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Transverse Vector Vertex Function and Transverse Ward–Takahashi Relations in QED*

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Abstract *The transverse vector vertex function in momentum space in four-dimensional QED is derived in terms of a set of transverse Ward–Takahashi relations for the vector and the axial-vector vertices in the case of massless fermion. It is demonstrated explicitly that the transverse vector vertex function derived this way to one-loop order leads to the same result as one obtained in perturbation theory. This provides a basic approach to determine the transverse part of basic vertex function from the symmetry relations of the system.*

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1 Introduction

Gauge symmetry imposes powerful constraints on the basic interaction vertex functions of gauge theories, leading to the exact relations among Green's functions — the Ward–Takahashi (WT) identities.[1] They play the important role in providing the consistency conditions in perturbation theory as well as in the nonperturbative studies of gauge theories. The well-known WT identity for the vector vertex function Γ_V^μ in momentum space is

$$q_\mu \Gamma_V^\mu(p_1, p_2) = S_F^{-1}(p_2) - S_F^{-1}(p_1)\,, \tag{1}$$

where $q = p_2 - p_1$, p_2 and p_1 are the incoming and outgoing momenta, respectively, and $S_F(p_1)$ is the full fermion propagator. But the normal WT identity for the vertex specifies only its longitudinal part, leaving the transverse part undetermined.

The knowledge of the three-point vertices is crucial in the nonperturbative studies of gauge theories through the use of the Dyson–Schwinger equations (DSEs),[2] where the transverse vertex, i.e. the transverse part of the vertex has long been known to play the crucial role in ensuring multiplicative renormalizability and also in determining the propagator.[2−4] Therefore, in the past years much effort has been devoted to constructing the transverse part of the vertex based on an ansatz guided by perturbative constraints through the one-loop evaluation of the vertex.[2,4] However, such a constructed vertex is not unique since it is not fixed by the symmetry of the system. The latter provides the key point in determining the transverse part of the vertex: Like the longitudinal part, the transverse part of the vertex should be determined also by the WT-type constraint relations called the transverse WT relations.[5−7]

In this work we provide a basic approach to determine the transverse part of basic vertex functions from the symmetry relations of the system. At first we present the complete expressions of a set of transverse WT relations for the vector and the axial-vector vertex functions in momentum space, where the transverse WT relations involve the integral terms arising from the four-point functions which were neglected in earlier works.[6−8] Then we derive the transverse part of the vector vertex function in four-dimensional Abelian gauge theory — QED by self-consistently solving this set of transverse WT relations for the vector and axial-vector vertex functions in momentum space in the case of massless fermion without any ansatz. We demonstrate explicitly that this transverse vector vertex function to one-loop order leads to the same result as one obtained in perturbation theory.

2 Transverse Ward–Takahashi Relations for Vector and Axial Vector Vertex Functions

We begin with the transverse WT relation for the vector vertex function. The normal (longitudinal) WT identity (1) in coordinate space is related to the divergence of the time-ordered products of the three-point Green function involving the vector current operator,[1,6] while the transverse WT relation for the vector vertex is related to the curl of the time-ordered products of the three-point function involving the vector current operator.[6,7] By carefully computing

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the Fourier transformation in terms of the standard definition for the Fourier transformation of the three-point function, we obtain the transverse WT relation for the vector vertex function,

$$\begin{aligned} \mathrm{i}q^\nu\Gamma_V^\mu(p_1,p_2) - \mathrm{i}q^\mu\Gamma_V^\nu(p_1,p_2) &= S_F^{-1}(p_1)\sigma^{\mu\nu} + \sigma^{\mu\nu}S_F^{-1}(p_2) + 2m\Gamma_{\mathrm{T}}^{\mu\nu}(p_1,p_2) \\ &+ (p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2) - \int\frac{\mathrm{d}^4k}{(2\pi)^4}2k_\lambda\varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2;k)\,, \end{aligned} \tag{2}$$

where $q = p_2 - p_1$, $\Gamma_T^{\mu\nu}$ is the tensor vertex function in momentum space, and $U_P(x',x) = P\exp(-\mathrm{i}g\int_x^{x'}\mathrm{d}y^\rho A_\rho(y))$ is the Wilson line, which is introduced in order to keep the local gauge invariance, where A_μ is the gauge field. In the QED case, $g = e$ and A_ρ is the photon field. Here the integral term, which involves $\Gamma_{A\rho}(p_1,p_2;k)$ with the internal momentum k of the gauge boson appearing in the Wilson line, appear. $\Gamma_{A\rho}(p_1,p_2;k)$ is defined by

$$\begin{aligned} &\int \mathrm{d}^4x\,\mathrm{d}^4x'\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}(p_1\cdot x_1 - p_2\cdot x_2 + (k_1-k)\cdot x - (k_2-k)\cdot x')}\langle 0|T\bar\psi(x')\gamma_\rho\gamma_5 U_P(x',x)\psi(x)\psi(x_1)\bar\psi(x_2)|0\rangle \\ &\quad = (2\pi)^4\delta^4(p_1-p_2+k_1-k_2)\mathrm{i}S_F(p_1)\Gamma_{A\rho}(p_1,p_2;k)\mathrm{i}S_F(p_2)\,. \end{aligned} \tag{3}$$

Hence the integral term is the four-point function, which was neglected in Refs. [6] ∼ [8].

The transverse WT relation for the axial-vector vertex in coordinate space is related to the curl of the time-ordered products of the three-point function involving the axial vector current operator.[8] By carefully computing the Fourier transformation, we obtain the transverse WT relation for the axial-vector vertex function in momentum space,

$$\begin{aligned} \mathrm{i}q^\nu\Gamma_A^\mu(p_1,p_2) - \mathrm{i}q^\mu\Gamma_A^\nu(p_1,p_2) &= S_F^{-1}(p_1)\sigma^{\mu\nu}\gamma_5 - \sigma^{\mu\nu}\gamma_5 S_F^{-1}(p_2) + (p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2) \\ &- \int\frac{\mathrm{d}^4k}{(2\pi)^4}2k_\lambda\varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2;k)\,. \end{aligned} \tag{4}$$

Substituting $\gamma_\rho\gamma_5$ by γ_ρ in the matrix element of Eq. (3), we obtain the definition for $\Gamma_{V\rho}$ $(p_1,p_2;k)$. The integral term in Eq. (4) was neglected in Ref. [8].

Equations (2) and (4) show that the transverse parts of the vector and the axial-vector vertex functions are coupled with each other. It implies that the transverse parts of the vector and axial-vector vertex functions are not independent of each other in four-dimensional space-time. In the limit with zero fermion mass, the contribution from $m\Gamma_T^{\mu\nu}$ disappears and hence Eqs. (2) and (4) form formally a complete set of transverse WT relations for the vector and the axial-vector vertices. Then we can derive the transverse vector and axial-vector vertex functions in terms of this set of WT relations.

3 Transverse Vector Vertex Function

We now derive the transverse vector vertex function by consistently solving the set of WT relations for the vector and the axial-vector vertex functions in the case of massless fermion. To do this, we multiply both sides of Eqs. (2) and (4) by $-\mathrm{i}q_\nu$, and move the terms proportional to $q_\nu\Gamma_V^\nu$ and $q_\nu\Gamma_A^\nu$ into the right-hand side of the equations, and then substituting one equation obtained from Eq. (4) into another one and using Eq. (1). After lengthy computations we obtain the following full vector vertex function:

$$\Gamma_V^\mu(p_1,p_2) = \Gamma_{V(L)}^\mu(p_1,p_2) + \Gamma_{V(T)}^\mu(p_1,p_2) \tag{5}$$

with

$$\Gamma_{V(L)}^\mu(p_1,p_2) = q^{-2}q^\mu[S_F^{-1}(p_2) - S_F^{-1}(p_1)]\,, \tag{6}$$

and

$$\begin{aligned} \Gamma_{V(T)}^\mu(p_1,p_2) = {}& [q^2 + (p_1+p_2)^2 - ((p_1+p_2)\cdot q)^2q^{-2}]^{-1}\{-\mathrm{i}S_F^{-1}(p_1)\sigma^{\mu\nu}q_\nu - \mathrm{i}\sigma^{\mu\nu}q_\nu S_F^{-1}(p_2) \\ &+ \mathrm{i}[S_F^{-1}(p_1)\sigma^{\mu\lambda} - \sigma^{\mu\lambda}S_F^{-1}(p_2)](p_{1\lambda}+p_{2\lambda}) \\ &+ \mathrm{i}[S_F^{-1}(p_1)\sigma^{\lambda\nu} - \sigma^{\lambda\nu}S_F^{-1}(p_2)]q_\nu(p_{1\lambda}+p_{2\lambda})q^\mu q^{-2} - \mathrm{i}[S_F^{-1}(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu}S_F^{-1}(p_2)]q_\nu(p_1+p_2)\cdot q q^{-2} \\ &- \mathrm{i}[S_F^{-1}(p_1)\sigma^{\lambda\nu} + \sigma^{\lambda\nu}S_F^{-1}(p_2)]q_\nu(p_{1\lambda}+p_{2\lambda})[p_1^\mu + p_2^\mu - q^\mu(p_1+p_2)\cdot q q^{-2}]q^{-2} \\ &+ \mathrm{i}q_\nu C_A^{\mu\nu} + q_\nu q_\alpha q^{-2}(p_{1\lambda}+p_{2\lambda})\varepsilon^{\lambda\mu\nu\rho}C_V^{\rho\alpha} \\ &+ \mathrm{i}q_\nu(p_{1\lambda}+p_{2\lambda})[p_1^\mu + p_2^\mu - q^\mu(p_1+p_2)\cdot q q^{-2}]q^{-2}C_A^{\lambda\nu}\}\,, \end{aligned} \tag{7}$$

where

$$C_A^{\mu\nu} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2;k)\,, \tag{8}$$

$$C_V^{\mu\nu} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho}\Gamma_{V\rho}(p_1,p_2;k)\,. \tag{9}$$

Here $\Gamma^\mu_{V(L)}(p_1,p_2)$ denotes the longitudinal part of the vertex, which is a natural result of the normal WT relation (1), while $\Gamma^\mu_{V(T)}(p_1,p_2)$ is the transverse part of the vertex, which is derived from the transverse WT relations for the vector and axial-vector vertices, Eqs. (2) and (4). All of them are deduced from the symmetry relations. Consequently, the longitudinal vertex (6) as well as the transverse vertex (7), and then the full vertex function should be satisfied both perturbatively and nonperturbatively.

In the following, let us demonstrate explicitly that the transverse vector vertex given by Eq. (7) is satisfied to one-loop order in perturbation theory. We notice that the transverse vertex $\Gamma^\mu_{V(T)}$ includes two parts of contributions from the full fermion propagator and four-point functions, which may be denoted as $\Gamma^{\mu(I)}_{V(T)}$ and $\Gamma^{\mu(II)}_{V(T)}$, respectively. To demonstrate $\Gamma^\mu_{V(T)}$ given by Eqs. (7) being satisfied to one-loop order, we require to calculate the contributions from four-point functions. In perturbation theory the four-point function (3) and then the integral terms in Eqs. (2) and (4) can be calculated order by order. For example, the integral term at one-loop order in Eq. (3) can be written straightforwardly,[9,10]

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho}\Gamma_{A\rho}(p_1,p_2;k) &= g^2\int \frac{\mathrm{d}^4k}{(2\pi)^4} 2k_\lambda \varepsilon^{\lambda\mu\nu\rho}\gamma^\alpha \frac{1}{\not p_1-\not k-m}\gamma^\rho\gamma_5\frac{1}{\not p_2-\not k-m}\gamma^\beta\frac{-\mathrm{i}}{k^2}\Big[g_{\alpha\beta}+(\xi-1)\frac{k_\alpha k_\beta}{k^2}\Big]\\ &+g^2\int \frac{\mathrm{d}^4k}{(2\pi)^4} 2\varepsilon^{\alpha\mu\nu\rho}\Big[\gamma^\beta\frac{1}{\not p_1-\not k-m}\gamma^\rho\gamma_5+\gamma^\rho\gamma_5\frac{1}{\not p_2-\not k-m}\gamma^\beta\Big]\\ &\times\frac{-\mathrm{i}}{k^2}\Big[g_{\alpha\beta}+(\xi-1)\frac{k_\alpha k_\beta}{k^2}\Big]\,,\end{aligned} \tag{10}$$

where $\not k=\gamma_\mu k^\mu$, and ξ is the covariant-gauge parameter. The last two terms in the right-hand side of Eq. (10) are the one-loop self-energy contributions, which are required to maintain the gauge invariance, accompanying the vertex correction. Instead of $\gamma^\rho\gamma_5$ by γ^ρ in Eq. (10) then the integral term in Eq. (4) can be written.

The integral term (10) was calculated in Refs. [9] and [10] where the transverse WT relation (2) was demonstrated to be satisfied at one-loop order. We have

$$C_A^{\mu\nu} = -\Sigma(p_1)\sigma^{\mu\nu} - \sigma^{\mu\nu}\Sigma(p_2) - Q_V^{\mu\nu}\,, \tag{11}$$

where $\Sigma(p_i)$ ($i=1,2$) is the one-loop fermion self-energy and

$$\begin{aligned}Q_V^{\mu\nu} = -\frac{\mathrm{i}\alpha}{4\pi^3}\{&\gamma_\alpha\not p_1(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)\not p_2\gamma^\alpha J^{(0)} - \gamma_\alpha[\not p_1(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)\gamma^\lambda\\ &+\gamma^\lambda(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)\not p_2]\gamma^\alpha J^{(1)}_\lambda + \gamma_\alpha\gamma^\lambda(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)\gamma^\eta\gamma^\alpha J^{(2)}_{\lambda\eta}\\ &+(\xi-1)[(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)K^{(0)} - (p_1^2\gamma^\lambda\sigma^{\mu\nu}+p_2^2\sigma^{\mu\nu}\gamma^\lambda+\not p_1\sigma^{\mu\nu}\not p_2\gamma^\lambda\\ &+\gamma^\lambda\not p_1\sigma^{\mu\nu}\not p_2)J^{(1)}_\lambda + \gamma^\lambda(p_1^2\sigma^{\mu\nu}\not p_2+p_2^2\not p_1\sigma^{\mu\nu})\gamma^\eta I^{(2)}_{\lambda\eta}]\}\,.\end{aligned} \tag{12}$$

Here $\alpha=g^2/4\pi$, $J^{(0)}$, $J^{(1)}_\lambda$, $J^{(2)}_{\lambda\eta}$, $K^{(0)}$, and $I^{(2)}_{\lambda\eta}$ are some integrals,[4,10] for instance,

$$J^{(0)} = \int_M \mathrm{d}^4k\frac{1}{k^2[(p_1-k)^2+\mathrm{i}\epsilon][(p_2-k)^2+\mathrm{i}\epsilon]}\,. \tag{13}$$

These integrals can be carried out in the cutoff regularization scheme or in the dimensional regularization scheme.[4]

The integral term (9) at one-loop order can be performed similarly. We find

$$C_V^{\mu\nu} = -\Sigma(p_1)\sigma^{\mu\nu}\gamma_5 + \sigma^{\mu\nu}\gamma_5\Sigma(p_2) - Q_A^{\mu\nu}\,, \tag{14}$$

where $Q_A^{\mu\nu}$ can be obtained from $Q_V^{\mu\nu}$ if instead of the factors $(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)$, $\gamma^\lambda\sigma^{\mu\nu}$ and $\sigma^{\mu\nu}\gamma^\lambda$ in Eq. (14) by $(\not p_1\sigma^{\mu\nu}+\sigma^{\mu\nu}\not p_2)\gamma_5$, $\gamma^\lambda\sigma^{\mu\nu}\gamma_5$ and $\sigma^{\mu\nu}\gamma^\lambda\gamma_5$, respectively.

Now substituting the fermion propagator to one-loop order $S_F^{-1}(p_i)=\not p_i-\Sigma(p_i)$, $i=1,2$ together with Eqs. (11) $\sim$ (14) into Eq. (7), after some algebraic calculations we obtain

$$\Gamma_V^\mu = \Gamma^\mu_{V(T)} + \Gamma^\mu_{V(L)} = \gamma^\mu + \Lambda^\mu_V\,, \tag{15}$$

where

$$\Lambda_V^\mu(p_1,p_2) = -\frac{i\alpha}{4\pi^3}\{\gamma^\alpha \not{p}_1\gamma^\mu\not{p}_2\gamma_\alpha J^{(0)} - (\gamma^\alpha\not{p}_1\gamma^\mu\gamma^\lambda\gamma_\alpha + \gamma^\alpha\gamma^\lambda\gamma^\mu\not{p}_2\gamma_\alpha)J_\lambda^{(1)} + \gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^f\gamma_\alpha J_{\lambda f}^{(2)}$$
$$+ (\xi - 1)[\gamma^\mu K^{(0)} - (\gamma^\lambda\not{p}_1\gamma^\mu + \gamma^\mu\not{p}_2\gamma^\lambda)J_\lambda^{(1)} + \gamma^\lambda\not{p}_1\gamma^\mu\not{p}_2\gamma^f I_{\lambda f}^{(2)}]\}\,, \tag{16}$$

which is the familiar expression of one-loop vector vertex function in perturbation theory.[4] This shows that the full vector vertex function given by Eqs. (5) $\sim$ (7) to one-loop order leads to the same result as one obtained in perturbation theory by using Feynman rules. Since the longitudinal part of the vertex (6) is satisfied both perturbatively and nonperturbatively, thus the transverse part of the vertex (7) is satisfied to one-loop in perturbation theory. We notice that the four-point functions, $\Gamma_{A\rho}(p_1,p_2;k)$ and $\Gamma_{V\rho}(p_1,p_2;k)$, can be calculated order by order in perturbation theory, thus we may demonstrate that the full vector vertex as well as the transverse part of the vertex function from symmetry relations should be satisfied order by order in perturbation theory.

4 Conclusion and Discussion

In this paper we have derived the transverse part of the vector vertex function in momentum space in four-dimensional QED in terms of a set of transverse WT relations for the vector and the axial-vector vertex functions in the case of massless fermion. Combining the transverse part of the vertex with the longitudinal part of the vertex specified by the normal WT relation for the vertex, we obtain the full vector vertex function. Such a transverse vertex function should be satisfied perturbatively and nonperturbatively because it is determined by the symmetry relations which should be satisfied both perturbatively and nonperturbatively. We have demonstrated explicitly that such a derived vertex function is satisfied indeed to one-loop order. Thus this provides a basic approach to determine the transverse part of the basic vertex function from the symmetry relations of the system.

We notice that the transverse part of the vector vertex from the symmetry relations, $\Gamma^\mu_{V(T)}$, includes two parts of contributions from the full fermion propagator and the four-point functions, $\Gamma^{\mu(I)}_{V(T)}$ and $\Gamma^{\mu(II)}_{V(T)}$, respectively. In the application of the vertex function to the DSE for the fermion propagator, the second part contribution of the vertex from the four-point functions, $\Gamma^{\mu(II)}_{V(T)}$, needs to be considered further. The simplest way is to neglect $\Gamma^{\mu(II)}_{V(T)}$, which corresponds to the cutoff of four-point functions. In this case, it is easy to check that $\Gamma^\mu_{V(L)}(p_1,p_2)+\Gamma^{\mu(I)}_{V(T)}(p_1,p_2) \to \gamma^\mu$ when the fermion propagator is taken as bare one. It shows that $\Gamma^{\mu(I)}_{V(T)}$ gives the leading contribution to the transverse part of the vertex. In principle, the four-point functions might be deduced by considering the corresponding WT relations for them, which needs to be studied further.

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