Commun. Theor. Phys. (Beijing, China) **44** (2005) pp. 821–826
© International Academic Publishers Vol. 44, No. 5, November 15, 2005

A Generalized Variable-Coefficient Algebraic Method Exactly Solving (3+1)-Dimensional Kadomtsev–Petviashvilli Equation*

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(Received January 19, 2005)

Abstract *A generalized variable-coefficient algebraic method is applied to construct several new families of exact solutions of physical interest for (3+1)-dimensional Kadomtsev–Petviashvilli (KP) equation. Among them, the Jacobi elliptic periodic solutions exactly degenerate to the soliton solutions at a certain limit condition. Compared with the existing tanh method, the extended tanh method, the Jacobi elliptic function method, and the algebraic method, the proposed method gives new and more general solutions.*

PACS numbers: 03.65.Fd, 02.30.Jr, 05.45.Yv
Key words: generalized variable-coefficient algebraic method, (3+1)-dimensional KP equation, exact explicit solutions

1 Introduction

Many phenomena in physics and other sciences are often described by nonlinear partial differential equations (PDEs). When we want to understand the physical mechanism of phenomena in nature, described by nonlinear PDEs, exact solutions for the nonlinear PDEs have to be explored. Thus the methods for deriving exact solutions for the governing equations have to be developed. To study exact solutions of nonlinear PDEs has become one of the most important topics in mathematical physics. In the past decades, various powerful methods have been proposed, namely the inverse scattering method, Hirota's bilinear method, the Bäcklund transformation method, the Darboux transformation method, tanh method, variable separation approach, and homogeneous balance method. Among those, the tanh method[1] provides a particularly straightforward and effective algorithm to obtain solutions for a large number of nonlinear PDEs. Based on the fact that soliton solutions are essentially of a localized nature, one can write the solitary wave solution of a nonlinear equation as a polynomial of hyperbolic functions and transform it into a nonlinear system of algebraic equations. In recent years, much research work has been concentrated on the various extensions and applications of the tanh method.[2–5] Recently Bai presented an effective extension to the tanh method and developed a new generalized variable-coefficient algebraic method to uniformly construct a series of exact solutions for many nonlinear equations.[6]

In this paper, we shall extend this method to high-dimensional nonlinear evolution equations. For illustration, we apply this method to the (3+1)-dimensional Kadomtsev–Petviashvilli (KP) equation,

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0\,. \qquad (1)$$

The (3+1)-dimensional KP equation (1) describes the dynamics of solitons and nonlinear waves in plasmas and superfluids. When u is z-independent, equation (1) is completely integrable and then many kinds of solution can be obtained by some different approaches like inverse scattering transformation, bilinear method, etc. Because of the nonintegrability of Eq. (1), it is difficult to give some exact solutions of Eq. (1). In Ref. [7], Wang and Lou studied its special type of exact solutions. Here the proposed method will give a series of exact solutions.

2 Generalized Variable-Coefficient Algebraic Method

Let us recall our proposed method, whose main steps are outlined as follows.[6]

For a given system of nonlinear PDE with independent variables $x = (x_0 \equiv t, x_1, x_2, \ldots, x_n)$ and dependent variable u, v,

$$P(x_0 \equiv t, x_1, x_2, \ldots, x_n, u, v, u_{x_i}, v_{x_i}, u_{x_i x_j}, v_{x_i x_j}, \ldots, u_{x_{i_1} x_{i_2} \cdots x_{i_N}}, v_{x_{i_1} x_{i_2} \cdots x_{i_N}}) = 0\,, \qquad (2a)$$

$$Q(x_0 \equiv t, x_1, x_2, \ldots, x_n, u, v, u_{x_i}, v_{x_i}, u_{x_i x_j}, v_{x_i x_j}, \ldots, u_{x_{i_1} x_{i_2} \cdots x_{i_N}}, v_{x_{i_1} x_{i_2} \cdots x_{i_N}}) = 0\,. \qquad (2b)$$

In general, the left-hand sides of Eqs. (2a) and (2b) are polynomials in u, v and their various derivatives. Our method for solving Eqs. (2) proceeds in the following three steps.

Step 1 We assume that equations (2) have solutions

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*The project supported by National Natural Science Foundation of China and the Natural Science Foundation of Shandong Province of China under Grant No. 2004ZX16

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of the form

$$u = A_0(x) + \sum_{i=1}^{M} \left(A_i(x)P^i + B_i(x)P^{-i}\right), \tag{3a}$$

$$v = a_0(x) + \sum_{j=1}^{N} \left(a_j(x)P^j + b_j(x)P^{-j}\right), \tag{3b}$$

where M and N are integers to be determined by balancing the highest order nonlinear terms and the highest order partial derivative terms in Eqs. (2), $P = P(\xi)$ and $\xi = \xi(x)$ are all differentiable functions and $P(\xi)$ is a solution of the following first-order nonlinear ordinary differential equation (ODE),

$$P'(\xi) = \varepsilon\sqrt{\sum_{i=0}^{n} c_i P^i(\xi)}\,, \tag{4}$$

where $\varepsilon = \pm 1$. The positive integer n and the constants c_i $(i = 0, 1, \dots, n)$ are to be determined. Then the derivatives with respect to the variable ξ become the derivatives with respect to the variable P as

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \to \varepsilon\sqrt{\sum_{i=0}^{n} c_i P^i}\,\frac{\mathrm{d}}{\mathrm{d}P}\,, \quad \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \to \varepsilon^2\Big[\frac{1}{2}\sum_{i=1}^{n} i c_i P^{i-1}\frac{\mathrm{d}}{\mathrm{d}P} + \sum_{i=0}^{n} c_i P^i \frac{\mathrm{d}^2}{\mathrm{d}P^2}\Big], \dots$$

Step 2 Substituting Eqs. (3) into the given PDEs (2) and making use of Eq. (4), collecting all the terms with the same order of P^k and $P^k\sqrt{\sum_{i=0}^{n} c_i P^i}$ (where $i = 0, 1, \dots, n$), and setting the coefficients of each order of P^k and $P^k\sqrt{\sum_{i=0}^{n} c_i P^i}$ to zero yields a set of over-determined partial differential equations with respect to the differential functions $A_i(x)$, $B_i(x)$, $a_j(x)$, $b_j(x)$ (where $i = 0, 1, 2, \dots, M$; $j = 0, 1, 2, \dots, N$) and $\xi(x)$.

Step 3 Solving the over-determined partial differential equations obtained in step 2 by using symbolic computation packages like *Maple* or *Mathematica* will lead to the explicit expressions for $A_i(x)$, $B_i(x)$, $a_j(x)$, $b_j(x)$ (where $i = 0, 1, 2, \dots, M$; $j = 0, 1, 2, \dots, N$), and $\xi(x)$ or the constraints among them. We remark here that the exact solutions of Eqs. (2) depend on the explicit solvability of Eq. (4). The solution of the system of algebraic equations will become tedious with the increase of the values of M, N, and n. In the case when $n = 4$, equation (4) gives a series of fundamental solutions such as polynomial, exponential, soliton, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. We consider only the case $n = 4$ in this paper and take

$$P'(\xi) = \varepsilon\sqrt{c_0 + c_1 P(\xi) + c_2 P^2(\xi) + c_3 P^3(\xi) + c_4 P^4(\xi)}\,. \tag{5}$$

By considering the different values of c_0, c_1, c_2, c_3, and c_4, we find that equation (5) admits a series of fundamental solutions, which are displayed in the Appendix.

Remark 1 The algorithm is more powerful than the tanh method,[1] the extended tanh method,[2,3] the Jacobi function expansion method,[4] and the algebraic method.[5] To demonstrate this, we argue here that the various previously introduced methods are included in our more general solution, i.e. Eq. (3). Let us consider the special cases $A_i(x) = A_i$, $a_j(x) = a_j$, $B_i(x) = b_j(x) = 0$ (where A_i and a_j are constants and $i = 0, 1, 2, \dots, M$; $j = 0, 1, 2, \dots, N$) and

$$\xi(x) = k_0 t + k_1 x_1 + k_2 x_2 + \cdots + k_n x_n + \varphi\,,$$

where $k_0, k_1, k_2, \dots, k_n$, and φ are constants.

Then, our approach reduces to the algebraic method proposed by Fan.[5] Also, for the case $c_1 = c_3 = 0$, $c_0 = c_4 = 1$, and $c_2 = 2$, equation (5) has the solution $\tanh\xi$ and our method reduces to the tanh method.[1] In the case when $c_1 = c_3 = 0$, $c_0 = c_2^2/4$, and $c_4 = 1$, equation (5) degenerates to a Riccati equation. In this case our proposed method becomes the extended tanh method.[2,3] If $c_1 = c_3 = 0$, the results of Eq. (5) readily cover the results of the Jacobi function expansion method.[4]

Remark 2 The proposed method not only gives a unified formulation to construct a series of exact solutions, but also provides a guideline to classify the types of solutions according to the given parameters. Furthermore, the proposed method is computerizable in solving nonlinear equations by using symbolic software like *Maple* or *Mathematica*.

3 Application to (3+1)-Dimensional KP Equation

According to the proposed method, we expand the solution of Eq. (1) as

$$u(x,y,z,t) = A_0(x,y,z,t) + \sum_{i=1}^{M} (A_i(x,y,z,t)P^i(\xi(x,y,z,t)) + B_i(x,y,z,t)P^{-i}(\xi(x,y,z,t)))\,, \tag{6}$$

and $P(\xi)$ satisfies Eq. (5). Balancing the highest order partial derivative term with the highest order nonlinear terms in Eq. (1) leads to $M = 2$ and $n = 4$, hence we have

$$u(x,y,z,t) = A_0(x,y,z,t) + A_1(x,y,z,t)P(\xi(x,y,z,t)) + \frac{B_1(x,y,z,t)}{P(\xi(x,y,z,t))}$$

$$+ A_2(x,y,z,t)P^2(\xi(x,y,z,t)) + \frac{B_2(x,y,z,t)}{P^2(\xi(x,y,z,t))}\,, \tag{7}$$

where $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $P(\xi(x,y,z,t))$, and $\xi(x,y,z,t) = p(y,z,t)x + q(y,z,t)$ are all differentiable functions and $P(\xi)$ satisfies the ODE (5).

Substituting Eq. (7) along with Eq. (5) into Eq. (1), then setting the coefficients of each order of P^k and $P^k\sqrt{\sum_{i=0}^4 c_i P^i}$ (where $k = 0, 1, \ldots$) to zero, we get a set of over-determined partial differential equations with respect to the unknown functions $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $p(y,z,t)$, and $q(y,z,t)$. Solving the obtained system leads to the following three cases.

Case 1 $c_1 = c_3 = 0$

$$A_0(x,y,z,t) = \frac{1}{6}\frac{q_z^2(y,z,t) + 4p^4c_2 - pq_t(y,z,t) + q_y^2(y,z,t)}{p^2}\,, \quad A_1(x,y,z,t) = 0\,,$$

$$B_1(x,y,z,t) = 0\,, \quad A_2(x,y,z,t) = 2c_4p^2\,, \quad B_2(x,y,z,t) = 2c_0p^2\,, \quad p(y,z,t) = p\,, \tag{8}$$

where c_0, c_2, c_4, and p are arbitrary constants, and $q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$.

Case 2 $c_4 = 0$

$$A_0(x,y,z,t) = \frac{1}{24}\frac{4c_0q_z^2(y,z,t) + 16c_0p^4c_2 - 3p^4c_1^2 + 4c_0q_y^2(y,z,t) - 4c_0pq_t(y,z,t)}{p^2c_0}\,,$$

$$A_1(x,y,z,t) = \frac{p^2(c_1^3 + 8c_0^2c_3 - 4c_0c_1c_2)}{16c_0^2}\,, \quad B_1(x,y,z,t) = c_1p^2\,,$$

$$A_2(x,y,z,t) = \frac{1}{512}\frac{p^2(-c_1^2 + 4c_2c_0)(4c_2c_0 - 3c_1^2)(c_1^3 + 8c_0^2c_3 - 4c_0c_1c_2)}{c_0^4(c_1c_2 + 2c_3c_0)}\,,$$

$$B_2(x,y,z,t) = 2c_0p^2\,, \quad p(y,z,t) = p\,, \tag{9}$$

where c_0, c_1, c_2, c_3, and p are arbitrary constants, and $q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$.

Case 3 $c_0 = c_1 = 0$

$$A_0(x,y,z,t) = \frac{1}{24}\frac{4c_4q_z^2(y,z,t) + 16c_4p^4c_2 - 3p^4c_3^2 + 4c_4q_y^2(y,z,t) - 4c_4pq_t(y,z,t)}{p^2c_4}\,,$$

$$A_1(x,y,z,t) = c_3p^2\,, \quad B_1(x,y,z,t) = -\frac{p^2c_3(4c_4c_2 - c_3^2)}{16c_4^2}\,, \quad A_2(x,y,z,t) = 2c_4p^2\,,$$

$$B_2(x,y,z,t) = -\frac{1}{512}\frac{p^2(-c_3^2 + 4c_2c_4)^2(4c_2c_4 - 3c_3^2)}{c_2c_4^4}\,, \quad p(y,z,t) = p\,, \tag{10}$$

where c_2, c_3, c_4, and p are arbitrary constants, and $q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$.

According to Case 1, we obtain the following exact solutions of Eq. (1):

$$u_1(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2}{c_4}A_2(x,y,z,t)\,\mathrm{sech}^2(\sqrt{c_2}\xi)\,, \quad c_0 = 0\,, \quad c_2 > 0\,, \quad c_4 < 0\,, \tag{11a}$$

$$u_2(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2}{c_4}A_2(x,y,z,t)\sec^2(\sqrt{-c_2}\xi)\,, \quad c_0 = 0\,, \quad c_2 < 0\,, \quad c_4 > 0\,, \tag{11b}$$

$$u_3(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2}{2c_4}A_2(x,y,z,t)\tanh^2\Bigl(\sqrt{-\frac{c_2}{2}}\xi\Bigr) - \frac{2c_4}{c_2}\frac{B_2(x,y,z,t)}{\tanh^2(\sqrt{-c_2/2}\xi)}\,, \tag{11c}$$

where $c_2 < 0$, $c_4 > 0$, $c_0 = c_2^2/4c_4$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $q(y,z,t)$ are determined by Eqs. (8).

$$u_4(x,y,z,t) = A_0(x,y,z,t) + \frac{c_2}{c_4}A_2(x,y,z,t)\tan^2\Bigl(\sqrt{\frac{c_2}{2}}\xi\Bigr) + \frac{c_4}{c_2}\frac{B_2(x,y,z,t)}{\tan^2(\sqrt{c_2/2}\xi)}\,, \tag{11d}$$

where $c_2 > 0$, $c_4 > 0$, $c_0 = c_2^2/4c_4$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $q(y,z,t)$ are determined by Eqs. (8).

$$u_5(x,y,z,t) = A_0(x,y,z,t) + A_2(x,y,z,t)\frac{1}{c_4\xi^2}\,, \quad c_0 = 0\,, \quad c_2 = 0\,, \quad c_4 > 0\,, \tag{11e}$$

$$u_6(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2m^2}{c_4(2m^2-1)}A_2(x,y,z,t)\mathrm{cn}^2\Bigl(\sqrt{\frac{c_2}{2m^2-1}}\xi\Bigr) - \frac{c_4(2m^2-1)}{c_2m^2}\frac{B_2(x,y,z,t)}{\mathrm{cn}^2\bigl(\sqrt{c_2\xi/2m^2-1}\bigr)}\,, \tag{11f}$$

where $c_2 > 0$, $c_4 < 0$, $c_0 = -c_2^2 m^2(1-m^2)/c_4(2m^2-1)^2$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_2(x,y,z,t)$, $q(y,z,t)$, $B_2(x,y,z,t)$ are determined by Eqs. (8).

$$u_7(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2 m^2}{c_4(m^2+1)} A_2(x,y,z,t)\mathrm{sn}^2\Big(\sqrt{-\frac{c_2}{m^2+1}}\xi\Big) - \frac{c_4(m^2+1)}{c_2 m^2}\frac{B_2(x,y,z,t)}{\mathrm{sn}^2\big(\sqrt{-c_2/(m^2+1)}\xi\big)}\,, \tag{11g}$$

where $c_2 < 0$, $c_4 > 0$, $c_0 = c_2^2 m^2/c_4(m^2+1)^2$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_2(x,y,z,t)$, $q(y,z,t)$, $B_2(x,y,z,t)$ are determined by Eqs. (8).

$$u_8(x,y,z,t) = A_0(x,y,z,t) - \frac{c_2}{c_4(2-m^2)} A_2(x,y,z,t)\mathrm{dn}^2\Big(\sqrt{\frac{c_2}{2-m^2}}\xi\Big) - \frac{c_4(2-m^2)}{c_2}\frac{B_2(x,y,z,t)}{\mathrm{dn}^2\big(\sqrt{c_2/(2-m^2)}\xi\big)}\,, \tag{11h}$$

where $c_2 > 0$, $c_4 < 0$, $c_0 = c_2^2(1-m^2)/c_4(2-m^2)^2$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_2(x,y,z,t)$, $q(y,z,t)$, $B_2(x,y,z,t)$ are determined by Eqs. (8).

Remark 3 The Jacobi periodic solution degenerates to the soliton solution when $m \to 1$. It is well known that soliton solutions and Jacobi doubly periodic solutions are interesting and of physical relevance. Here we take the solutions (11a) and (11f) as samples to further analyze their properties and find that when t goes from $-\infty$ to $+\infty$, they asymptotically tend to their limit positions at $t = +\infty$ from their limit positions at $t = -\infty$. The properties are similar to the asymptotically standing soliton.

Case 2 gives the following solutions:

$$u_9(x,y,z,t) = \frac{1}{6}\frac{q_z^2 + p^4 c_2 + q_y^2 - pq_t}{p^2} - \frac{p^2(2c_3 - c_2)c_2}{4c_3}\,\mathrm{sech}^2\Big(\frac{\sqrt{c_2}}{2}\,\xi\Big)\,; \tag{12a}$$

where $c_2 > 0$, $\xi(x,y,z,t) = px + q(y,z,t)$, and $q = q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$.

$$u_{10}(x,y,z,t) = \frac{1}{6}\frac{q_z^2 + 4p^4 c_2 + q_y^2 - pq_t}{p^2} - \frac{p^2(2c_3 - c_2)c_2}{4c_3}\sec^2\Big(\frac{\sqrt{-c_2}}{2}\,\xi\Big)\,; \tag{12b}$$

where $c_2 < 0$, $\xi(x,y,z,t) = px + q(y,z,t)$, $q = q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$.

$$u_{11}(x,y,z,t) = \frac{1}{6}\frac{q_z^2 + q_y^2 - pq_t}{p^2} + \frac{p^2}{2}\frac{1}{\xi^2}\,, \tag{12c}$$

where $\xi(x,y,z,t) = px + q(y,z,t)$, $q = q(y,z,t)$ is an arbitrary function of $\{y,z,t\}$;

$$\begin{aligned} u_{12}(x,y,z,t) = {} & A_0(x,y,z,t) + A_1(x,y,z,t)\wp\big(\sqrt{c_3}\xi/2, g_2, g_3\big) + \frac{B_1(x,y,z,t)}{\wp\big(\sqrt{c_3}\xi/2, g_2, g_3\big)} \\ & + A_2(x,y,z,t)\wp^2\big(\sqrt{c_3}\xi/2, g_2, g_3\big) + \frac{B_2(x,y,z,t)}{\wp^2\big(\sqrt{c_3}\xi/2, g_2, g_3\big)}\,, \end{aligned} \tag{12d}$$

where $c_2 = 0$, $c_3 > 0$, $g_2 = -4c_1/c_3$, $g_3 = -4c_0/c_3$, $\xi(x,y,z,t) = px + q(y,z,t)$, $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $q(y,z,t)$ are determined by Eqs. (9);

$$u_{13}(x,y,z,t) = \frac{1}{6}\frac{q_z^2(y,z,t) + q_y^2(y,z,t) - pq_t(y,z,t)}{p^2} + \frac{2p^2}{\xi^2}\,, \tag{12e}$$

$$u_{14}(x,y,z,t) = A_0(x,y,z,t) + A_1(x,y,z,t)\Big(-\frac{c_0}{c_1} + \frac{c_1\xi^2}{4}\Big) + \frac{B_1(x,y,z,t)}{-c_0/c_1 + c_1\xi^2/4} + \frac{B_2(x,y,z,t)}{\big(-c_0/c_1 + c_1\xi^2/4\big)^2}\,, \tag{12f}$$

where $c_2 = 0$, $c_3 = 0$, $c_1 \neq 0$, $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $B_2(x,y,z,t)$, $q(y,z,t)$ are determined by Eqs. (9);

$$\begin{aligned} u_{15}(x,y,z,t) = {} & A_0(x,y,z,t) + A_1(x,y,z,t)\left(-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}\xi)\right) + \frac{B_1(x,y,z,t)}{-c_1/2c_2 + \exp(\varepsilon\sqrt{c_2}\xi)} \\ & + A_2(x,y,z,t)\Big(-\frac{c_1}{2c_2} + \exp(\varepsilon\sqrt{c_2}\xi)\Big)^2 + \frac{B_2(x,y,z,t)}{\big(-c_1/2c_2 + \exp(\varepsilon\sqrt{c_2}\xi)\big)^2}\,, \end{aligned} \tag{12g}$$

where $c_0 = c_1^2/4c_2$, $c_2 > 0$, $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, $q(y,z,t)$ are determined by Eqs. (9).

Finally, Case 3 leads to

$$\begin{aligned} u_{16}(x,y,z,t) = {} & A_0(x,y,z,t) - A_1(x,y,z,t)\frac{c_2\sec^2(\sqrt{-c_2}\xi/2)}{2\varepsilon\sqrt{-c_2c_4}\tan(\sqrt{-c_2}\xi/2) + c_3} \\ & - B_1(x,y,z,t)\frac{2\varepsilon\sqrt{-c_2c_4}\tan(\sqrt{-c_2}\xi/2) + c_3}{c_2\sec^2(\sqrt{-c_2}\xi/2)} + A_2(x,y,z,t)\left(\frac{c_2\sec^2(\sqrt{-c_2}\xi/2)}{2\varepsilon\sqrt{-c_2c_4}\tan(\sqrt{-c_2}\xi/2) + c_3}\right)^2 \\ & + B_2(x,y,z,t)\left(\frac{2\varepsilon\sqrt{-c_2c_4}\tan(\sqrt{-c_2}\xi/2) + c_3}{c_2\sec^2(\sqrt{-c_2}\xi/2)}\right)^2\,, \quad c_2 < 0\,, \end{aligned} \tag{13a}$$

$$
\begin{aligned}
u_{17}(x,y,z,t) &= A_0(x,y,z,t) - A_1(x,y,z,t)\frac{c_2\operatorname{sech}^2(\sqrt{c_2}\xi/2)}{2\varepsilon\sqrt{c_2c_4}\tanh(\sqrt{c_2}\xi/2)-c_3} \\
&\quad - B_1(x,y,z,t)\frac{2\varepsilon\sqrt{c_2c_4}\tanh(\sqrt{c_2}\xi/2)-c_3}{c_2\operatorname{sech}^2(\sqrt{c_2}\xi/2)} + A_2(x,y,z,t)\left(\frac{c_2\operatorname{sech}^2(\sqrt{c_2}\xi/2)}{2\varepsilon\sqrt{c_2c_4}\tanh(\sqrt{c_2}\xi/2)-c_3}\right)^2 \\
&\quad + B_2(x,y,z,t)\left(\frac{2\varepsilon\sqrt{c_2c_4}\tanh(\sqrt{c_2}\xi/2)-c_3}{c_2\operatorname{sech}^2(\sqrt{c_2}\xi/2)}\right)^2, \quad c_2>0\,, \qquad (13b)
\end{aligned}
$$

where $\xi(x,y,z,t)=px+q(y,z,t)$. The function $A_0(x,y,z,t)$, $A_1(x,y,z,t)$, $B_1(x,y,z,t)$, $A_2(x,y,z,t)$, $B_2(x,y,z,t)$, and $q(y,z,t)$ are given in Case 3, Eq. (10).

4 Conclusion

In summary, a generalized variable-coefficient algebraic method with computerized symbolic computation is developed to deal with the (3+1)-dimensional KP equation. Then, a rich variety of exact explicit solutions are obtained. As indicated before, in Remark 1, the proposed algorithm is more general than the tanh method,[1] the extended tanh method,[2,3] the Jacobi function expansion method,[4] and the algebraic method.[5] Except the equations considered in this paper, the proposed method also is readily applicable to other high-dimensional nonlinear evolution equations including (3+1)-dimensional Jimbo–Miwa equation, (3+1)-dimensional breaking soliton equation, (3+1)-dimensional Nizhnik–Novikov–Veselov equation, and (3+1)-dimensional potential-Yu–Toda–Sasa–Fukuyama equation, etc. We omit their solutions here.

Appendix

Table 1 A series of fundamental solutions of Eq. (5).

c_0	c_1	c_2	c_3	c_4	P
0	0	>0	0	<0	$\varepsilon\sqrt{-c_2/c_4}\operatorname{sech}\left(\sqrt{c_2}\,\xi\right)$
$c_2^2/4c_4$	0	<0	0	>0	$\varepsilon\sqrt{-c_2/2c_4}\,\tanh\left(\sqrt{-c_2/2}\,\xi\right)$
0	0	<0	0	>0	$\varepsilon\sqrt{-c_2/c_4}\sec\left(\sqrt{-c_2}\,\xi\right)$
$c_2^2/4c_4$	0	>0	0	>0	$\varepsilon\sqrt{c_2/2c_4}\,\tan\left(\sqrt{c_2/2}\,\xi\right)$
0	0	0	0	>0	$-\varepsilon/\sqrt{c_4}\xi$
$-c_2^2m^2(1-m^2)/c_4(2m^2-1)^2$	0	>0	0	<0	$\varepsilon\sqrt{-c_2m^2/c_4(2m^2-1)}\operatorname{cn}\left(\sqrt{c_2/(2m^2-1)}\xi\right)$
$c_2^2(1-m^2)/c_4(2-m^2)^2$	0	>0	0	<0	$\varepsilon\sqrt{-c_2/c_4(2-m^2)}\operatorname{dn}\left(\sqrt{c_2/(2-m^2)}\xi\right)$
$c_2^2m^2/c_4(m^2+1)^2$	0	<0	0	>0	$\varepsilon\sqrt{-c_2m^2/c_4(m^2+1)}\operatorname{sn}\left(\sqrt{-c_2/(m^2+1)}\xi\right)$
0	0	>0	$\neq 0$	0	$-(c_2/c_3)\operatorname{sech}^2\left(\sqrt{c_2}\xi/2\right)$
0	0	<0	$\neq 0$	0	$-(c_2/c_3)\sec^2\left(\sqrt{-c_2}\xi/2\right)$
0	0	0	$\neq 0$	0	$1/c_3\xi^2$
$\neq 0$	$\neq 0$	0	>0	0	$\wp\left(\sqrt{c_3}\xi/2, g_2, g_3\right)$
0	0	<0			$-c_2\sec^2\left(\sqrt{-c_2}\xi/2\right)/2\varepsilon\sqrt{-c_2c_4}\tan\left(\sqrt{-c_2}\xi/2\right)+c_3$
0	0	>0			$-c_2\operatorname{sech}^2\left(\sqrt{c_2}\xi/2\right)/2\varepsilon\sqrt{c_2c_4}\tanh\left(\sqrt{c_2}\xi/2\right)-c_3$
>0	0	0	0	0	$\varepsilon\sqrt{c_0}\xi$
	$\neq 0$	0	0	0	$-c_0/c_1+c_1\xi^2/4$
$c_1^2/4c_2$		>0	0	0	$-c_1/2c_2+\exp\left(\varepsilon\sqrt{c_2}\xi\right)$
0		<0	0	0	$-c_1/2c_2+\varepsilon c_1\sin(\sqrt{-c_2}\xi)/2c_2$
0		>0	0	0	$-c_1/2c_2+\varepsilon\sqrt{-c_1^2}\sinh(2\sqrt{c_2}\xi)/2c_2$

Note that $g_2=-4c_1/c_3$ and $g_3=-4c_0/c_3$ are called invariants of the Weierstrass elliptic functions, m is the modulus of the Jacobi elliptic functions; the more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Ref. [8].

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