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# Collective Excitations in Spin-2 Bose–Einstein Condensates*

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***Abstract*** *The Green's functions and the correlation functions in spin-2 Bose–Einstein condensates at finite temperature are defined and the generalized Dyson–Beliaev equations are introduced. We discuss the spin conservation in z direction and decouple the Green's functions and the generalized Dyson–Beliaev equations according to different spin conservations in z direction. The anomalous vertex functions are introduced and the self-energies are separated into the proper self-energies and the improper self-energies. The generalized Dyson–Beliaev equations are decoupled according to separation of the self-energies. We calculate the Green's functions step by step in the Bogoliubov approximation and discuss the collective excitations in spin-2 Bose–Einstein condensates in the polar, ferromagnetic, and cyclic cases, respectively.*

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**Key words:** Bose–Einstein condensates, collective excitations, Green's function

## 1 Introduction

The observation of Bose–Einstein condensation in a remarkable series of experiments on trapped alkali metal gases[1,2] has given rise to extensive experimental and theoretical researches. For example, matter wave coherence properties,[3−6] vortices,[7−9] effects of superfluid flow,[10,11] solitons,[12−14] collective excitations,[15−18] and so on have been explored.

Since trapping a Bose condensate by purely optical means was realized several years ago,[9] which liberates the internal degrees of freedom of spinor bosons that are frozen in a magnetic trap, much attention has been attracted on the study of spinor Bose condensate, which has multiple components. Ho[20] and Ohmi and Machida[21] have researched a spin-1 BEC and obtained the general theoretical frameworks respectively. Ciobanu, Yip, and Ho[22] have studied an $F = 2$ spinor Bose condenate and drawn the phase diagrams. Koashi and Ueda have obtained the exact eigenstates of spin-1 and spin-2 vertical BEC and discussed their magnetic response.[23,24] In Refs. [25] and [26], collective excitations in spin-1 Bose–Einstein condensates are studied with the Green's function method. However, the collective excitations in spin-2 Bose–Einstein condensates have not been investigated with the Green's function method, which is the main task of our paper.

In this paper, we define the Green's function in a spin-2 Bose–Einstein condensate and calculate the spectra of excitations in a spin-2 Bose–Einstein condensate in the Bogoliubov approximation. The outline of this paper is as follows. In Sec. 2, we give the effective Hamiltonian of the spin-2 Bose gas with a canonical transformation. In Sec. 3, we define the Green's functions and the correlation functions and obtain the generalized Dyson–Beliaev equations. We also discuss the spin conservation in $z$ direction and spin transfer decomposition of Green's functions and correlation functions. We separate the self-energies into the proper parts and the improper parts and obtain the coupled Green's functions in different cases. In Sec. 4, we calculate the Green's functions step by step in the Bogoliubov approximation and discuss the collective excitations in spin-2 Bose–Einstein condensates in the polar, ferromagnetic, and cyclic cases, respectively.

## 2 Effective Hamiltonian of Spin-2 Bose Gas

We consider bosons with hyperfine spin $F = 2$, such as $^{23}$Na, $^{87}$Rb, or $^{85}$Rb, which have five components. For simplicity, we just discuss the homogeneous case in this paper. The Hamiltonian can be written in the second quantized form as

$$\begin{aligned} H = \int \mathrm{d}\boldsymbol{r} \Big[ & \frac{\hbar}{2m} \nabla \Psi_\alpha^\dagger \cdot \nabla \Psi_\alpha - \mu \Psi_\alpha^\dagger \Psi_\alpha \\ & + \frac{\bar{c}_0}{2} \Psi_\alpha^\dagger \Psi_\beta^\dagger \Psi_\beta \Psi_\alpha + \frac{\bar{c}_1}{2} (\Psi_\alpha^\dagger (\boldsymbol{F})_{\alpha\beta} \Psi_\beta)^2 \\ & + \bar{c}_2 \Psi_\alpha^\dagger \Psi_{\alpha'}^\dagger \langle 2\alpha; 2\alpha' | 00 \rangle \langle 00 | 2\beta; 2\beta' \rangle \Psi_\beta \Psi_{\beta'} \Big], \end{aligned} \tag{1}$$

where $m$ is the atomic mass, $\Psi_{+2}, \ldots, \Psi_{-2}$ are the five-component field operators corresponding to the sublevels $m_F = +2, \ldots, -2$ of the hyperfine state $F = 2$, $\mu$ is

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the chemical potential, $\bar{c}_0$, $\bar{c}_1$, and $\bar{c}_2$ are related to $s$-wave scattering length $a_0$, $a_2$, and $a_4$ of the two colliding bosons with total angular momenta 0, 2 and 4 by $\bar{c}_0 = 4\pi\hbar^2(3a_4+4a_2)/7m$, $\bar{c}_1 = 4\pi\hbar^2(a_4-a_2)/7m$ and $\bar{c}_2 = 4\pi\hbar^2(3a_4-10a_2+7a_0)/7m$, $\langle 2\alpha; 2\alpha'|00\rangle$ and $\langle 00|2\beta; 2\beta'\rangle$ are Clebsch–Gordan coefficients. $F_\alpha$ ($\alpha = x, y, z$) are $5\times 5$ spin matrices satisfying the commutation relation $[F_\alpha, F_\beta] = \varepsilon_{\alpha\beta\gamma}F_\gamma$. In this representation the spin operators are

$$F_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$F_y = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & -\sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$F_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \tag{2}$$

and the corresponding raising and lowering operators are

$$F_+ = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_- = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}. \tag{3}$$

We take a Fourier transformation as

$$\Psi(\boldsymbol{r}) = \sum_s a_s(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{4}$$

$$\Psi^\dagger(\boldsymbol{r}) = \sum_s a_s^\dagger(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{5}$$

where $a_s(\boldsymbol{k})$ and $a_s^\dagger(\boldsymbol{k})$ are annihilation and creation operators, which destroy and create one particle state of plane waves with momentum $\boldsymbol{k}$ and spin projection $s$, respectively. These operators are bosonic in our case, so they satisfy the commutation relations:

$$[a_r(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k}')] = \delta_{r,s}\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{6}$$

$$[a_r(\boldsymbol{k}), a_s(\boldsymbol{k}')] = [a_r^\dagger(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k}')] = 0. \tag{7}$$

In this formalism the Hamiltonian (1) can be rewritten as

$$H = \sum_{\boldsymbol{k}}(e_{\boldsymbol{k}}-\mu)a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k}) + \frac{1}{2}\sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4} a_{r'}^\dagger(\boldsymbol{k}_1)a_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}a_s(\boldsymbol{k}_3)a_{s'}(\boldsymbol{k}_4), \tag{8}$$

where $e_{\boldsymbol{k}} = \hbar^2k^2/2m$ stands for the kinetic energy of a particle and $\mu$ is the chemical potential of the system and

$$V_{rs}^{r's'} = c_0\delta_{rs}\delta_{r's'} + c_1(\boldsymbol{F})_{rs}(\boldsymbol{F})_{r's'} + c_2(S_+)_{rr'}(S_-)_{ss'}, \tag{9}$$

where $c_0 = \bar{c}_0$, $c_1 = \bar{c}_1$, and $c_2 = 4\bar{c}_2/5$. Here $S_+$ and $S_-$ are matrixes as follows:

$$S_+ = S_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

which correspond to creating and destroying a spin-singlet pairs respectively. In the formula (9) and in the following the convention of summing over repeated indices is applied except when stated otherwise.

Spin-2 Bose gas has one more interaction parameter than that of spin-1 Bose gas. As a result, there are three possible phases in zero magnetic field. These phases are characterized by a pair of order parameters $\langle\boldsymbol{F}\rangle$ and $\langle S_-\rangle$ describing the ferromagnetic order and the formation of singlet pairs respectively. The three phases, characterized by ($|\langle\boldsymbol{F}\rangle| = 2$, $\langle S_-\rangle$=0), ($|\langle\boldsymbol{F}\rangle| = 0$, $\langle S_-\rangle$=1), and ($|\langle\boldsymbol{F}\rangle| = 0$, $\langle S_-\rangle$=0) in zero field, are called ferromagnetic, polar, and cyclic phases, respectively.

When the temperature is below the critical temperature of Bose–Einstein condensation, the Bose gas is condensed. In the Bose–Einstein condensed phases gauge symmetry is broken and some field operators have anomalous averages. $\langle a_s(0)\rangle = \sqrt{N_0}\zeta_s$ and $\langle a_s^\dagger(0)\rangle = \sqrt{N_0}\zeta_s^\dagger$, where $N_0$ is the number of particles in the condensate and $\zeta_s$ is the normalized spinor of the condensate. For the polar, ferromagnetic and cyclic cases we can take $\zeta_r$ as $(0,0,1,0,0)^{\mathrm{T}}$, $(1,0,0,0,0)^{\mathrm{T}}$, and $(1/2,0,\sqrt{2}/2,0,1/2)^{\mathrm{T}}$, where the superscript T denotes the operation of transposition. To consider this symmetry breaking, we introduce a new set of bosonic annihilation and creation operators with a canonical transformation:

$$b_s(\boldsymbol{k}) = a_s(\boldsymbol{k}) - \delta_{\boldsymbol{k},0}\sqrt{N_0}\zeta_s, \tag{11}$$

$$b_s^\dagger(\boldsymbol{k}) = a_s^\dagger(\boldsymbol{k}) - \delta_{\boldsymbol{k},0}\sqrt{N_0}\zeta_s^\dagger. \tag{12}$$

The relation between the chemical potential and condensate density can be derived from the requirement that $\langle b_s(\boldsymbol{k})\rangle = \langle b_s^\dagger(\boldsymbol{k})\rangle = 0$. Substituting the canonical transformation into Eq. (8), one obtains the Hamiltonian in terms of the new operators:

$$
\begin{aligned}
H = & \sum_{\boldsymbol{k}}(e_{\boldsymbol{k}}-\mu_0)b_r^\dagger(\boldsymbol{k})b_r(\boldsymbol{k}) - \mu\sqrt{N_0}\left[\zeta_r^\dagger b_r(0)+b_r^\dagger(0)\zeta_r\right] - \mu N_0 + \frac{1}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}b_s(\boldsymbol{k}_3)b_{s'}(\boldsymbol{k}_4) \\
& + \frac{\sqrt{N_0}}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}b_s(\boldsymbol{k}_3)\zeta_{s'}\delta_{\boldsymbol{k}_4,0} + \frac{\sqrt{N_0}}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}\zeta_s\delta_{\boldsymbol{k}_3,0}b_{s'}(\boldsymbol{k}_4) \\
& + \frac{\sqrt{N_0}}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)\zeta_r^\dagger\delta_{\boldsymbol{k}_2,0}V_{rs}^{r's'}b_s(\boldsymbol{k}_3)b_{s'}(\boldsymbol{k}_4) + \frac{\sqrt{N_0}}{2}\sum \zeta_{r'}^\dagger\delta_{\boldsymbol{k}_1,0}b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}b_s(\boldsymbol{k}_3)b_{s'}(\boldsymbol{k}_4) \\
& + \frac{N_0}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}\zeta_s\delta_{\boldsymbol{k}_3,0}\zeta_{s'}\delta_{\boldsymbol{k}_4,0} + \frac{N_0}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)\zeta_r^\dagger\delta_{\boldsymbol{k}_2,0}V_{rs}^{r's'}b_s(\boldsymbol{k}_3)\zeta_{s'}\delta_{\boldsymbol{k}_4,0} \\
& + \frac{N_0}{2}\sum \zeta_{r'}^\dagger\delta_{\boldsymbol{k}_1,0}b_s^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}b_s(\boldsymbol{k}_3)\zeta_{s'}\delta_{\boldsymbol{k}_4,0} + \frac{N_0}{2}\sum b_{r'}^\dagger(\boldsymbol{k}_1)\zeta_r^\dagger\delta_{\boldsymbol{k}_2,0}V_{rs}^{r's'}\zeta_s\delta_{\boldsymbol{k}_3,0}b_{s'}(\boldsymbol{k}_4) \\
& + \frac{N_0}{2}\sum \zeta_{r'}^\dagger\delta_{\boldsymbol{k}_1,0}b_r^\dagger(\boldsymbol{k}_2)V_{rs}^{r's'}\zeta_s\delta_{\boldsymbol{k}_3,0}b_{s'}(\boldsymbol{k}_4) + \frac{N_0}{2}\sum \zeta_{r'}^\dagger\delta_{\boldsymbol{k}_1,0}\zeta_r^\dagger\delta_{\boldsymbol{k}_2,0}V_{rs}^{r's'}b_s(\boldsymbol{k}_3)b_{s'}(\boldsymbol{k}_4) \\
& + \frac{N_0^{3/2}}{2}\zeta_{r'}^\dagger\zeta_r^\dagger V_{rs}^{r's'}\zeta_s b_{s'}(0) + \frac{N_0^{3/2}}{2}\zeta_{r'}^\dagger\zeta_r^\dagger V_{rs}^{r's'}b_s(0)\zeta_{s'} + \frac{N_0^{3/2}}{2}\zeta_{r'}^\dagger b_r^\dagger(0)V_{rs}^{r's'}\zeta_s\zeta_{s'} \\
& + \frac{N_0^{3/2}}{2}b_{r'}^\dagger(0)\zeta_r^\dagger V_{rs}^{r's'}\zeta_s\zeta_{s'} + \frac{N_0^2}{2}\zeta_{r'}^\dagger\zeta_r^\dagger V_{rs}^{r's'}\zeta_s\zeta_{s'} + \sum_{\boldsymbol{k}}(\mu_0-\mu)b_r^\dagger(\boldsymbol{k})b_r(\boldsymbol{k})\,, \qquad (13)
\end{aligned}
$$

where the term $\sum_{\boldsymbol{k}}\mu_0 b_r^\dagger(\boldsymbol{k})b_r(\boldsymbol{k})$ ($\mu_0 \le 0$) is added and subtracted to avoid the difficulty of $\mu$ being positive in the condensed phases, which would lead to a singularity in the unperturbed propagator.

## 3 Green's Functions and Perturbation Theory

### 3.1 *Green's Functions and Dyson–Beliaev Equations*

We will generalize the method in Ref. [25] to spin-2 case. Similarly, we define the Green's function like that of the spin-1 case,

$$\mathcal{G}_{\gamma\delta}^{rs}(\boldsymbol{k},\tau) = -\left\langle T_\tau\left[b_r^\gamma(\boldsymbol{k},\tau)b_s^{\delta^\dagger}(\boldsymbol{k},0)\right]\right\rangle, \qquad (14)$$

where $\tau$ stands for the imaginary time and $T_\tau$ is the $\tau$ ordering operator. The bracket $\langle\cdots\rangle$ means that the thermodynamic average is made. The Greek indices are introduced for abbreviation with $b_s^\dagger(\boldsymbol{k}) = b_s(\boldsymbol{k})$ and $b_s^-(\boldsymbol{k}) = b_s^\dagger(-\boldsymbol{k})$

$$\mathcal{G}_{\gamma\delta}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n) = \frac{1}{\beta\hbar}\int_0^{\beta\hbar}\mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}\omega_n\tau}\mathcal{G}_{\gamma\delta}^{rs}(\boldsymbol{k},\tau)\,. \qquad (15)$$

If we do not consider the interactions between the particles, the Hamiltonian of the system can be written as

$$H_0 = \sum_{\boldsymbol{k}}(e_{\boldsymbol{k}}-\mu_0)b_r^\dagger(\boldsymbol{k})b_r(\boldsymbol{k})\,, \qquad (16)$$

which is just the first term in Eq. (13). From the noninteracting Hamiltonian (16) of the system we can define the free propagator as

$$\mathcal{G}_{(0)\gamma\delta}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n) = \frac{\delta_{rs}\delta_{\gamma\delta}}{\gamma\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}}-\mu_0)}\,. \qquad (17)$$

The Green's functions can be expanded perturbatively. After taking partial summation over the perturbation series, one obtains the generalized Dyson–Beliaev equations,

$$\mathcal{G}_{\gamma\delta}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n) = \mathcal{G}_{(0)\gamma\delta}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n) + \mathcal{G}_{(0)\gamma\rho}^{rr'}(\boldsymbol{k},\mathrm{i}\omega_n) \times \Sigma_{\rho\sigma}^{r's'}(\boldsymbol{k},\mathrm{i}\omega_n)\mathcal{G}_{\sigma\delta}^{s's}(\boldsymbol{k},\mathrm{i}\omega_n)\,, \qquad (18)$$

where $\Sigma_{\rho\sigma}^{r's'}(\boldsymbol{k},\mathrm{i}\omega_n)$ is the self-energy, the contribution of those graphs that are one-particle *irreducible* and connect to the external lines with indices $(r,\gamma)$ and $(s,\delta)$. Here *one-particle irreducible graph means the graph that cannot be split into two by cutting a single one-particle line.*

Similarly, one can introduce the correlation function such as

$$D_{r's'}^{sr}(\boldsymbol{k},\tau) = -\left\langle T_\tau\left[\sigma_{rs}(\boldsymbol{k},\tau)\sigma_{s'r'}(-\boldsymbol{k},0)\right]\right\rangle \qquad (19)$$

with

$$\sigma_{rs}(\boldsymbol{k}) = \sum_{\boldsymbol{q}} a_r^\dagger(\boldsymbol{k}+\boldsymbol{q})a_s(\boldsymbol{q})\,. \qquad (20)$$

Similar rearrangements can be carried out for the perturbation series of the correlation functions (19). One can introduce their *proper* parts (the polarization parts), *the contribution of those graphs that cannot be split into two by cutting a single interaction line representing the pseudopotential* (9). Then the equations determining the correlation function can be written as

$$D_{r's'}^{sr}(\boldsymbol{k},\mathrm{i}\omega_n) = \hbar\Pi_{r's'}^{sr}(\boldsymbol{k},\mathrm{i}\omega_n) + \Pi_{ab}^{sr}(\boldsymbol{k},\mathrm{i}\omega_n) \times V_{cd}^{ba}D_{r's'}^{dc}(\boldsymbol{k},\mathrm{i}\omega_n)\,. \qquad (21)$$

We define the anomalous correlation functions,

$$A_{a\alpha}^{sr}(\boldsymbol{k},\tau) = -\left\langle T_\tau\left[\sigma_{rs}(\boldsymbol{k},\tau)b_a^{\alpha^\dagger}(\boldsymbol{k},0)\right]\right\rangle, \qquad (22)$$

$$A_{r's'}^{a\alpha}(\boldsymbol{k},\tau) = -\left\langle T_\tau\left[b_a^\alpha(\boldsymbol{k},\tau)\sigma_{s'r'}(-\boldsymbol{k},0)\right]\right\rangle, \qquad (23)$$

which have zero value in the symmetric phase, i.e., for a noncondensed system. It is easily understood from the perturbation series of the anomalous correlation functions that they can be decomposed in such a way that,[25]

$$A_{a\alpha}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n) = \Lambda_{c\gamma}^{rs}(\boldsymbol{k},\mathrm{i}\omega_n)\mathcal{G}_{\gamma\alpha}^{ca}(\boldsymbol{k},\mathrm{i}\omega_n)\,, \qquad (24)$$

where $\mathcal{G}^{ca}_{\gamma\alpha}$ is the one-particle Green's function and $\Lambda^{rs}_{c\gamma}$ is the anomalous vertex, which is the sum of the irreducible contributions of those graphs with one incoming interaction and one incoming particle line. These anomalous vertex functions can be expressed with their proper parts and the irreducible and proper parts of the density correlation functions as

$$\Lambda^{sr}_{a\alpha}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\Lambda}^{sr}_{a\alpha}(\boldsymbol{k}, \mathrm{i}\omega_n) + \Pi^{(r)sr}_{cd}(\boldsymbol{k}, \mathrm{i}\omega_n) \times V^{dc}_{ef}\Lambda^{fe}_{a\alpha}(\boldsymbol{k}, \mathrm{i}\omega_n)\,, \tag{25}$$

where $\tilde{\Lambda}$ is the irreducible and proper anomalous vertex function and $\Pi^{(r)}$ is the irreducible and proper part of the density correlation function. With the introduction of the new set of creation and destruction operators with Eqs. (11) and (12), one can derive the relation between the condensate density and the chemical potential from the requirement that $\langle b_s(\boldsymbol{k})\rangle = \langle b^\dagger_s(\boldsymbol{k})\rangle = 0$. In Ref. [25], the authors obtain the following equation by using perturbation theory,

$$\langle b^\gamma_r(0)\rangle = \Sigma^s_{0\delta}(0,0)\mathcal{G}^{sr}_{\delta\gamma}(0,0)\,, \tag{26}$$

where $\Sigma^s_{0\delta}$ is the sum of those irreducible graphs that have only one incoming (outgoing) line. Using the symmetry properties of the Green's functions,[25] one can easily verify that the consistency condition is equivalent to

$$\Sigma^s_{0\delta} = 0\,, \tag{27}$$

from which the value of the chemical potential can be determined.

### 3.2 *Rotational Symmetry and Spin Transfer Decomposition*

The $z$ component of the total angular momentum is conserved because rotational symmetry along $z$ axis is not broken in the system. Thus, for $\mathcal{G}^{rs}_{\gamma\delta}$ and $\Sigma^{rs}_{\gamma\delta}$ the equation $\gamma r - \delta s = (\gamma - \delta)\zeta^\dagger_r(F_z)_{rs}\zeta_s$ is satisfied[25] as shown in Fig. 1. Due to the conservation of spin-$z$ component, equation (18) decouples to independent matrix equations according to the number $n = [\gamma(c-r) + \delta(c-s)]/2$, with $c = \zeta^\dagger_r(F_z)_{rs}\zeta_s$ specifying the spin carried by the propagator relative to the spin of the atom in the condensate. Following Ref. [25], we define the following matrices for the polar case similarly,

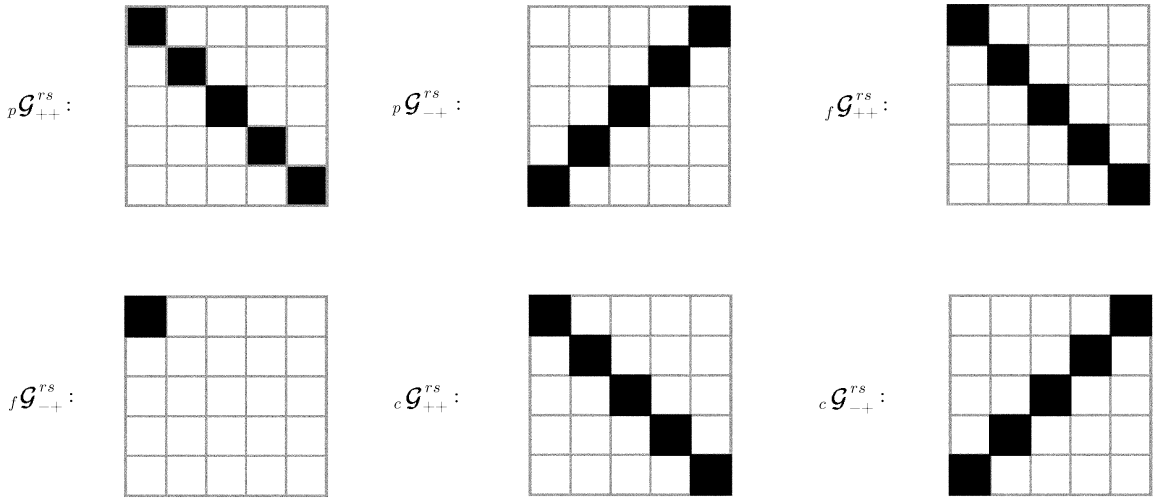


**Fig. 1** The spin conservation property of the Green's functions. Here ${}_p\mathcal{G}$ denotes the Green's functions of the polar case, ${}_f\mathcal{G}$ denotes the Green's functions of the ferromagnetic case, and ${}_c\mathcal{G}$ denotes the Green's functions of the cyclic case. A black box refers to a non-zero element and we used $\zeta = (0,0,1,0,0)^{\mathrm{T}}$ for the polar case, $\zeta = (1,0,0,0,0)^{\mathrm{T}}$ for the ferromagnetic case and $\zeta = (1,0,\sqrt{2},0,-1)^{\mathrm{T}}/2$ for the cyclic case specifying the condensate.

$$ {}^0\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{0,0}_{+,+} & \mathcal{G}^{0,0}_{+,-} \\ \mathcal{G}^{0,0}_{-,+} & \mathcal{G}^{0,0}_{-,-} \end{bmatrix}_{\gamma\delta}, \quad {}^0\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{0,0}_{+,+} & \Sigma^{0,0}_{+,-} \\ \Sigma^{0,0}_{-,+} & \Sigma^{0,0}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{28}$$

$$ {}^1\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{-1,-1}_{+,+} & \mathcal{G}^{-1,1}_{+,-} \\ \mathcal{G}^{1,-1}_{-,+} & \mathcal{G}^{1,1}_{-,-} \end{bmatrix}_{\gamma\delta}, \quad {}^1\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{-1,-1}_{+,+} & \Sigma^{-1,1}_{+,-} \\ \Sigma^{1,-1}_{-,+} & \Sigma^{1,1}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{29}$$

$$ {}^{-1}\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{1,1}_{+,+} & \mathcal{G}^{1,-1}_{+,-} \\ \mathcal{G}^{-1,1}_{-,+} & \mathcal{G}^{-1,-1}_{-,-} \end{bmatrix}_{\gamma\delta}, \quad {}^{-1}\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{1,1}_{+,+} & \Sigma^{1,-1}_{+,-} \\ \Sigma^{-1,1}_{-,+} & \Sigma^{-1,-1}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{30}$$

$$ {}^2\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{-2,-2}_{+,+} & \mathcal{G}^{-2,2}_{+,-} \\ \mathcal{G}^{2,-2}_{-,+} & \mathcal{G}^{2,2}_{-,-} \end{bmatrix}_{\gamma\delta}, \quad {}^2\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{-2,-2}_{+,+} & \Sigma^{-2,2}_{+,-} \\ \Sigma^{2,-2}_{-,+} & \Sigma^{2,2}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{31}$$

$$ {}^{-2}\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{2,2}_{+,+} & \mathcal{G}^{2,-2}_{+,-} \\ \mathcal{G}^{-2,2}_{-,+} & \mathcal{G}^{-2,-2}_{-,-} \end{bmatrix}_{\gamma\delta}, \quad {}^{-2}\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{2,2}_{+,+} & \Sigma^{2,-1}_{+,-} \\ \Sigma^{-2,2}_{-,+} & \Sigma^{-2,-2}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{32} $$

where the indices in the upper left corner correspond to the number $n$. For the ferromagnetic case, we define the similar matrices,

$$ {}^{0}\mathcal{G}_{\gamma\delta} = \begin{bmatrix} \mathcal{G}^{2,2}_{+,+} & \mathcal{G}^{2,2}_{+,-} \\ \mathcal{G}^{2,2}_{-,+} & \mathcal{G}^{2,2}_{-,-} \end{bmatrix}_{\gamma\delta}, $$

$$ {}^{0}\Sigma_{\gamma\delta} = \begin{bmatrix} \Sigma^{2,2}_{+,+} & \Sigma^{2,2}_{+,-} \\ \Sigma^{2,2}_{-,+} & \Sigma^{2,2}_{-,-} \end{bmatrix}_{\gamma\delta}, \tag{33} $$

$$ {}^{1}\mathcal{G} = \mathcal{G}^{1,1}_{+,+}, \quad {}^{1}\Sigma = \Sigma^{1,1}_{+,+}, \tag{34} $$

$$ {}^{-1}\mathcal{G} = \mathcal{G}^{-1,-1}_{-,-}, \quad {}^{-1}\Sigma = \Sigma^{-1,-1}_{-,-}, \tag{35} $$

$$ {}^{2}\mathcal{G} = \mathcal{G}^{0,0}_{+,+}, \quad {}^{2}\Sigma = \Sigma^{0,0}_{+,+}, \tag{36} $$

$$ {}^{-2}\mathcal{G} = \mathcal{G}^{0,0}_{-,-}, \quad {}^{-2}\Sigma = \Sigma^{0,0}_{-,-}, \tag{37} $$

$$ {}^{3}\mathcal{G} = \mathcal{G}^{-1,-1}_{+,+}, \quad {}^{3}\Sigma = \Sigma^{0,0}_{+,+}, \tag{38} $$

$$ {}^{-3}\mathcal{G} = \mathcal{G}^{-1,-1}_{-,-}, \quad {}^{-3}\Sigma = \Sigma^{0,0}_{-,-}, \tag{39} $$

$$ {}^{4}\mathcal{G} = \mathcal{G}^{-2,-2}_{+,+}, \quad {}^{4}\sigma = \Sigma^{0,0}_{+,+}, \tag{40} $$

$$ {}^{-4}\mathcal{G} = \mathcal{G}^{-2,-2}_{-,-}, \quad {}^{-4}\sigma = \Sigma^{0,0}_{-,-}. \tag{41} $$

For the cyclic case, we can define the same matrices as that in the polar case, because their values of $c = \zeta_r^\dagger (F_z)_{rs} \zeta_s$ are identical, i.e. they both equal zero.

For different values of $n$, there is a corresponding Dyson–Beliaev equation obtained by substituting the above definitions into Eq. (18). After calculating, one can obtain the following Green's functions straightforwardly. For all the polar and cyclic cases and for the first mode of the ferromagnetic case ($n = 0$), the Green's function can be written as[25]

$$ {}^{n}\mathcal{G}_{\alpha\gamma} = \frac{\delta_{\alpha,\gamma}(\alpha \mathrm{i}\omega_n + \hbar^{-1}\epsilon_{\boldsymbol{k}}) + \alpha\gamma\Sigma_{-\gamma,-\alpha}}{(\mathrm{i}\omega_n - \hbar^{-1}\epsilon_{\boldsymbol{k}} - \Sigma_{1,1})(\mathrm{i}\omega_n + \hbar^{-1}\epsilon_{\boldsymbol{k}} + \Sigma_{-1,-1}) + \Sigma_{-1,1}\Sigma_{1,-1}} \equiv \frac{{}^{n}N_{\alpha,\gamma}}{{}^{n}\Delta}, \tag{42} $$

where $\epsilon_{\boldsymbol{k}} = e_k - \mu_0$ and the quantities ${}^{n}N_{\alpha,\gamma}$ and ${}^{n}\Delta$ are defined. For the other modes of the ferromagnetic case ($n = \pm1, \pm2, \pm3, \pm4$), the Green's function is

$$ {}^{\pm q}\mathcal{G} = \frac{1}{\pm \mathrm{i}\omega_q - \hbar^{-1}\epsilon_{\boldsymbol{k}} - {}^{\pm q}\Sigma}, \tag{43} $$

where $q = 1, 2, 3, 4$.

The conservation of the $z$ component of the spin for these $D^{sr}_{r's'}$ correlation functions means that $r - s = r' - s'$, which holds for the proper parts as well. Thus equation (21) also decouples according to the specific spin transfer. For different spin transfers, we define the matrices as follows and use letters with double underlines to represent matrixes.

For 0 spin transfer,

$$ ({}^{0}\underline{\underline{D}})_{ab} := D^{aa}_{bb} = \begin{bmatrix} D^{2,2}_{2,2} & D^{2,2}_{1,1} & D^{2,2}_{0,0} & D^{2,2}_{-1,-1} & D^{2,2}_{-2,-2} \\ D^{1,1}_{2,2} & D^{1,1}_{1,1} & D^{1,1}_{0,0} & D^{1,1}_{-1,-1} & D^{1,1}_{-2,-2} \\ D^{0,0}_{2,2} & D^{0,0}_{1,1} & D^{0,0}_{0,0} & D^{0,0}_{-1,-1} & D^{0,0}_{-2,-2} \\ D^{-1,-1}_{2,2} & D^{-1,-1}_{1,1} & D^{-1,-1}_{0,0} & D^{-1,-1}_{-1,-1} & D^{-1,-1}_{-2,-2} \\ D^{-2,-2}_{2,2} & D^{-2,-2}_{1,1} & D^{-2,-2}_{0,0} & D^{-2,-2}_{-1,-1} & D^{-2,-2}_{-2,-2} \end{bmatrix}_{ab}, \tag{44} $$

$$ ({}^{0}\underline{\underline{\Pi}})_{ab} := \Pi^{aa}_{bb} = \begin{bmatrix} \Pi^{2,2}_{2,2} & \Pi^{2,2}_{1,1} & \Pi^{2,2}_{0,0} & \Pi^{2,2}_{-1,-1} & \Pi^{2,2}_{-2,-2} \\ \Pi^{1,1}_{2,2} & \Pi^{1,1}_{1,1} & \Pi^{1,1}_{0,0} & \Pi^{1,1}_{-1,-1} & \Pi^{1,1}_{-2,-2} \\ \Pi^{0,0}_{2,2} & \Pi^{0,0}_{1,1} & \Pi^{0,0}_{0,0} & \Pi^{0,0}_{-1,-1} & \Pi^{0,0}_{-2,-2} \\ \Pi^{-1,-1}_{2,2} & \Pi^{-1,-1}_{1,1} & \Pi^{-1,-1}_{0,0} & \Pi^{-1,-1}_{-1,-1} & \Pi^{-1,-1}_{-2,-2} \\ \Pi^{-2,-2}_{2,2} & \Pi^{-2,-2}_{1,1} & \Pi^{-2,-2}_{0,0} & \Pi^{-2,-2}_{-1,-1} & \Pi^{-2,-2}_{-2,-2} \end{bmatrix}_{ab}, \tag{45} $$

$$ ({}^{0}\underline{\underline{C}})_{ab} := V^{aa}_{bb} = \begin{bmatrix} c_0 + 4c_1 & c_0 + 2c_1 & c_0 & c_0 - 2c_1 & c_0 - 4c_1 + c_2 \\ c_0 + 2c_1 & c_0 + c_1 & c_0 & c_0 - c_1 + c_2 & c_0 - 2c_1 \\ c_0 & c_0 & c_0 + c_2/2 & c_0 & c_0 \\ c_0 - 2c_1 & c_0 - c_1 + c_2 & c_0 & c_0 + c_1 & c_0 + 2c_1 \\ c_0 - 4c_1 + c2 & c_0 - 2c_1 & c_0 & c_0 + 2c_1 & c_0 + 4c_1 \end{bmatrix}_{ab}. \tag{46} $$

For +1 spin transfer,

$$ ({}^{+1}\underline{\underline{D}})_{ab} := D^{a,a+1}_{b+1,b} = \begin{bmatrix} D^{1,2}_{2,1} & D^{1,2}_{1,0} & D^{1,2}_{0,-1} & D^{1,2}_{-1,-2} \\ D^{0,1}_{2,1} & D^{0,1}_{1,0} & D^{0,1}_{0,-1} & D^{0,1}_{-1,-2} \\ D^{-1,0}_{2,1} & D^{-1,0}_{1,0} & D^{-1,0}_{0,-1} & D^{-1,0}_{-1,-2} \\ D^{-2,-1}_{2,1} & D^{-2,-1}_{1,0} & D^{-2,-1}_{0,-1} & D^{-2,-1}_{-1,-2} \end{bmatrix}_{ab}, \tag{47} $$

$$({}^{+1}\underline{\underline{\Pi}})_{ab} := \Pi^{a,a+1}_{b+1,b} = \begin{bmatrix} \Pi^{1,2}_{2,1} & \Pi^{1,2}_{1,0} & \Pi^{1,2}_{0,-1} & \Pi^{1,2}_{-1,-2} \\ \Pi^{0,1}_{2,1} & \Pi^{0,1}_{1,0} & \Pi^{0,1}_{0,-1} & \Pi^{0,1}_{-1,-2} \\ \Pi^{-1,0}_{2,1} & \Pi^{-1,0}_{1,0} & \Pi^{-1,0}_{0,-1} & \Pi^{-1,0}_{-1,-2} \\ \Pi^{-2,-1}_{2,1} & \Pi^{-2,-1}_{1,0} & \Pi^{-2,-1}_{0,-1} & \Pi^{-2,-1}_{-1,-2} \end{bmatrix}_{ab}, \tag{48}$$

$$({}^{+1}\underline{\underline{C}})_{ab} := V^{a,a+1}_{b+1,b} = \begin{bmatrix} 2c_1 & \sqrt{6}c_1 & \sqrt{6}c_1 & 2c_1 \\ \sqrt{6}c_1 & 3c_1 & 3c_1 & \sqrt{6}c_1 \\ \sqrt{6}c_1 & 3c_1 & 3c_1 & \sqrt{6}c_1 \\ 2c_1 & \sqrt{6}c_1 & \sqrt{6}c_1 & 2c_1 \end{bmatrix}_{ab}. \tag{49}$$

For $-1$ spin transfer,

$$({}^{-1}\underline{\underline{D}})_{ab} := D^{a+1,a}_{b,b+1} = \begin{bmatrix} D^{2,1}_{1,2} & D^{2,1}_{0,1} & D^{2,1}_{-1,0} & D^{2,1}_{-2,-1} \\ D^{1,0}_{1,2} & D^{1,0}_{0,1} & D^{1,0}_{-1,0} & D^{1,0}_{-2,-1} \\ D^{0,-1}_{1,2} & D^{0,-1}_{0,1} & D^{0,-1}_{-1,0} & D^{0,-1}_{-2,-1} \\ D^{-1,-2}_{1,2} & D^{-1,-2}_{0,1} & D^{-1,-2}_{-1,0} & D^{-1,-2}_{-2,-1} \end{bmatrix}_{ab}, \tag{50}$$

$$({}^{-1}\underline{\underline{\Pi}})_{ab} := \Pi^{a+1,a}_{b,b+1} = \begin{bmatrix} \Pi^{2,1}_{1,2} & \Pi^{2,1}_{0,1} & \Pi^{2,1}_{-1,0} & \Pi^{2,1}_{-2,-1} \\ \Pi^{1,0}_{1,2} & \Pi^{1,0}_{0,1} & \Pi^{1,0}_{-1,0} & \Pi^{1,0}_{-2,-1} \\ \Pi^{0,-1}_{1,2} & \Pi^{0,-1}_{0,1} & \Pi^{0,-1}_{-1,0} & \Pi^{0,-1}_{-2,-1} \\ \Pi^{-1,-2}_{1,2} & \Pi^{-1,-2}_{0,1} & \Pi^{-1,-2}_{-1,0} & \Pi^{-1,-2}_{-2,-1} \end{bmatrix}_{ab}, \tag{51}$$

$$({}^{-1}\underline{\underline{C}})_{ab} := V^{a+1,a}_{b,b+1} = \begin{bmatrix} 2c_1 & \sqrt{6}c_1 & \sqrt{6}c_1 & 2c_1 \\ \sqrt{6}c_1 & 3c_1 & 3c_1 & \sqrt{6}c_1 \\ \sqrt{6}c_1 & 3c_1 & 3c_1 & \sqrt{6}c_1 \\ 2c_1 & \sqrt{6}c_1 & \sqrt{6}c_1 & 2c_1 \end{bmatrix}_{ab}. \tag{52}$$

For 0, $\pm1$ spin transfer cases, according to different spin transfers equation (21) decouples as follows:

$${}^{n}\underline{\underline{D}} = \hbar\, {}^{n}\underline{\underline{\Pi}} + {}^{n}\underline{\underline{\Pi}}\, {}^{n}\underline{\underline{C}}\, {}^{n}\underline{\underline{D}}. \tag{53}$$

For other spin transfer cases, since the matrix ${}^{n}\underline{\underline{C}} = \underline{\underline{0}}$, the result ${}^{n}\underline{\underline{D}} = \hbar\, {}^{n}\underline{\underline{\Pi}}$ can be obtained.

For anomalous vertex, the allowed spin projection of the incoming (outgoing) one particle is determined by the role of spin conservation and spin projection of the condensate. Here, we introduce the anomalous vertex vectors $\Lambda_\alpha$ for the polar, the ferromagnetic and the cyclic cases. In the following, we use letters with a single underline to represent vectors.

For the polar case (where $c = \zeta_r(F_z)_{rs}\zeta_s = 0$) the anomalous vertex vectors are

$${}^{0}_{p}\underline{\Lambda}_{\alpha} = \begin{pmatrix} \Lambda^{2,2}_{0,\alpha} \\ \Lambda^{1,1}_{0,\alpha} \\ \Lambda^{0,0}_{0,\alpha} \\ \Lambda^{-1,-1}_{0,\alpha} \\ \Lambda^{-2,-2}_{0,\alpha} \end{pmatrix}, \tag{54}$$

$${}^{1}_{p}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{1,2}_{-1,+} \\ \Lambda^{0,1}_{-1,+} \\ \Lambda^{-1,0}_{-1,+} \\ \Lambda^{-2,-1}_{-1,+} \end{pmatrix}, \quad {}^{1}_{p}\underline{\Lambda}_{-} = \begin{pmatrix} \Lambda^{1,2}_{1,-} \\ \Lambda^{0,1}_{1,-} \\ \Lambda^{-1,0}_{1,-} \\ \Lambda^{-2,-1}_{1,-} \end{pmatrix}, \tag{55}$$

$${}^{-1}_{p}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{2,1}_{1,+} \\ \Lambda^{1,0}_{1,+} \\ \Lambda^{0,-1}_{1,+} \\ \Lambda^{-1,-2}_{1,+} \end{pmatrix}, \quad {}^{-1}_{p}\underline{\Lambda}_{-} = \begin{pmatrix} \Lambda^{2,1}_{-1,-} \\ \Lambda^{1,0}_{-1,-} \\ \Lambda^{0,-1}_{-1,-} \\ \Lambda^{-1,-2}_{-1,-} \end{pmatrix}, \tag{56}$$

$${}^{2}_{p}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{0,2}_{-2,+} \\ \Lambda^{-1,1}_{-2,+} \\ \Lambda^{-2,0}_{-2,+} \end{pmatrix}, \quad {}^{2}_{p}\underline{\Lambda}_{-} = \begin{pmatrix} \Lambda^{0,2}_{2,-} \\ \Lambda^{-1,1}_{2,-} \\ \Lambda^{-2,0}_{2,-} \end{pmatrix}, \tag{57}$$

$${}^{-2}_{p}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{2,0}_{2,+} \\ \Lambda^{1,-1}_{2,+} \\ \Lambda^{0,-2}_{2,+} \end{pmatrix}, \quad {}^{-2}_{p}\underline{\Lambda}_{-} = \begin{pmatrix} \Lambda^{2,0}_{-2,-} \\ \Lambda^{1,-1}_{-2,-} \\ \Lambda^{0,-2}_{-2,-} \end{pmatrix}. \tag{58}$$

For the ferromagnetic case (where $c = \zeta_r(F_z)_{rs}\zeta_s = 2$) the anomalous vertex vectors are

$${}^{0}_{f}\underline{\Lambda}_{\alpha} = \begin{pmatrix} \Lambda^{2,2}_{2,\alpha} \\ \Lambda^{1,1}_{2,\alpha} \\ \Lambda^{0,0}_{2,\alpha} \\ \Lambda^{-1,-1}_{2,\alpha} \\ \Lambda^{-2,-2}_{2,\alpha} \end{pmatrix}, \tag{59}$$

$${}^{1}_{f}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{1,2}_{1,+} \\ \Lambda^{0,1}_{1,+} \\ \Lambda^{-1,0}_{1,+} \\ \Lambda^{-2,-1}_{1,+} \end{pmatrix}, \quad {}^{1}_{f}\underline{\Lambda}_{-} = 0, \tag{60}$$

$${}^{-1}_{f}\underline{\Lambda}_{+} = 0, \quad {}^{-1}_{f}\underline{\Lambda}_{-} = \begin{pmatrix} \Lambda^{2,1}_{1,-} \\ \Lambda^{1,0}_{1,-} \\ \Lambda^{0,-1}_{1,-} \\ \Lambda^{0,-2}_{1,-} \end{pmatrix}, \tag{61}$$

$${}^{2}_{f}\underline{\Lambda}_{+} = \begin{pmatrix} \Lambda^{0,2}_{0,+} \\ \Lambda^{-1,1}_{0,+} \\ \Lambda^{-2,0}_{0,+} \end{pmatrix}, \quad {}^{2}_{f}\underline{\Lambda}_{-} = 0, \tag{62}$$

$$ {}^{-2}_{f}\underline{\Lambda}_{+}=0\,,\quad {}^{-2}_{f}\underline{\Lambda}_{-}=\begin{pmatrix}\Lambda^{2,0}_{0,-}\\ \Lambda^{1,-1}_{0,-}\\ \Lambda^{0,-2}_{0,-}\end{pmatrix}\,, \tag{63}$$

$$ {}^{3}_{f}\underline{\Lambda}_{+}=\begin{pmatrix}\Lambda^{-1,2}_{-1,+}\\ \Lambda^{-2,1}_{-1,+}\end{pmatrix}\,,\quad {}^{3}_{f}\underline{\Lambda}_{-}=0\,, \tag{64}$$

$$ {}^{-3}_{f}\underline{\Lambda}_{+}=0\,,\quad {}^{-3}_{f}\underline{\Lambda}_{-}=\begin{pmatrix}\Lambda^{2,-1}_{-1,-}\\ \Lambda^{1,-2}_{-1,-}\end{pmatrix}\,, \tag{65}$$

$$ {}^{4}_{f}\underline{\Lambda}_{+}=\Lambda^{-2,2}_{-2,+}\,,\quad {}^{4}_{f}\underline{\Lambda}_{-}=0\,, \tag{66}$$

$$ {}^{-4}_{f}\underline{\Lambda}_{+}=0\,,\quad {}^{-4}_{f}\underline{\Lambda}_{-}=\Lambda^{2,-2}_{-2,-}\,. \tag{67}$$

For the cyclic case (where $c=\zeta_r(F_z)_{rs}\zeta_s=0$) the anomalous vertex vectors are identical with those in the polar case, since $c=\zeta_r(F_z)_{rs}\zeta_s$ is zero in both the two cases.

According to different spin transfers, equation (25) decouples,

$$ {}^{n}\underline{\Lambda}_{\alpha}={}^{n}\underline{\tilde{\Lambda}}_{\alpha}+{}^{n}\underline{\underline{\Pi}}^{(r)}\,{}^{n}\underline{\underline{C}}\,{}^{n}\underline{\Lambda}_{\alpha}\,, \tag{68}$$

where $\underline{\underline{\Pi}}^{(r)}$ is the matrix for the irreducible and proper part of the density correlation function and $\underline{\tilde{\Lambda}}_{\alpha}$ is the irreducible and proper anomalous vertex vector.

### 3.3 *Proper and Improper Self-energy*

For the Bose system, the anomalous averages $\langle a_s(\boldsymbol{k})\rangle=0$ for all $\boldsymbol{k}$ above the critical temperature. Thus, the anomalous correlation functions (22) and (23) are zero. In this case, the self-energies are proper and the polarization parts are irreducible. Below the critical temperature, the system is in Bose condensed phase. In this case the anomalous averages $\langle a_s(\boldsymbol{k})\rangle\neq 0$, so the anomalous correlation functions do not vanish. Therefore, the self-energies are no longer proper and the polarization parts are no longer irreducible. They can be separated as

$$ \Sigma^{rs}_{\gamma\delta}=\widetilde{\Sigma}^{rs}_{\gamma\delta}+M^{rs}_{\gamma\delta}\,, \tag{69}$$

$$ \Pi^{sr}_{r's'}=\Pi^{(r)sr}_{r's'}+\Pi^{(s)sr}_{r's'}\,, \tag{70}$$

where $\widetilde{\Sigma}$ is the contribution of those self-energy graphs that are proper, while $M$ is the contribution of the graphs that are improper, and similarly $\Pi^{(r)}$ is the contribution of those polarization graphs that are irreducible and $\Pi^{(s)}$ is the contribution of the reducible polarization graphs. Substituting Eq. (69) into Eq. (18), one can obtain the following equations:

$$ \tilde{\mathcal{G}}^{rs}_{\gamma\delta}=\mathcal{G}^{rs}_{(0)\gamma\delta}+\mathcal{G}^{rr'}_{(0)\gamma\sigma}\tilde{\Sigma}^{r's'}_{\sigma\rho}\tilde{\mathcal{G}}^{s's}_{\rho\delta}\,, \tag{71}$$

$$ \mathcal{G}^{rs}_{\gamma\delta}=\tilde{\mathcal{G}}^{rs}_{\gamma\delta}+\tilde{\mathcal{G}}^{rr'}_{\gamma\sigma}M^{r's'}_{\sigma\rho}\mathcal{G}^{s's}_{\rho\delta}\,, \tag{72}$$

where the proper Green's function $\widetilde{\mathcal{G}}$ is introduced. Then, we can calculate the Green's function in two steps. Firstly, one can obtain the proper Green's function from the free particle Green's function and the proper self-energies. In the next step, one calculates the Green's function from the proper Green's functions and the improper self-energies.

Similarly, the interaction propagator can decouple as follows:

$$ W^{rs}_{r's'}=V^{rs}_{r's'}+W^{rs}_{ab}\Pi^{(r)ba}_{cd}V^{dc}_{r's'}\,, \tag{73}$$

$$ \mathcal{W}^{rs}_{r's'}=W^{rs}_{r's'}+\mathcal{W}^{rs}_{ab}\Pi^{(s)ba}_{cd}W^{dc}_{r's'}\,, \tag{74}$$

where the effective potential $W$ is introduced.

The improper self-energies can be represented with the reducible and proper vertex functions and the effective potential such as[25]

$$ \begin{aligned}\hbar M^{ab}_{\alpha\beta}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\Lambda}^{a\alpha}_{cd}(\boldsymbol{k},\mathrm{i}\omega_n)W^{dc}_{ef}(\boldsymbol{k},\mathrm{i}\omega_n)\\ \times\tilde{\Lambda}^{fe}_{b\beta}(\boldsymbol{k},\mathrm{i}\omega_n)\,.\end{aligned} \tag{75}$$

Substituting the decomposition (69) into Eq. (42) and with a straightforward calculation, one obtains the expression

$$ {}^{n}\mathcal{G}_{\alpha\gamma}\equiv\frac{{}^{n}N_{\alpha\gamma}}{{}^{n}\Delta}=\frac{{}^{n}\tilde{N}_{\alpha\gamma}+\alpha\gamma\,{}^{n}M_{-\gamma,-\alpha}}{{}^{n}\tilde{\Delta}-{}^{n}\tilde{N}_{\sigma\tau}\,{}^{n}M_{\tau\sigma}-\det{}^{n}M}\,, \tag{76}$$

where we introduced the quantities

$$ {}^{n}\tilde{N}_{\alpha\gamma}=\delta_{\alpha\gamma}(\alpha\mathrm{i}\omega_n+\hbar^{-1}\epsilon_{\boldsymbol{k}})+\alpha\gamma\,{}^{n}\tilde{\Sigma}_{-\gamma,-\alpha}\,, \tag{77}$$

$$ \begin{aligned}{}^{n}\tilde{\Delta}=(\mathrm{i}\omega_n-\hbar^{-1}\epsilon_{\boldsymbol{k}}-{}^{n}\tilde{\Sigma}_{11})(\mathrm{i}\omega_n+\hbar^{-1}\epsilon_{\boldsymbol{k}}+{}^{n}\tilde{\Sigma}_{-1,-1})\\ +{}^{n}\tilde{\Sigma}_{-1,1}\,{}^{n}\tilde{\Sigma}_{1,-1}\end{aligned} \tag{78}$$

with ${}^{n}\tilde{\mathcal{G}}_{\alpha\gamma}={}^{n}\tilde{N}_{\alpha\gamma}/{}^{n}\tilde{\Delta}$ and $\det{}^{n}M={}^{n}M_{11}\,{}^{n}M_{-1,-1}-{}^{n}M_{1,-1}\,{}^{n}M_{-1,1}$.

## 4 Collective Excitations in Bogoliubov Approximation

In this section, we shall calculate the Green's functions in the Bogoliubov approximation. This approximation is valid at very low temperature, where all the terms coming from noncondensate density can be neglected. The proper self-energies in the Bogoliubov approximation are[25]

$$ \tilde{\Sigma}^{rs}_{\alpha\gamma}=\hbar^{-1}\big[(\mu_0-\mu)\delta_{rs}\delta_{\alpha\gamma}+N_0\zeta_{r'}\dagger V^{r's'}_{rs}\zeta_{s'}\delta_{\alpha\gamma}\big]\,, \tag{79}$$

$$ \tilde{\Sigma}^{r}_{01}=\hbar^{-1}\sqrt{N_0}\big[(-\mu)\zeta_r+N_0\zeta_{s'}\dagger\zeta_s\dagger V^{s'r'}_{sr}\zeta_{r'}\big]\,. \tag{80}$$

In the Bogoliubov approximation, the proper anomalous vertex is

$$ \tilde{\Lambda}^{sr}_{a\alpha}=\sqrt{N_0}\big[\delta_{ra}\delta_{\alpha,-1}\zeta_s+\delta_{sa}\delta_{\alpha,1}\zeta^{\dagger}_{r}\big]\,. \tag{81}$$

The regular polarization $\Pi^{(r)}$ is zero in the Bogoliubov approximation, so equation (73) becomes

$$ W^{rs}_{r's'}=V^{rs}_{r's'}\,, \tag{82}$$

i.e. the effective potential is just the pseudopotential (9). In the following, we shall calculate the Green's functions and discuss the collective excitations in condensates in the polar, ferromagnetic, and cyclic cases.

### 4.1 *Polar Case*

In the polar case, the condensate spinor can be chosen as $\zeta_s=\delta_{s,0}$. From Eqs. (80) and Eq. (27), the chemical potential $\mu=N_0(c_0+c_2/2)$ can be derived. After substituting the chemical potential and the condensate spinor

into Eq. (79), one obtains the proper self-energies as follows:

$$ {}^{0}\underline{\underline{\tilde{\Sigma}}} = \hbar^{-1} \begin{bmatrix} \mu_0 & 0 \\ 0 & \mu_0 \end{bmatrix}, \tag{83} $$

$$ {}^{\pm1}\underline{\underline{\tilde{\Sigma}}} = {}^{\pm2}\underline{\underline{\tilde{\Sigma}}} = \hbar^{-1} \begin{bmatrix} \mu_0 - N_0 c_2/2 & 0 \\ 0 & \mu_0 - N_0 c_2/2 \end{bmatrix}. \tag{84} $$

Here the proper self-energies matrixes are diagonal, the proper self-energies matrixes for $n = \pm1$ and the proper self-energies matrixes for $n = \pm2$ are identical, the proper self-energies matrix for $n = 0$ is independent of any interaction parameter, and the proper self-energies matrixes for $n = \pm1$ and $n = \pm2$ are dependent on interaction parameter $c_2$.

With the proper self-energies, one can arrive at the proper Green's functions. For $n = 0$, we have

$$ \tilde{\mathcal{G}}^{00}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1} e_{\boldsymbol{k}}}, \tag{85} $$

where the $\mu_0$ parameter has cancelled out. For the $n = +1$, the proper Green's functions are

$$ \begin{aligned} \tilde{\mathcal{G}}^{-1,-1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - N_0 c_2/2)}, \\ \tilde{\mathcal{G}}^{1,1}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - N_0 c_2/2)}, \\ \tilde{\mathcal{G}}^{-1,1}_{+-}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \tilde{\mathcal{G}}^{1,-1}_{-+}(\boldsymbol{k}, \mathrm{i}\omega_n) = 0. \end{aligned} \tag{86} $$

For $n = -1, \pm2$, the proper Green's functions are equal to the corresponding ones for $n = +1$ respectively, i.e.

$$ \begin{aligned} &\tilde{\mathcal{G}}^{1,1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-2,-2}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{2,2}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-1,-1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n), \\ &\tilde{\mathcal{G}}^{-1,-1}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{2,2}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-2,-2}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\nabla}^{1,1}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n), \\ &\tilde{\mathcal{G}}^{1,-1}_{+-}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-1,1}_{-+}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-2,2}_{+-}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{2,-2}_{-+}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{2,-2}_{+-}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-2,2}_{-+}(\boldsymbol{k}, \mathrm{i}\omega_n) = 0. \end{aligned} \tag{87} $$

Before we calculate the improper self-energies, we must know the anomalous vertex vectors. From Eq. (81) we can obtain the anomalous vertex vectors such as

$$ {}^{0}_{p}\underline{\tilde{\Lambda}}_{\alpha} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{N_0} \\ 0 \\ 0 \end{pmatrix}, \tag{88} $$

$$ {}^{1}_{p}\underline{\tilde{\Lambda}}_{+} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{N_0} \\ 0 \\ 0 \end{pmatrix}, \quad {}^{1}_{p}\underline{\tilde{\Lambda}}_{-} = \begin{pmatrix} 0 \\ \sqrt{N_0} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{89} $$

$$ {}^{-1}_{p}\underline{\tilde{\Lambda}}_{+} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{N_0} \\ 0 \\ 0 \end{pmatrix}, \quad {}^{-1}_{p}\underline{\tilde{\Lambda}}_{-} = \begin{pmatrix} 0 \\ \sqrt{N_0} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{90} $$

$$ {}^{2}_{p}\underline{\tilde{\Lambda}}_{+} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{N_0} \\ 0 \\ 0 \end{pmatrix}, \quad {}^{2}_{p}\underline{\tilde{\Lambda}}_{-} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{91} $$

$$ {}^{-2}_{p}\underline{\tilde{\Lambda}}_{+} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad {}^{-2}_{p}\underline{\tilde{\Lambda}}_{-} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{N_0} \\ 0 \\ 0 \end{pmatrix}. \tag{92} $$

In addition, in the Bogoliubov approximation, substituting them into Eq. (75), one can obtain the improper self-energies as

$$ {}^{0}\underline{\underline{M}} = \hbar^{-1} \begin{bmatrix} N_0(c_0 + c_2/2) & N_0(c_0 + c_2/2) \\ N_0(c_0 + c_2/2) & N_0(c_0 + c_2/2) \end{bmatrix}, \quad {}^{\pm1}\underline{\underline{M}} = \hbar^{-1} \begin{bmatrix} 3N_0 c_1 & 3N_0 c_1 \\ 3N_0 c_1 & 3N_0 c_1 \end{bmatrix}, \quad {}^{\pm2}\underline{\underline{M}} = 0. \tag{93} $$

Putting all together to Eq. (76), the Green's functions will be

$$ {}^{0}\mathcal{G}_{\alpha\gamma} = \frac{\delta_{\alpha\gamma}(\alpha \mathrm{i}\omega_n + \hbar^{-1} e_{\boldsymbol{k}} + \alpha\gamma\hbar^{-1} N_0(c_0 + c_2/2)}{(\mathrm{i}\omega_n)^2 - \hbar^{-2} e_{\boldsymbol{k}}(e_{\boldsymbol{k}} + 2N_0 c_0 + N_0 c_2)}, \tag{94} $$

$$ {}^{\pm1}\mathcal{G}_{\alpha\gamma} = \frac{\delta_{\alpha\gamma}[\alpha \mathrm{i}\omega_n + \hbar^{-1}(e_{\boldsymbol{k}} - N_0 c_2/2)] + 3\alpha\gamma\hbar^{-1} N_0 c_1}{(\mathrm{i}\omega_n)^2 - \hbar^{-2}(e_{\boldsymbol{k}} - N_0 c_2/2)(e_{\boldsymbol{k}} - N_0 c_2/2 + 6N_0 c_1)}, \tag{95} $$

$$ {}^{\pm 2}\mathcal{G}_{\alpha\gamma} = \frac{\delta_{\alpha\gamma}[\alpha \mathrm{i}\omega_n + \hbar^{-1}(e_{\boldsymbol{k}} - N_0 c_2/2)]}{(\mathrm{i}\omega_n)^2 - \hbar^{-2}(e_{\boldsymbol{k}} - N_0 c_2/2)^2} \,. \tag{96} $$

The spectra of collective excitations can be obtained by changing the complex frequency $\mathrm{i}\omega$ into a real one $\omega$ and setting the denominators of the Green's functions to zero. For the polar case, we arrive at

$$ {}^{0}\omega = \hbar^{-1}\sqrt{e_{\boldsymbol{k}}(e_{\boldsymbol{k}} + 2N_0c_0 + N_0c_2)} \underset{k\to 0}{\to} \sqrt{\frac{N_0(2c_0+c_2)}{2M}}k \,, \tag{97} $$

$$ {}^{\pm 1}\omega = \hbar^{-1}\sqrt{(e_{\boldsymbol{k}} - N_0c_2/2)(e_{\boldsymbol{k}} - N_0c_2/2 + 6N_0c_1)} \,, \tag{98} $$

$$ {}^{\pm 2}\omega = \hbar^{-1}(e_{\boldsymbol{k}} - N_0c_2/2) \,. \tag{99} $$

The ${}^{0}\omega$ mode, corresponding to the Green's function ${}^{0}\mathcal{G}$, has linear spectrum and is Goldstone mode. The ${}^{\pm 1}\omega$ and ${}^{\pm 2}\omega$ modes correspond to ${}^{\pm 1}\mathcal{G}$ and ${}^{\pm 2}\mathcal{G}$ respectively, which are non-Goldstone modes and start with a gap.

### 4.2 *Ferromagnetic Case*

In the ferromagnetic case, we choose the condensate spinor as $\zeta_s = \delta_{s,2}$. From Eqs. (80) and (27), the chemical potential $\mu = N_0(c_0 + 4c_1)$ can be derived. After substituting the chemical potential and the condensate spinor into Eq. (79), the proper self-energies are obtained, such as

$$ {}^{0}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}\begin{bmatrix} \mu_0 & 0 \\ 0 & \mu_0 \end{bmatrix} , \tag{100} $$

$$ {}^{\pm 1}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}(\mu_0 - 2N_0c_1) \,, \tag{101} $$

$$ {}^{\pm 2}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}(\mu_0 - 4N_0c_1) \,, \tag{102} $$

$$ {}^{\pm 3}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}(\mu_0 - 6N_0c_1) \,, \tag{103} $$

$$ {}^{\pm 4}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}(\mu_0 - 8N_0c_1) \,. \tag{104} $$

Here the proper self-energies matrix ${}^{0}\tilde{\underline{\underline{\Sigma}}}$ is diagonal. The matrixes for $n = \pm 1, \pm 2, \pm 3, \pm 4$ are $1 \times 1$ matrixes, i.e. they have only one element. The proper self-energies matrix for $n = 0$ is independent of any interaction parameter and those for $n = \pm 1, \pm 2, \pm 3, \pm 4$ are just dependent on interaction parameter $c_1$.

With the proper self-energies, one can arrive at the proper Green's functions,

$$ \tilde{\mathcal{G}}^{2,2}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1}e_{\boldsymbol{k}}} \,, $$

$$ \tilde{\mathcal{G}}^{1,1}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 2N_0c_1)} \,, $$

$$ \tilde{\mathcal{G}}^{0,0}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 4N_0c_s)} \,, $$

$$ \tilde{\mathcal{G}}^{-1,-1}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 6N_0c_1)} \,, $$

$$ \tilde{\mathcal{G}}^{-2,-2}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha \mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 8N_0c_1)} \,. \tag{105} $$

The anomalous vertex vectors are obtained from Eq. (81) as

$$ {}^{0}_{f}\tilde{\underline{\Lambda}}_{\alpha} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{106} $$

$$ {}^{1}_{f}\tilde{\underline{\Lambda}}_{+} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad {}^{1}_{f}\tilde{\underline{\Lambda}}_{-} = 0 \,, \tag{107} $$

$$ {}^{-1}_{f}\tilde{\underline{\Lambda}}_{+} = 0 \,, \quad {}^{-1}_{f}\tilde{\underline{\Lambda}}_{-} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{108} $$

$$ {}^{2}_{f}\tilde{\underline{\Lambda}}_{+} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad {}^{2}_{f}\tilde{\underline{\Lambda}}_{-} = 0 \,, \tag{109} $$

$$ {}^{-2}_{p}\tilde{\underline{\Lambda}}_{+} = 0 \,, \quad {}^{-2}_{p}\tilde{\underline{\Lambda}}_{-} = \begin{pmatrix} \sqrt{N_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{110} $$

$$ {}^{3}_{f}\tilde{\underline{\Lambda}}_{+} = \begin{pmatrix} \sqrt{N_0} \\ 0 \end{pmatrix} , \quad {}^{3}_{f}\tilde{\underline{\Lambda}}_{-} = 0 \,, \tag{111} $$

$$ {}^{-3}_{p}\tilde{\underline{\Lambda}}_{+} = 0 \,, \quad {}^{-3}_{p}\tilde{\Lambda}_{-} = \begin{pmatrix} \sqrt{N_0} \\ 0 \end{pmatrix} , \tag{112} $$

$$ {}^{4}_{f}\tilde{\underline{\Lambda}}_{+} = \sqrt{N_0} \,, \quad {}^{4}_{f}\tilde{\underline{\Lambda}}_{-} = 0 \,, \tag{113} $$

$$ {}^{-4}_{p}\tilde{\underline{\Lambda}}_{+} = 0 \,, \quad {}^{-4}_{p}\tilde{\underline{\Lambda}}_{-} = \sqrt{N_0} \,. \tag{114} $$

In addition, in the Bogoliubov approximation, substituting the above anomalous vertex vectors into Eq. (75), one can obtain the improper self-energies as

$$ {}^{0}\underline{\underline{M}} = \hbar^{-1}\begin{bmatrix} N_0(c_0+4c_1) & N_0(c_0+4c_1) \\ N_0(c_0+4c_1) & N_0(c_0+4c_1) \end{bmatrix} , \quad {}^{\pm 1}M = \hbar^{-1}N_0c_s \,, \quad {}^{\pm 2}M = {}^{\pm 3}M = {}^{\pm 4}M = 0 \,. \tag{115} $$

Putting all together to Eq. (76), the Green's functions will be

$$ {}^{0}\mathcal{G}_{\alpha\gamma} = \mathcal{G}^{2,2}_{\alpha\gamma} = \frac{\delta_{\alpha\gamma}(\alpha \mathrm{i}\omega_n + \hbar^{-1}e_{\boldsymbol{k}}) + \alpha\gamma\hbar^{-1}N_0(c_0+4c_1)}{(\mathrm{i}\omega_n)^2 - \hbar^{-2}e_{\boldsymbol{k}}(e_{\boldsymbol{k}} + 2N_0c_0 + 8N_0c_1)}\,, \tag{116} $$

$$ {}^{1}\mathcal{G}_{++} = \mathcal{G}^{1,1}_{++} = \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}e_{\boldsymbol{k}}}, \tag{117} $$

$$ {}^{-1}\mathcal{G}_{--} = \mathcal{G}^{1,1}_{--} = \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}e_{\boldsymbol{k}}}, \tag{118} $$

$$ {}^{2}\mathcal{G}_{++} = \mathcal{G}^{0,0}_{++} = \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 4N_0c_1)}\,, \tag{119} $$

$$ {}^{-2}\mathcal{G}_{--} = \mathcal{G}^{0,0}_{--} = \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 4N_0c_1)}\,, \tag{120} $$

$$ {}^{3}\mathcal{G}_{++} = \mathcal{G}^{-1,-1}_{++} = \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 6N_0c_1)}\,, \tag{121} $$

$$ {}^{-3}\mathcal{G}_{--} = \mathcal{G}^{-1,-1}_{--} = \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 6N_0c_1)}\,, \tag{122} $$

$$ {}^{4}\mathcal{G}_{++} = \mathcal{G}^{-2,-2}_{++} = \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 8N_0c_1)}\,, \tag{123} $$

$$ {}^{-4}\mathcal{G}_{--} = \mathcal{G}^{-2,-2}_{--} = \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - 8N_0c_1)}\,. \tag{124} $$

The spectra of collective excitations can be obtained by changing the complex frequency $\mathrm{i}\omega$ into a real one $\omega$ and setting the denominators of the Green's functions to zero. For the ferromagnetic case, we arrive at

$$ {}^{0}\omega = \hbar^{-1}\sqrt{e_{\boldsymbol{k}}(e_{\boldsymbol{k}} + 2N_0c_0 + 8N_0c_1)} \xrightarrow[k\to 0]{} \sqrt{\frac{N_0(c_0+4c_1)}{M}}k\,, \tag{125} $$

$$ {}^{\pm1}\omega = \pm\hbar^{-1}e_{\boldsymbol{k}}\,, \tag{126} $$

$$ {}^{\pm2}\omega = \pm\hbar^{-1}(e_{\boldsymbol{k}} - 4N_0c_1)\,, \tag{127} $$

$$ {}^{\pm3}\omega = \pm\hbar^{-1}(e_{\boldsymbol{k}} - 6N_0c_1)\,, \tag{128} $$

$$ {}^{\pm4}\omega = \pm\hbar^{-1}(e_{\boldsymbol{k}} - 8N_0c_1)\,. \tag{129} $$

The ${}^{0}\omega$ and ${}^{\pm1}\omega$ modes, corresponding to the Green's function ${}^{0}\mathcal{G}$ and ${}^{\pm1}\mathcal{G}$, respectively, are Goldstone modes. The ${}^{\pm2}\omega$, ${}^{\pm3}\omega$, and ${}^{\pm4}\omega$ modes correspond to ${}^{\pm2}\mathcal{G}$, ${}^{\pm3}\mathcal{G}$, ${}^{\pm4}\mathcal{G}$, respectively. They are non-Goldstone modes and start with a gap.

### 4.3 *Cyclic Case*

In the cyclic case, we chose $\zeta_2 = 1/2$, $\zeta_0 = \sqrt{2}$, $\zeta_{-2} = -1/2$, $\zeta_1 = \zeta_{-1} = 0$. We can derive the chemical potential $\mu = N_0(c_0 + c_2/2)$ from Eqs. (80) and (27). Substituting the chemical potential and the condensate spinor into Eq. (79), we obtain the proper self-energies as

$$ {}^{0}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}\begin{bmatrix} \mu_0 & 0 \\ 0 & \mu_0 \end{bmatrix}, \tag{130} $$

$$ {}^{\pm1}\tilde{\underline{\underline{\Sigma}}} = {}^{\pm2}\tilde{\underline{\underline{\Sigma}}} = \hbar^{-1}\begin{bmatrix} \mu_0 - N_0c_2/4 & 0 \\ 0 & \mu_0 - N_0c_2/4 \end{bmatrix}. \tag{131} $$

Here the proper self-energies matrixes are diagonal, the proper self-energies matrixes for $n = \pm1$ and the proper self-energies matrixes for $n = \pm2$ are identical, the proper self-energies matrix for $n = 0$ is independent of any interaction parameter, and the proper self-energies matrixes for $n = \pm1$ and $n = \pm2$ are dependent on interaction parameter $c_2$. However, the coefficients of $c_2$ are different from those in the polar cases.

With the proper self-energies, one can arrive at the proper Green's functions. For $n = 0$, we have

$$ \tilde{\mathcal{G}}^{00}_{\alpha\gamma}(\boldsymbol{k}, \mathrm{i}\omega_n) = \frac{\delta_{\alpha\gamma}}{\alpha\mathrm{i}\omega_n - \hbar^{-1}e_{\boldsymbol{k}}}\,. \tag{132} $$

For the $n = +1$, the proper Green's functions are

$$ \begin{aligned} \tilde{\mathcal{G}}^{-1,-1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \frac{1}{\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - N_0c_2/4)}\,, \\ \tilde{\mathcal{G}}^{1,1}_{--}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \frac{1}{-\mathrm{i}\omega_n - \hbar^{-1}(e_{\boldsymbol{k}} - N_0c_2/4)}\,, \\ \tilde{\mathcal{G}}^{-1,1}_{+-}(\boldsymbol{k}, \mathrm{i}\omega_n) &= \tilde{\mathcal{G}}^{1,-1}_{-+}(\boldsymbol{k}, \mathrm{i}\omega_n) = 0\,. \end{aligned} \tag{133} $$

For $n = -1, \pm2$, the proper Green's functions are equal to the corresponding ones for $n = +1$ respectively, i.e.

$$ \tilde{\mathcal{G}}^{1,1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-2,-2}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{2,2}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n) = \tilde{\mathcal{G}}^{-1,-1}_{++}(\boldsymbol{k}, \mathrm{i}\omega_n)\,, $$

$$\tilde{\mathcal{G}}^{-1,-1}_{--}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{2,2}_{--}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{-2,-2}_{--}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{1,1}_{--}(\boldsymbol{k},\mathrm{i}\omega_n)\,,$$
$$\tilde{\mathcal{G}}^{1,-1}_{+-}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{-1,1}_{-+}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{-2,2}_{+-}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{2,-2}_{-+}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{2,-2}_{+-}(\boldsymbol{k},\mathrm{i}\omega_n)=\tilde{\mathcal{G}}^{-2,2}_{-+}(\boldsymbol{k},\mathrm{i}\omega_n)=0\,. \tag{134}$$

The anomalous vertex vectors are obtained from Eq. (81) as

$$\overset{0}{c}\underline{\tilde{\Lambda}}_{\alpha}=\begin{pmatrix}0\\0\\\sqrt{N_0/2}\\0\\0\end{pmatrix}, \tag{135}$$

$$\overset{1}{c}\underline{\tilde{\Lambda}}_{+}=\begin{pmatrix}0\\0\\\sqrt{N_0/2}\\0\end{pmatrix},\quad \overset{1}{c}\underline{\Lambda}_{-}=\begin{pmatrix}0\\\sqrt{N_0/2}\\0\\0\end{pmatrix}, \tag{136}$$

$$\overset{-1}{c}\underline{\tilde{\Lambda}}_{+}=\begin{pmatrix}0\\0\\\sqrt{N_0/2}\\0\end{pmatrix},\quad \overset{-1}{c}\underline{\tilde{\Lambda}}_{-}=\begin{pmatrix}0\\\sqrt{N_0/2}\\0\\0\end{pmatrix}, \tag{137}$$

$$\overset{2}{c}\underline{\tilde{\Lambda}}_{+}=\begin{pmatrix}0\\0\\\sqrt{N_0/2}\end{pmatrix},\quad \overset{2}{c}\underline{\tilde{\Lambda}}_{-}=\begin{pmatrix}\sqrt{N_0/2}\\0\\0\end{pmatrix}, \tag{138}$$

$$\overset{-2}{c}\underline{\tilde{\Lambda}}_{+}=\begin{pmatrix}\sqrt{N_0/2}\\0\\0\end{pmatrix},\quad \overset{-2}{c}\underline{\tilde{\Lambda}}_{-}=\begin{pmatrix}0\\0\\\sqrt{N_0/2}\end{pmatrix}. \tag{139}$$

In addition, in the Bogoliubov approximation, substituting the above anomalous vertex vectors into Eq. (75), one can obtain the improper self-energies as

$$\overset{0}{\underline{\underline{M}}}=\hbar^{-1}\begin{bmatrix}\frac{N_0}{2}(c_0+c_2/2) & \frac{N_0}{2}(c_0+c_2/2)\\ \frac{N_0}{2}(c_0+c_2/2) & \frac{N_0}{2}(c_0+c_2/2)\end{bmatrix},\quad \overset{\pm1}{\underline{\underline{M}}}=\hbar^{-1}\begin{bmatrix}\frac{3}{2}N_0c_1 & \frac{3}{2}N_0c_1\\ \frac{3}{2}N_0c_1 & \frac{3}{2}N_0c_1\end{bmatrix},\quad \overset{\pm2}{\underline{\underline{M}}}=0\,. \tag{140}$$

Putting all together to Eq. (76), the Green's functions will be

$$\overset{0}{\mathcal{G}}_{\alpha\gamma}=\frac{\delta_{\alpha\gamma}(\alpha\mathrm{i}\omega_n+\hbar^{-1}e_{\boldsymbol{k}})+\frac{1}{2}\alpha\gamma\hbar^{-1}N_0(c_0+c_2/2)}{(\mathrm{i}\omega_n)^2-\hbar^{-2}e_{\boldsymbol{k}}(e_{\boldsymbol{k}}+N_0c_0+N_0c_2/2)}, \tag{141}$$

$$\overset{\pm1}{\mathcal{G}}_{\alpha\gamma}=\frac{\delta_{\alpha\gamma}[\alpha\mathrm{i}\omega_n+\hbar^{-1}(e_{\boldsymbol{k}}-N_0c_2/4)]+\frac{3}{2}\alpha\gamma\hbar^{-1}N_0c_1}{(\mathrm{i}\omega_n)^2-\hbar^{-2}(e_{\boldsymbol{k}}-N_0c_2/4)(e_{\boldsymbol{k}}-N_0c_2/4+3N_0c_1)}, \tag{142}$$

$$\overset{\pm2}{\mathcal{G}}_{\alpha\gamma}=\frac{\delta_{\alpha\gamma}[\alpha\mathrm{i}\omega_n+\hbar^{-1}(e_{\boldsymbol{k}}-N_0c_2/4)]}{(\mathrm{i}\omega_n)^2-\hbar^{-2}(e_{\boldsymbol{k}}-N_0c_2/4)^2}. \tag{143}$$

The spectra of collective excitations can be obtained by changing the complex frequency $\mathrm{i}\omega$ into a real one $\omega$ and setting the denominators of the Green's functions to zero. For the cyclic case, we arrive at

$$\overset{0}{\omega}=\hbar^{-1}\sqrt{e_{\boldsymbol{k}}(e_{\boldsymbol{k}}+N_0c_0+N_0c_2/2)}\underset{k\to0}{\to}\sqrt{\frac{N_0(2c_0+c_2)}{4M}}k\,, \tag{144}$$

$$\overset{\pm1}{\omega}=\hbar^{-1}\sqrt{(e_{\boldsymbol{k}}-N_0c_2/4)(e_{\boldsymbol{k}}-N_0c_2/4+3N_0c_1)}\,, \tag{145}$$

$$\overset{\pm2}{\omega}=\hbar^{-1}(e_{\boldsymbol{k}}-N_0c_2/4)\,. \tag{146}$$

The $\overset{0}{\omega}$ mode, corresponding to the Green's function $\overset{0}{\mathcal{G}}$, has linear spectrum and is Goldstone mode. The $\overset{\pm1}{\omega}$ and $\overset{\pm2}{\omega}$ modes correspond to $\overset{\pm1}{\mathcal{G}}$ and $\overset{\pm2}{\mathcal{G}}$ respectively. They are non-Goldstone modes and they start with a gap.

## 5 Conclusions

In this paper, we defined the Green's functions and the correlation functions in spin-2 Bose–Einstein condensates at finite temperature and introduced the generalized Dyson–Beliaev equations. We have discussed the spin conservation in $z$ direction and decoupled the Green's functions and the generalized Dyson–Beliaev equations according to different spin conservations in $z$ direction. In addition, we have introduced the anomalous vertex functions and separated the

self-energies into the proper self-energies and the improper self-energies. The generalized Dyson–Beliaev equations were decoupled according to separation of the self-energies. We have calculated the Green's functions step by step in the Bogoliubov approximation and discussed the collective excitations in spin-2 Bose–Einstein condensates.

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