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Interference Phase of Mass Neutrinos in Kerr Space-Time*

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Abstract *Along the geodesic we calculate the interference phase of the mass neutrinos in some special cases. Because of the rotation of the mass resource which induces the gravitational field, the angular momentum per unit mass, a, has a contribution to the phase, which is different from the case in Schwarzschild space-time.*

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1 Introduction

The mass neutrino oscillations have been hot topics in the high energy experimental and theoretical physics recently, in particular, with highly confident atmospheric neutrino experiment of super-Kamiokande to assure the neutrino mass.[1] As a natural extension of the theoretical consideration, the description of neutrino oscillation in the flat space-time should be replaced by that in the curved space-time if the gravitational background is taken into account. In 1996, N. Fornengo and his accompaniers studied the interference phase of the high energy mass neutrinos when they propagate in the radial and nonradial directions, respectively, in Schwarzschild field.[2] In 2001, Zhang and A. Beesham also studied the propagation of neutrinos in Schwarzschild field with a different method,[3] and they obtained that the phase calculated along the null is equivalent to the half phase along the geodesic in high energy limit, which means that the correct relative phase of the mass neutrinos is either the null phase or the half geodesic phase, and they also pointed out that the convenient use of the energy condition was very important in calculating the interference phase of the mass neutrinos.

In this paper, we will discuss the interference phase of the mass neutrinos along the geodesic in Kerr metric. The existence of parameter θ included in Kerr metric makes the discussion much more complicated, so we only present the interference phase in the radial direction and in the case of $\theta = 0$. By analysis we know that our result can be reduced to the case in Schwarzschild field[3] if we let the angular momentum per unit mass $a = 0$. So the paper is organized as follows. In Sec. 2, along the geodesic we calculate the mass neutrino interference phase in the radial direction. In Sec. 3, we present the phase formulas when $\theta = 0$ and compare it with that obtained in Schwarzschild field.[3] The conclusions of this paper are given in the last section. We set $G = \hbar = c = 1$ throughout this paper. Other gravitational effects are given in Refs. [4] $\sim$ [8].

2 Radial Neutrino Oscillation in Kerr Space-Time

At first, in the flat space-time, the phase factor can be written in a conventional manner[9]

$$\Phi = \int m\,\mathrm{d}s = \int (E\,\mathrm{d}t - P\,\mathrm{d}x) = \int \eta_{\mu\nu} P^{\mu}\,\mathrm{d}x^{\nu}\,, \quad (1)$$

where the phase factor Φ is also the classical Lagrange function for the particle motion, and the route is along the geodesic determined by the variational principle or Jacob-Hamilton equation.[10] $\mathrm{d}s$ is the interval in flat space-time with $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$, and the metric

$$\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)\,.$$

In the curved space-time, Zhang and A. Beesham have studied the phase along the geodesic in Schwarzschild space-time. Now we follow this way of calculation to study the phase of the neutrino oscillation in the radial direction in Kerr space-time, and the route is along the geodesic. In Kerr space-time, the line element can be written as

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{11}\,\mathrm{d}r^2 + g_{22}\,\mathrm{d}\theta^2 + g_{33}\,\mathrm{d}\varphi^2 + 2g_{03}\,\mathrm{d}t\,\mathrm{d}\varphi\,, \quad (2)$$

where

$$g_{00} = \frac{r^2 + a^2\cos^2\theta - 2Mr}{r^2 + a^2\cos^2\theta}\,,$$

$$g_{11} = -\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2Mr}\,,$$

$$g_{22} = -(r^2 + a^2\cos^2\theta)\,,$$

$$g_{33} = -\frac{\sin^2\theta(r^2 + a^2 + 2Mra^2\sin^2\theta)}{r^2 + a^2\cos^2\theta}\,,$$

$$g_{03} = g_{30} = \frac{2Mar\sin^2\theta}{r^2 + a^2\cos^2\theta}\,,$$

$$g^{00} = \frac{(r^2 + a^2\cos^2\theta)(r^2 + a^2) - 2Ma^2r\sin^2\theta}{(r^2 + a^2\cos^2\theta)(r^2 + a^2 - 2Mr)}\,,$$

$$g^{11} = -\frac{r^2 + a^2 - 2Mr}{r^2 + a^2\cos^2\theta}\,,$$

$$g^{22} = -\frac{1}{r^2 + a^2\cos^2\theta}\,,$$

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$$g^{33} = -\frac{a^2\cot^2\theta + (r^2-2Mr)\csc^2\theta}{(r^2+a^2\cos^2\theta)(r^2+a^2-2Mr)}\,,$$

$$g^{03} = g^{30} = \frac{2Mar}{(r^2+a^2\cos^2\theta)(r^2+a^2-2Mr)}\,. \tag{3}$$

Here M is the gravitational mass of the star, and a is its angular momentum per unit mass.

We consider the phase of the mass neutrinos in Kerr space-time by means of the particle geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0\,. \tag{4}$$

Along radial direction and with $\mathrm{d}\theta = \mathrm{d}\varphi = 0$, the non-trivial equation of timelike is

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + 2\Gamma^0_{01}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0\,. \tag{5}$$

According to $\Gamma^\lambda_{\mu\nu} = g^{\lambda\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau})/2$, we have

$$\Gamma^0_{01} = \frac{g^{00}}{2}\frac{\mathrm{d}g_{00}}{\mathrm{d}r} + \frac{g^{03}}{2}\frac{\mathrm{d}g_{03}}{\mathrm{d}r}\,. \tag{6}$$

Thus equation (5) can be rewritten as

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \Big(g^{00}\frac{\mathrm{d}g_{00}}{\mathrm{d}r} + g^{03}\frac{\mathrm{d}g_{03}}{\mathrm{d}r}\Big)\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0\,. \tag{7}$$

Letting

$$A(r,\theta) = g^{00}\frac{\mathrm{d}g_{00}}{\mathrm{d}r} + g^{03}\frac{\mathrm{d}g_{03}}{\mathrm{d}r} \quad \text{and}$$

$$F(r,\theta) = \mathrm{e}^{\int A(r,\theta)\,\mathrm{d}r}\,, \tag{8}$$

equation (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[F(r,\theta)\frac{\mathrm{d}t}{\mathrm{d}s}\Big] = 0\,. \tag{9}$$

Integrating Eq. (9), we obtain

$$F(r,\theta)\frac{\mathrm{d}t}{\mathrm{d}s} = \text{const}, \quad \text{or}$$

$$E = P_0 = mF(r,\theta)\frac{\mathrm{d}t}{\mathrm{d}s} = \text{const.} \tag{10}$$

Equation (10) represents that the covariant energy is the motion constant along the geodesic, by which the calculation is proceeded. It is stressed that it is the covariant energy P_0 (not P^0) that is the constant of motion. This fact is important in later calculation of the neutrino phase. Otherwise the ambiguous definition of the neutrino energy in the gravitational field would lead to the confusion in understanding the gravitationally induced neutrino oscillation. And also P_0 does not represent the neutrino energy measured by a locally inertial observer at rest at finite radius, but rather the energy of the neutrino measured by such an observer at rest at infinity.[9]

Considering $\mathrm{d}\theta/\mathrm{d}s = \mathrm{d}\varphi/\mathrm{d}s = 0$, equation (2) can be rewritten as

$$g_{00}\Big(\frac{\mathrm{d}t}{\mathrm{d}s}\Big)^2 + g_{11}\Big(\frac{\mathrm{d}r}{\mathrm{d}s}\Big)^2 = 1\,. \tag{11}$$

Substituting Eq. (10) into Eq. (11), we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \sqrt{\frac{g_{11}}{1 - g_{00}[P_0/mF(r,\theta)]^2}}\,. \tag{12}$$

Thus the accurate phase factor in Kerr space-time along the radial direction will be acquired by next calculation

$$\begin{aligned}\Phi(\text{geod}) &= \int m\Big(\frac{\mathrm{d}s}{\mathrm{d}r}\Big)\mathrm{d}r \\ &= m\int\sqrt{\frac{g_{11}}{1 - g_{00}[P_0/mF(r,\theta)]^2}}\;\mathrm{d}r\,.\end{aligned} \tag{13}$$

But the existence of parameter θ in Kerr metric makes the result of Eq. (13) so complicated that we hardly see its physical meaning directly. So next we will only discuss the simplest case.

3 Phase When $\theta = 0$

In the simplest case, we calculate the phase when $\theta = 0$ in Eq. (13). Thus we have

$$F(r,0) = \mathrm{e}^{\int A(r,0)\,\mathrm{d}r} = \frac{r^2+a^2-2Mr}{r^2+a^2}\,,$$

and

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}r} &= \frac{1}{\sqrt{(P_0/m)^2 - 1 + 2Mr/(r^2+a^2)}} \\ &= \frac{1}{\sqrt{k + 2Mr/(r^2+a^2)}}\,,\end{aligned} \tag{14}$$

where $k = (P_0/m)^2 - 1$. So according to Eq. (13) we can calculate the interference phase in the radial direction when $\theta = 0$, that is

$$\Phi = \int\frac{m\,\mathrm{d}r}{\sqrt{k + 2Mr/(r^2+a^2)}}\,. \tag{15}$$

But the direct integral of Eq. (15) is very complicated, so we first expand the integral factor before our integral is proceeded. That is

$$\begin{aligned}\Phi &= \int\frac{m\,\mathrm{d}r}{\sqrt{k + 2Mr/(r^2+a^2)}} \\ &= \int m\Big[\frac{1}{(k+2M/r)^{1/2}} + \frac{Ma^2}{(k+2M/r)^{3/2}r^3} + O(a^3)\Big]\mathrm{d}r \\ &= \frac{mr}{k}\sqrt{k + \frac{2M}{r}} - \frac{mr_0}{k}\sqrt{k + \frac{2M}{r_0}} \\ &\quad - \frac{Mm}{k^{3/2}}\ln\Big[kr + M + r\sqrt{k\Big(k + \frac{2M}{r}\Big)}\,\Big] \\ &\quad + \frac{Mm}{k^{3/2}}\ln\Big[kr_0 + M + r_0\sqrt{k\Big(k + \frac{2M}{r_0}\Big)}\,\Big] \\ &\quad - \frac{ma^2}{M}\Big[\frac{k + M/r}{\sqrt{k + 2M/r}} - \frac{k + M/r_0}{\sqrt{k + 2M/r_0}}\Big] \\ &\quad + O(a^3)\,.\end{aligned} \tag{16}$$

In the far place from the gravitational source, we can think $2M/kr$ and $2M/kr_0$ as small quantities. Then we can make the following approximation

$$\Big(1+\frac{2M}{kr}\Big)^{1/2}\doteq 1+\frac{M}{kr}\,,$$

$$\Big(1+\frac{2M}{kr_0}\Big)^{1/2}\doteq 1+\frac{M}{kr_0}\,,$$

$$\Big(1+\frac{2M}{kr}\Big)^{-1/2}\doteq 1-\frac{M}{kr}\,,$$

$$\Big(1+\frac{2M}{kr_0}\Big)^{-1/2}\doteq 1-\frac{M}{kr_0}\,,$$

then equation (16) will be rewritten as

$$\Phi\doteq\frac{m}{\sqrt{k}}(r-r_0)-\frac{Mm}{k^{3/2}}\ln\Big(\frac{kr+M}{kr_0+M}\Big)+\frac{Mma^2}{k^{3/2}}\Big(\frac{1}{r^2}-\frac{1}{r_0^2}\Big)+O(a^3)\,. \tag{17}$$

For electron neutrino, $k\sim 10^{12}\gg 1$, we obtain

$$\Phi\doteq\frac{m^2}{P_0}(r-r_0)-\frac{Mm^4}{P_0^3}\ln\Big(\frac{kr+M}{kr_0+M}\Big)+\frac{Mm^4a^2}{P_0^3}\Big(\frac{1}{r^2}-\frac{1}{r_0^2}\Big)+O(a^3)\,. \tag{18}$$

From Eq. (18) we can see that if the star which induces the gravitational field is rotating, the angular momentum per unit mass a will have an influence on the interference phase of the mass neutrinos. However, compared with the second term in Eq. (18), the contribution of a to the phase is very little. But both the second and the third terms are also very little compared to the first term, that is, Kerr gravitation has only little contribution to the neutrino oscillation phase. And if we let $a=0$, equation (16) will reduce to the case in Schwarzschild field.

Now let us follow the very routine phase calculation as proceeded in the standard treatment of the neutrino phase in a gravitational field,

$$\Phi=\int(g_{00}P^0\,\mathrm{d}t+g_{11}P^r\,\mathrm{d}r)\,. \tag{19}$$

From the mass shell condition in Kerr space-time,

$$g_{\mu\nu}P^\mu P^\nu=m^2,\quad g_{00}(P^0)^2+g_{11}(P^r)^2=m^2\,, \tag{20}$$

we can obtain

$$g_{11}P^r=-m\Big(\frac{r^2+a^2}{r^2+a^2-2Mr}\Big)\times\sqrt{\Big(\frac{P_0}{m}\Big)^2-1+\frac{2Mr}{r^2+a^2}}\,. \tag{21}$$

From Eqs. (10) and (14) we obtain

$$\frac{\mathrm{d}t}{\mathrm{d}r}=\frac{P_0}{m[1-2Mr/(r^2+a^2)]}\times\frac{1}{\sqrt{(P_0/m)^2-1+2Mr/(r^2+a^2)}}\,. \tag{22}$$

Substituting Eqs. (21) and (22) into Eq. (19), we have

$$\Phi=\int\Big(P_0\frac{\mathrm{d}t}{\mathrm{d}r}+g_{11}P^r\Big)\mathrm{d}r=\int\frac{m\mathrm{d}r}{\sqrt{(P_0/m)^2-1+2Mr/(r^2+a^2)}}=\int\frac{m\mathrm{d}r}{\sqrt{k+2Mr/(r^2+a^2)}}\,. \tag{23}$$

This result is equivalent to Eq. (15)

4 Conclusion

In this paper, we have given the interference phase of mass neutrinos propagating in the radial direction along the geodesic in the gravitational field of a rotating spherically symmetric object, which is described by Kerr metric. With different angle θ equation (13) has different results. And almost all of these results are very complicated. So we have only calculated the phase at the condition of $\theta=0$, and gotten equation (18). From Eq. (18) we can see that because of the existence of the angular momentum per unit mass, a, the formulas of the interference phase are different from those that appear in Schwarzschild spacetime. That is, angular momentum per unit mass a has a contribution to the neutrino oscillation phase. Also our results are different from those that appear in flat spacetime. This is not surprising since the gravitational redshift for radial neutrino propagation is well known.

Here our work is only confined to the condition of $\theta=0$, but the more meaningful work is to calculate the interference phase of the mass neutrinos in any direction. So we will go on caring for the work in the future.

References

[1] Y. Fukuda, *et al.*, Phys. Rev. Lett. **81** (1998) 1158.

[2] N. Fornengo, *et al.*, Phys. Rev. **D56** (1997) 1895.

[3] C.M. Zhang and A. Beesham, Gen. Rel. Grav. **33** (2001) 1011.

[4] Wang Yong-Jiu and Tang Zhi-Ming, Commun. Theor. Phys. (Beijing, China) **37** (2002) 667.

[5] Wang Yong-Jiu and Tang Zhi-Ming, Astrophysics and Space Science **281** (2002) 689.

[6] Wang Yong-Jiu and Tang Zhi-Ming, Science in China **A45** (2002) 528.

[7] Wang Yong-Jiu and Tang Zhi-Ming, Science in China **A44** (2001) 801.

[8] Huang Xiu-Ju and Wang Yong-Jiu, Chin. Phys. Lett. **19** (2002) 755.

[9] C.Y. Cardall and G.M. Fuller, Phys. Rev. **D55** (1997) 7960.

[10] D. Piriz, M. Roy, and J. Wudka, Phys. Rev. **D54** (1996) 1587.