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Quantum Statistical Entropy of Five-Dimensional Black Hole*

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Abstract *The generalized uncertainty relation is introduced to calculate quantum statistic entropy of a black hole. By using the new equation of state density motivated by the generalized uncertainty relation, we discuss entropies of Bose field and Fermi field on the background of the five-dimensional spacetime. In our calculation, we need not introduce cutoff. There is not the divergent logarithmic term as in the original brick-wall method. And it is obtained that the quantum statistic entropy corresponding to black hole horizon is proportional to the area of the horizon. Further it is shown that the entropy of black hole is the entropy of quantum state on the surface of horizon. The black hole's entropy is the intrinsic property of the black hole. The entropy is a quantum effect. It makes people further understand the quantum statistic entropy.*

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1 Introduction

Entropy of the black hole is one of the important subjects in theoretical physics. Since entropy has statistical meaning, the understanding of entropy involves the sense of the microscopic essence of the black hole. However, how does one measure the microscopic states of black hole by entropy? This problem is not solved.[1] On the other hand, since Bekenstein and Hawking[2−4] proposed that the entropy of black hole is proportional to the area of the horizon. Studying various black hole thermal properties is one of the important subjects in black hole physics.[5,6] Especially, there is a growing interest in the entropy of black hole indicated by the vast amount of existing literatures.[7−13] One frequently used method among them is the brick-wall model advanced by G't Hooft.[10] This model proposed a statistic method to calculate the black hole entropy by taking the entropy of quantum state out of black hole as the entropy. Using this method, we can derive Bekenstein–Hawking (B-H) entropy for each static state black hole.[14−18] But in calculation, we find that quantum state density is divergent near the black hole horizon. To derive B-H entropy, we have to introduce ultraviolet cutoff. This cutoff is unnatural. Subsequently, it is found that the black hole entropy mostly is the contribution of quantum state near the horizon surface. The brick-wall model is improved and the membrane model is proposed.[19,20] The membrane model only considers the quantum state within lamella near the horizon. The infrared cutoff and small mass approximation in original brick-wall model are avoided naturally. But ultraviolet cutoff remains.

Recently, references [21] ∼ [23] discussed the effect of generalized uncertainty relation on state density. They computed the quantum statistic entropy of the black hole in four-dimensional spacetime by considering the contribution of quantum state only within the range where its distance from horizon surface is less than Planck scale. It is shown that if we do not introduce cutoff, the quantum statistic entropy of black hole is only proportional to the area of horizon and there are not other divergent terms. So it provides a way for discussing the statistic origin of black hole entropy.

In this paper, we extend the method proposed in Ref. [22] and study the black hole entropy in five-dimensional spacetime. Because solving wave equation is difficult for higher-dimensional spacetime, the research of quantum statistic entropy of black hole is progressing slowly in the higher-dimensional spacetime. If we adopt quantum statistic method,[20,22] the difficulty to solve wave equation in the higher-dimensional spacetime is avoided. It is possible to derive quantum statistic entropy of black hole in the higher-dimensional spacetime. By introducing the generalized uncertainty relation to the calculation of the black hole quantum statistic entropy, we only need to discuss within the lamella with Planck scale near horizon. We need not introduce cutoff and can obtain the conclusion that the quantum statistic entropy of black hole is proportional to the area of horizon. The problem that the state density is divergent near the horizon is solved. The entropy is the number of quantum

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state near the black hole horizon within Planck scale. It is not only the intrinsic property of the black hole but also a quantum effect. The calculation result makes people further understand the quantum statistic entropy of the black hole. By using quantum statistical method, we not only obtain the conclusion that the entropy of Bose system is proportional to the area of horizon but also that the entropy of Fermi system is proportional to the area of horizon. The method given in this paper has a general meaning.

2 Entropy of Bose Field

The topological AdS black hole in five-dimensional spacetime is given by[24]

$$\mathrm{d}S^2_{\mathrm{TAdS}} = -g(r)\mathrm{d}t^2 + \frac{1}{g(r)}\mathrm{d}r^2 + r^2[\mathrm{d}\chi^2 + f_k(\chi)^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]\,, \tag{1}$$

where k describes the horizon geometry with a constant curvature, $g(r)$ and $f_k(\chi)$ are given by

$$g(r) = k - \frac{m}{r^2} + \frac{r^2}{l^2}\,, \quad f_0(\chi) = \chi\,, \quad f_1(\chi) = \sin\chi\,, \quad f_{-1}(\chi) = \sinh\chi\,. \tag{2}$$

Here we have defined $k = 1, 0$, and -1 cases as the Schwarzschild-AdS (SAdS) black hole,[25] flat-AdS (FAdS) black hole, and hyperbolic-Ads (HAdS) black hole,[26] respectively. In the case of $k = 1$, $m = 0$, we have an exact AdS_5-space with its curvature radius l. However, $m \neq 0$ generates the topological AdS black holes. The only event horizon is given by

$$r^2_{\mathrm{EH}} = \frac{l^2}{2}\Big(-k + \sqrt{k^2 + \frac{4m}{l^2}}\,\Big)\,. \tag{3}$$

According to the theory of general relativity, an observer at rest at the infinity distance obtains the frequency shift of the particles from the surface of a star as follows:

$$\nu = \nu_0\sqrt{g(r)}\,, \tag{4}$$

where ν_0 is the natural frequency of the atoms on the surface of star and ν is the one obtained by the observer at rest at the infinity distance.

Radiation temperature of the black hole is given by (equilibrium temperature)

$$T_H = \frac{k}{2\pi r_{\mathrm{EH}}} + \frac{r_{\mathrm{EH}}}{\pi l^2}\,. \tag{5}$$

From Refs. [27] and [28], the inhere temperature obtained by the observer at rest at the infinity distance is

$$T = \frac{T_H}{\sqrt{g(r)}}\,, \tag{6}$$

where $\sqrt{g(r)}$ is the red-shift factor.

Under Planck scale the quantum system must consider the effect of quantum gravitation. The standard Heisenberg uncertainty principle should be modified to the generalized uncertainty principle.[29,30] The simplest formula corresponding to the generalized coordinate-momentum uncertainty principle is[21]

$$\Delta x\Delta p \geq \frac{1}{2}\Big[\hbar + \frac{\lambda}{\hbar}(\Delta p)^2\Big]\,, \tag{7}$$

where λ is the valve of Planck length. It is a correction factor of gravitation. Equation (7) shows that the degree of location uncertainty is not arbitrarily small. Its minimum is

$$\Delta x_{\min} = 2\sqrt{\lambda}\,. \tag{8}$$

Meanwhile, reference [30] points out that according to the generalized uncertainty principle, in phase space $\mathrm{d}^n x\mathrm{d}^n p$, the number of quantum state is

$$\mathrm{d}N = \frac{\mathrm{d}^n x\mathrm{d}^n p}{(2\pi\hbar)^n(1+\lambda p^2)^n}\,, \tag{9}$$

where n is the number of freedom degree. We take the simplest functional from of the Planck unit system ($c = \hbar = K_B = 1$).

For a bosonic system, we calculate the normal canonical partition function as follows:

$$\ln Z = -\sum_i g_i\ln(1 - \mathrm{e}^{-\beta\varepsilon_i})\,. \tag{10}$$

For five-dimensional spacetimes, the area of a three-dimensional curved surface at an arbitrary point r is

$$A(r) = r^3V_3\,, \tag{11}$$

where V_3 is the volume of unit three-dimensional hypersurface. Thus, the partition function of the system at the lamella with arbitrary thickness at the point outside the horizon is

$$\ln Z = -j\int\frac{A(r)\mathrm{d}r}{\sqrt{g(r)}}\sum_i g_i\ln(1 - \mathrm{e}^{\beta\varepsilon_i}) = -j\int\frac{A(r)\mathrm{d}r}{\sqrt{g(r)}}\int\frac{V_3\nu^3\mathrm{d}\nu}{(1+\lambda(2\pi\nu)^2)^4}\ln(1 - \mathrm{e}^{\beta h\nu}) \approx j\int\frac{A(r)\mathrm{d}r}{\sqrt{g(r)}}\int_0^\infty\frac{V_3\beta h}{4(1+\lambda(2\pi\nu)^2)^4(\mathrm{e}^{\beta h\nu}-1)}\nu^4\mathrm{d}\nu\,, \tag{12}$$

where j is the spinning degeneracy of particles, $\beta = \beta_0\sqrt{g(r)}$, $p = 2\pi\nu$. The free energy of the system is

$$F = -\frac{1}{\beta_0}\ln Z = -j\int A(r)\mathrm{d}r\int_0^\infty\frac{V_3 h}{4(1+\lambda(2\pi\nu)^2)^4(\mathrm{e}^{\beta h\nu}-1)}\nu^4\mathrm{d}\nu\,. \tag{13}$$

Thus the entropy of the system is

$$S_{\mathrm{BE}} = \beta_0^2\frac{\partial F}{\partial\beta_0} = j\beta_0\int A(r)\mathrm{d}r\int_0^\infty\frac{V_3\beta h^2\,\mathrm{e}^{\beta h\nu}}{4(1+\lambda(2\pi\nu)^2)^4(\mathrm{e}^{\beta h\nu}-1)^2}\nu^5\mathrm{d}\nu$$

$$= j\frac{V_3}{4(2\pi)^4\beta_0^4}\int\frac{A(r)\,\mathrm{d}r}{g^{5/2}(r)}\int_0^\infty\frac{\mathrm{e}^x x^5\,\mathrm{d}x}{(1+\lambda x^2/\beta_0^2 g(r))^4(\mathrm{e}^x-1)^2}\,, \tag{14}$$

where $x=\beta h\nu$ and

$$I(\lambda)=\int_0^\infty\frac{\mathrm{e}^x x^5\,\mathrm{d}x}{(1+\lambda x^2/\beta_0^2 g(r))^4(\mathrm{e}^x-1)^2}\approx\int_0^\infty\frac{(x^3+x^4)\,\mathrm{d}x}{(1+\lambda x^2/\beta_0^2 g(r))^4}=\frac{1}{12}\beta_0^4\Big(\frac{g(r)}{\lambda}\Big)^2+\frac{\pi}{32}\beta_0^5\Big(\frac{g(r)}{\lambda}\Big)^{5/2}. \tag{15}$$

In view of Ref. [29], we do the integral near the horizon of the black hole and take the integral region $[r_{\rm EH}, r_{\rm EH}+\varepsilon]$, where $r_{\rm EH}$ is the location of horizon. Then

$$S_{\rm BE}=j\frac{V_3}{4(2\pi)^4\beta_0^4}\int_{r_{\rm EH}}^{r_{\rm EH}+\varepsilon}\frac{A(r)\,\mathrm{d}r}{g^{5/2}(r)}\Big[\frac{1}{12}\beta_0^4\Big(\frac{g(r)}{\lambda}\Big)^2+\frac{\pi}{32}\beta_0^5\Big(\frac{g(r)}{\lambda}\Big)^{5/2}\Big]$$

$$= j\frac{V_3}{16(2\pi)^4}\frac{A(r_{\rm EH})}{\lambda^2}\Big[\frac{1}{3}\sqrt{\frac{2\varepsilon}{\kappa}}+\frac{\pi}{8}\beta_0\frac{\varepsilon}{\lambda^{1/2}}\Big]\,. \tag{16}$$

We are only interested in the contribution from the area region near the horizon, $[r_{\rm EH}, r_{\rm EH}+\varepsilon]$, which corresponds to a proper distance of the order of the minimal length, $2\sqrt{\lambda}$. This is because the entropy approaches the upper bound only in this vicinity. Furthermore, it is just the vicinity neglected by brick-wall model. We have

$$2\sqrt{\lambda}=\int_{r_{\rm EH}}^{r_{\rm EH}+\varepsilon}\frac{\mathrm{d}r}{\sqrt{g(r)}}\approx\int_{r_{EH}}^{r_{\rm EH}+\varepsilon}\frac{\mathrm{d}r}{\sqrt{2\kappa(r-r_{\rm EH})}}=\sqrt{\frac{2\varepsilon}{\kappa}}\,, \tag{17}$$

where κ is the surface gravity at the horizon of black hole and it is identified as $\kappa=2\pi\beta_0^{-1}$. Thus we naturally derive that the entropy is proportional to the horizon area,

$$S_{\rm BE}=j\frac{V_3}{16(2\pi)^4}\frac{A(r_{\rm EH})}{\lambda^{3/2}}\Big[\frac{2}{3}+\frac{\pi^2}{2}\Big]=\alpha A(r_{\rm EH})\,, \tag{18}$$

where

$$\alpha=j\frac{V_3}{16(2\pi)^4}\frac{1}{\lambda^{3/2}}\Big[\frac{2}{3}+\frac{\pi^2}{2}\Big]\,,$$

and $A(r_{\rm EH})$ is the surface area of the black hole.

3 Fermionic Entropy

For a fermionic system, the partition function is as follows:

$$\ln Z=\sum_i g_i\ln(1+\mathrm{e}^{-\beta\varepsilon_i})\,. \tag{19}$$

Using the calculation method that is similar to the method in Sec. 2, we derive that the entropy of the system is

$$S_{\rm FE}=\beta_0^2\frac{\partial F}{\partial\beta_0}=\mathrm{i}\beta_0\int A(r)\,\mathrm{d}r\int_0^\infty\frac{V_3\beta h^2\,\mathrm{e}^{\beta h\nu}}{4(1+\lambda(2\pi\nu)^2)^4(\mathrm{e}^{\beta h\nu}+1)^2}\nu^5\,\mathrm{d}\nu$$

$$=\mathrm{i}\frac{V_3}{4(2\pi)^4\beta_0^4}\int\frac{A(r)\,\mathrm{d}r}{g^{5/2}(r)}\int_0^\infty\frac{\mathrm{e}^x x^5\,\mathrm{d}x}{(1+\lambda x^2/\beta_0^2 g(r))^4(\mathrm{e}^x+1)^2}\,. \tag{20}$$

Suppose

$$I(\mu)=\int_0^\infty\frac{\mathrm{e}^x x^5\,\mathrm{d}x}{(1+\lambda x^2/\beta_0^2 g(r))^4(\mathrm{e}^x+1)^2}=\int_0^\infty\frac{\mathrm{e}^x x^5\,\mathrm{d}x}{(1+\mu x^2)^4(\mathrm{e}^x+1)^2}=\frac{1}{6}\frac{\partial^2}{\partial\mu^2}\int_0^\infty\frac{\mathrm{e}^x x\,\mathrm{d}x}{(1+\mu x^2)^2(\mathrm{e}^x+1)^2}$$

$$=\frac{1}{6}\frac{\partial^2}{\partial\mu^2}\int_0^\infty\Big[\frac{4}{(1+\mu x^2)^3(x+2)}-\frac{3}{(1+\mu x^2)^2(x+2)}\Big]\mathrm{d}x=\frac{2}{3}\frac{\partial^2}{\partial\mu^2}\Big\{\frac{1}{(4\mu+1)^3}\int_0^\infty\Big[\frac{1}{x+2}-\frac{\mu(4\mu+1)^2x}{(1+\mu x^2)^3}$$

$$+\frac{2\mu(4\mu+1)^2}{(1+\mu x^2)^3}-\frac{\mu(4\mu+1)x}{(1+\mu x^2)^2}+\frac{2\mu(4\mu+1)}{(1+\mu x^2)^2}-\frac{\mu x}{1+\mu x^2}+\frac{2\mu}{1+\mu x^2}\Big]\mathrm{d}x\Big\}$$

$$-\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\frac{1}{(4\mu+1)^2}\int_0^\infty\Big[\frac{1}{x+2}-\frac{\mu(4\mu+1)x}{(1+\mu x^2)^2}+\frac{2\mu(4\mu+1)}{(1+\mu x^2)^2}-\frac{\mu x}{1+\mu x^2}+\frac{2\mu}{1+\mu x^2}\Big]\mathrm{d}x$$

$$=\frac{2}{3}\frac{\partial^2}{\partial\mu^2}\Big\{\frac{1}{(4\mu+1)^3}\Big[-\ln\sqrt{\mu}-\frac{(4\mu+1)^2}{4}+\frac{3\pi}{8}\sqrt{\mu}(4\mu+1)^2-\frac{4\mu+1}{2}+\frac{\sqrt{\mu}(4\mu+1)\pi}{2}+\frac{\pi}{2}\sqrt{\mu}\Big]\Big\}$$

$$-\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\Big\{\Big(\frac{1}{(4\mu+1)^2}\Big)\Big[-\ln\sqrt{\mu}-\frac{4\mu+1}{2}+\frac{\pi}{2}\sqrt{\mu}(4\mu+1)+\frac{\pi}{2}\sqrt{\mu}\Big]\Big\}\approx\frac{1}{12}\mu^{-3}=\frac{1}{12}\beta_0^6\Big(\frac{g(r)}{\lambda}\Big)^3. \tag{21}$$

So the entropy of the black hole is

$$S_{\rm EF}=i\frac{V_3\beta_0^2}{48(2\pi)^4}\int\frac{A(r)g^{1/2}(r)\,\mathrm{d}r}{\lambda^3}=i\frac{V_3}{72\pi^2\lambda^{3/2}}A(r_{\rm EH})\,, \tag{22}$$

where i is the degeneracy of fermion.

4 Discussion

In this paper, we introduce the generalized uncertainty relation to study the entropy of the black hole horizon in higher-spacetime. By considering the effect of the generalized uncertainty relation on density, the divergence appearing in the brick wall model is removed, without any cutoff. It is derived that the quantum statistic entropy is proportional to the area of the horizon. The calculation results show that the quantum statistic entropy of the black hole is the number of quantum state near the horizon under Planck scale. It is the intrinsic property of the black hole. The entropy is a quantum effect. Both Bose system and Fermi system have this property. The calculation result makes people further understand the quantum statistic entropy of the black hole. From Eqs. (18) and (22), the quantum statistic entropy corresponding to the black hole horizon is related to λ introduced in the generalized uncertainty relation. From Eq. (8), we know that $2\sqrt{\lambda}$ is the least uncertainty degree of location under Planck scale. The research on the quantum statistic entropy corresponding to the black hole horizon is a very interesting physics subject. Through analysis, we obtain that the value of the quantum statistic entropy corresponding to the black hole horizon can be determined by the value of λ. Because the value of λ can be obtained by experiment, we will derive the value of the quantum statistic entropy corresponding to the black hole horizon by experiment.

As early as 1992, Li and Liu phenomenally proposed the state equations motivated by gravity and gave the state equations of the thermal radiation field near the horizon of black hole.[31] Using the Li–Liu equation, Wang calculated the entropy of a black hole and obtained that the entropy of the black hole is proportional to the area of the horizon.[32] Using the new equation of state density motivated by the generalized uncertainty relation, we calculate the quantum statistic entropy of black hole and obtain that the quantum statistic entropy is proportional to the area of the horizon. In our calculation the divergent logarithmic term in the original brick-wall method does not exist. It is shown that the introduction of the generalized uncertainty relation can eliminate the divergence of state density near the horizon surface. We start with different consideration, but obtain the same conclusion. So there must be an inherent relationship between the Li–Liu equation and generalized uncertainty relation, which is an important subject of theoretic physics we need to research.

Using the quantum statistic method, we discuss the five-dimensional spacetime. The difficulty to solve the wave equations is avoided. We provide a way to study the entropy in the higher-dimensional spacetime.

References

[1] S. Liberati, Il Nuovo Cimento B **112** (1997) 405.

[2] J.D. Bekenstein, Phys. Rev. D **7** (1973) 2333.

[3] S.W. Hawking, Commun. Math. Phys. **43** (1975) 199.

[4] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15** (1977) 2738.

[5] Z. Zhao, *Thermal Properties of Black Hole and Singularities of Space-time*, Beijing Normal University Press, Beijing (1999) (in Chinese).

[6] W.P. Amanda, Class. Quant. Grav. **15** (1998) 3291.

[7] D. Hochberg, T.W. Kephart, and J.W. York, Phys. Rev. D **48** (1993) 479.

[8] T. Padmanaban, Phys. Lett. A **136** (1989) 203.

[9] X. Li and Z. Zhao, Phys. Rev. D **62** (2000) 104001.

[10] G't Hooft, Nucl. Phys. B **256** (1985) 727.

[11] G. Cognola and P. Lecca, Phys. Rev. D **57** (1998) 1108.

[12] R.G. Cai, J.Y. Ji, and K.S. Soh, Class. Quant. Grav. **15** (1998) 2783.

[13] D. Chang, C.S. Chu, and F.L. Lin, Phys. Lett. B **583** (2004) 192.

[14] S.N. Solodukhin, Phys. Rev. D **51** (1995) 609.

[15] R. Zhao, Y.Q. Wu, and L.C. Zhang, Gen. Rel. Grav. **36** (2004) 2539.

[16] M.H. Lee and J.K. Kim, Phys. Rev. D **54** (1996) 3904.

[17] J.L. Jing and M.L. Yan, Phys. Rev. D **63** (2001) 084028.

[18] G.Y. Shen and D.M. Chen, Gen. Rel. Grav. **32** (2000) 2269.

[19] X. Li and Z. Zhao, Phys. Rev. D **62** (2000) 3904.

[20] R. Zhao, J.F. Zhang, and L.C. Zhang, Nucl. Phys. B **609** (2001) 247.

[21] X. Li, Phys. Letts. B **540** (2002) 9.

[22] R. Zhao, Y.Q. Wu, and L.C. Zhang, Class. Quant. Grav. **20** (2003) 4885.

[23] C.Z. Liu, X. Li, and Z. Zhao, Int. J. Theor. Phys. **42** (2003) 2081.

[24] Y.S. Myung, Phys. Lett. B **574** (2003) 289; Class. Quant. Grav. **21** (2004) 1279.

[25] S. Mukherji and S.S. Pal, JHEP **0205** (2002) 026.

[26] R.G. Cai, Phys. Rev. D **63** (2001) 124018.

[27] M.M. Akbar and S. Das, Class. Quant. Grav. **21** (2004) 1383.

[28] R.C. Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford University Press, Oxford (1934).

[29] A. Kempt, G. Mangano, and R.B. Mann, Phys. Rev. D **52** (1995) 1108.

[30] L.N. Cheng, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65** (2002) 125028.

[31] L.X. Li and L. Liu, Phys. Rev. D **46** (1992) 3296.

[32] D.X. Wang, Phys. Rev. D **50** (1994) 7385.