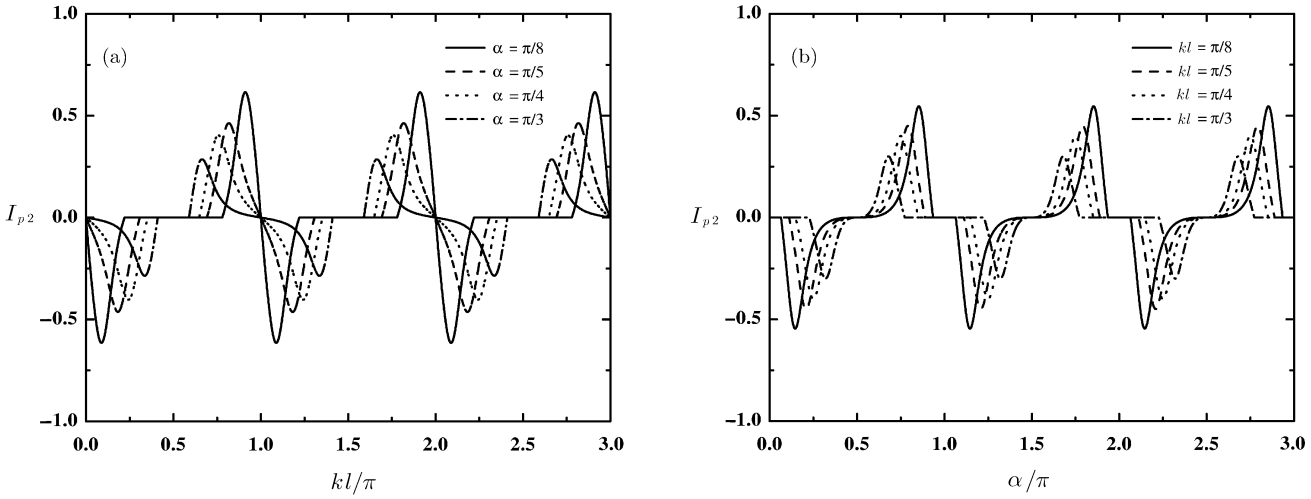


Fig. 4 Persistent current in the second ring under the condition of $f_2(kl, \alpha) < 0$ (formula (16)) at $\alpha_1 = \alpha_2 = \alpha$ for the given α (a) and kl (b).

When $3\cos^2 kl + 1 = 0$, equation (19) will defines two poles $kl = (1.5708, -0.5493\mathrm{i})$ and $(1.5708, 0.5493\mathrm{i})$, whatever α_1 or α_2 is.

When $\alpha_1 = \alpha_2 = 0$, $kl = (0, 0.7954\mathrm{i})$ will be a pole.

When $\alpha_1 = \alpha_2 = \alpha \neq 0$, the poles will satisfy

$$3\cos^4 kl + (1 - 8\cos^2\alpha)\cos^2 kl + 4\cos^4\alpha = 0\,. \tag{20}$$

More detailed discussion of the Fano resonance in two connected AB rings was given in Ref. [26].

In summary, we have solved analytically the persistent current and transmission of two connected AB rings coupled to external leads. It is found that persistent currents in different rings have different behaviors. The conditions on which one will find persistent current, zero and perfect transmission as well as Fano resonance have been discussed. The theoretical study here may be helpful for doing experiments on persistent current of multiple AB rings.

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