Commun. Theor. Phys. (Beijing, China) **37** (2002) pp 567–570
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Vol. 37, No. 5, May 15, 2002

Quantization of the O(N) Nonlinear Sigma Model

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(Received October 15, 2001)

Abstract *The Hamilton–Jacobi method of quantizing singular systems is discussed. The equations of motion are obtained as total differential equations in many variables. It is shown that if the system is integrable, one can obtain the canonical phase space coordinates and set of canonical Hamilton–Jacobi partial differential equations without any need to introduce unphysical auxiliary fields. As an example we quantize the O(2) nonlinear sigma model using two different approaches: the functional Schrödinger method to obtain the wave functionals for the ground and the exited state and then we quantize the same model using the canonical path integral quantization as an integration over the canonical phase-space coordinates.*

PACS numbers: 11.10.Ef
Key words: Hamiltonian and Lagrangian approach, Hamilton-Jacobi method, nonlinear sigma model, quantization of field systems

1 Introduction

Following the Dirac's conjecture, a critical point in the study of a gauge model is the presence of the first-class constraints.[1,2] The first-class constraints are related to symmetries while the second-class ones may imply some ambiguities when treated as quantum operators. The physical status of a theory is chosen by imposing complementary conditions which are given by the first-class constraints. The presence of the second-class constraints is in general avoided.

Recently the canonical path integral quantization was initiated in Refs [3]–[5] and it is based on the Hamilton–Jacobi method[6−9] to investigate singular systems. In this method the equations of motion are obtained as total differential equations in many variables. If the system is integrable one can obtain the canonical reduced phase-space coordinates and then the path integral representation as an integration over the canonical phase-space coordinates. The advantage using the canonical path integral method is that there is not difference between the first- and second-class constraints and we do not need the gauge fixing term because the gauge variables are separated in the process of constructing an integrable system of total differential equation. Besides the canonical path integral formalism has been employed in different physical systems, for example, the free electromagnetic theory,[4] the Proca model, the harmonic oscillator,[5] the parametrization invariant theories,[3] relativistic particle,[10] Yang–Mills theory,[11] and most recently the Einstein gravitational field theory.[12]

On the other side, two-dimensional models have played an important role in theoretical physics as a laboratory where many interesting phenomena can be studied in a fashion which is usually easier to handle more realistic than four-dimensional theories.[13] One well-known model is the O(N) nonlinear sigma model. The O(3) nonlinear sigma model has interesting physical content in its phenomenological aspect. For instance, in the Euclidean space it describes classical antiferromagnetic spin system at their critical points,[14] while in the Minkowski one it delineates the long wavelength limit of quantum antiferromagnets.[15] Also the model exhibits solitons, Hopf instanton and novel spin and statistics in 2 + 1 space-time dimensions with inclusion of the Chern–Simons term.[16,17]

In this paper, we quantize the O(2) nonlinear sigma model using the functional Schrödinger representation and the canonical path integral formalism.

2 The Hamilton–Jacobi Formalism for Fields

The canonical formulation[6−9] gives the set of Hamilton–Jacobi partial differential equations (HJPDE) as

$$H'_\alpha\Big(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\alpha}\Big) = 0\,,$$
$$\alpha, \beta = 0, n-r+1, \dots, n\,, \qquad a = 1, \dots, n-r\,, \tag{1}$$

where

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha\,, \tag{2}$$

and H_0 is defined as

$$H_0 = p_a w_a + p_\mu \dot{q}_a \,\big|_{p_\nu = -H_\nu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a)\,,$$
$$\mu, \nu = n-r+1, \dots, n\,. \tag{3}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$\mathrm{d}q_a = \frac{\delta H'_\alpha}{\delta p_a}\mathrm{d}t_\alpha\,, \qquad \mathrm{d}p_a = -\frac{\delta H'_\alpha}{\delta q_a}\mathrm{d}t_\alpha\,, \tag{4}$$

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$$\mathrm{d}p_\beta = -\frac{\delta H'_\alpha}{\delta t_\beta}\,\mathrm{d}t_\alpha\,, \tag{5}$$

$$\mathrm{d}z = \Big(-H_\alpha + p_a\frac{\delta H'_\alpha}{\delta p_a}\Big)\,\mathrm{d}t_\alpha\,, \tag{6}$$

$$\alpha,\beta = 0, n-r+1,\ldots,n\,,\quad a=1,\ldots,n-r\,,$$

where $z = S(t_\alpha; q_a)$, and $\delta H/\delta x$ represents the variation of H with respect to x. The set of Eqs (4) $\sim$ (6) is integrable,[8,9] if

$$\mathrm{d}H'_0 = 0\,, \tag{7}$$

$$\mathrm{d}H'_\mu = 0\,,\quad \mu = n-r+1,\ldots,n\,. \tag{8}$$

If conditions (7) and (8) are not satisfied identically, one considers them as new constraints and again testes the consistency conditions. Hence, the canonical formulation leads to obtaining the set of canonical phase space coordinates q_a and p_a as functions of t_α, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians H'_α are considered as the infinitesimal generators of canonical transformations given by parameters t_α respectively.

3 Quantization of Constrained Systems

For the quantization of constrained systems we can use the operator quantization method,[1,2] or the path integral quantization method.[3−5]

Now we shall give a brief information about these two methods.

3.1 *Operator Quantization*

For the operator quantization method we have

$$H'_\alpha\Psi = 0\,,\qquad \alpha = 0, n-r+1,\ldots,n\,, \tag{9}$$

where Ψ is the wavefunction. The consistency conditions are

$$[H'_\mu, H'_\nu]\Psi = 0\,,\qquad \mu,\nu = 1,\ldots,r\,, \tag{10}$$

where $[\,,]$ is the commutator. The constraints H'_α are called the first-class constraints if they satisfy

$$[H'_\mu, H'_\nu] = C^\gamma_{\mu\nu}H'_\gamma\,. \tag{11}$$

In the case that the Hamiltonians H'_μ satisfy

$$[H'_\mu, H'_\nu] = C_{\mu\nu} \tag{12}$$

with $C_{\mu\nu}$ depending on q_i and p_i, then from Eq. (9) there arise naturally Dirac's brackets, and the canonical quantization will be performed by taking Dirac's brackets into commutators.

3.2 *The Canonical Path Integral Quantization Method*

The path integral quantization is an alternative method to perform the quantization of constrained systems.

Now we shall give a brief review of the canonical path integral formulation of constrained systems.[3−5]

If the set of Eqs (4) and (5) is integrable, one can solve them to obtain the canonical phase-space coordinates,

$$q_a \equiv q_a(t,t_\mu)\,,\quad p_a \equiv p_a(t,t_\mu)\,,\quad \mu = 1,\ldots,r\,. \tag{13}$$

In this case, the path integral representation may be written as[3−5]

$$Z(q'_a, t'_\alpha; q_a, t_\alpha) = \int_{q_a}^{q'_a} \mathrm{D}q^a\mathrm{D}p^a \times \exp\mathrm{i}\Big\{\int_{t_\alpha}^{t'_\alpha}\Big[-H_\alpha + p_a\frac{\delta H'_\alpha}{\delta p_a}\Big]\mathrm{d}t_\alpha\Big\}\,,$$
$$a = 1,\ldots,n-r,\quad \alpha = 0,n-r+1,\ldots,n. \tag{14}$$

One should notice that the integral (14) is an integration over the canonical phase-space coordinates (q_a, p_a).

4 Quantization of O(N) Nonlinear Sigma Model

The O(N) nonlinear sigma model is described by the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a\partial^\mu\phi^a) + \frac{1}{2}\lambda(\phi^a\phi^a - 1)\,, \tag{15}$$

where $\mu = 0,1$, a is an index related to the O(N) symmetry group and λ is a Lagrange multiplier.

Following the canonical method,[6−9] we obtain the set of Hamilton–Jacobi partial differential equations as

$$H'_0 = \int \mathrm{d}x\Big[p_0 + \frac{1}{2}\pi^a\pi^a - \frac{1}{2}(\partial_i\phi^a)(\partial^i\phi^a) - \frac{1}{2}\lambda(\phi^a\phi^a - 1)\Big] = 0\,, \tag{16}$$

$$H'_\lambda = \int \mathrm{d}x(\pi_\lambda) = 0\,, \tag{17}$$

here $\pi_a = \dot\phi^a/2$ and π_λ represents the momentum conjugated to λ. To check whether this system is integrable or not, let us consider the total variation of Eq. (17). In fact

$$\mathrm{d}H'_\lambda = \int \mathrm{d}x\frac{1}{2}(\phi^a\phi^a - 1)\,\mathrm{d}t = H'_2 = 0\,. \tag{18}$$

The integral condition $\mathrm{d}H'_2 = 0$ gives

$$\phi^a\pi^a = 0\,. \tag{19}$$

The total variation of the constraint $\phi^a\pi^a = 0$ leads to the constraint

$$\lambda - \frac{1}{\phi^a\phi^a}\Big[\frac{1}{2}\pi^a\pi^a - \frac{1}{2}(\partial_i\phi^a\partial_i\phi^a)\Big] = 0\,. \tag{20}$$

Making use of Eqs (18) and (19) and inserting the value of λ from Eq. (20) into Eq. (16), we find the canonical Hamiltonians for our model,

$$H'_{c_0} = \int \mathrm{d}x\Big[p_0 + \frac{1}{2}(\pi^a\pi^a)(\phi^a\phi^a) - \frac{1}{2}(\phi^a\pi_a)(\phi^b\pi_b) - \frac{1}{2}\phi^a\partial_i\partial_i\phi^a\frac{1}{\phi^b\phi^b}\Big] = 0\,, \tag{21}$$

$$H'_\lambda = \int \mathrm{d}x(\pi_\lambda) = 0\,. \tag{22}$$

We would like to stress on the fact that we have two involution canonical Hamiltonians: H'_{c_0} and $H'{}_\lambda$, which are considered as infinitesimal generators of canonical transformations given by parameters t and λ respectively.

As was clarified previously that we use the integrability conditions to obtain the canonical reduced phase space coordinates (ϕ^a, π^a) as functions of parameters (t, λ) and then we can use the quantization procedures discussed in Sec. 3. Here we identify two main differences between Ref. [18] and the present work. Firstly, we have only the two canonical Hamiltonians which are the canonical set of Hamilton–Jacobi partial differential equations (21) and (22), and not the three Hamiltonians. Secondly, our model is integrable if the condition (20) is satisfied, and hence we obtain the reduced phase space coordinates and the canonical Hamiltonian without any need to introduce nonphysical fields by enlarging the initial phase space and using the improved Batalin–Fradkin–Tyutin formalism[19] (for more details see Ref. [20]).

For the sake of simplicity, we restrict our attention to the case for two fields ($N = 2$). Therefore, we rewrite H'_{c_0} in terms of the fields ϕ_1 and ϕ_2 and their respective momenta π_1 and π_2 as

$$H'_{c_0} = \int \Big\{ p_0 + \frac{1}{2}\Big[(\pi_1\phi_2 - \pi_2\phi_1)^2 - \frac{1}{\phi_1^2+\phi_2^2}(\phi_1\partial_i\partial_i\phi_1 + \phi_2\partial_i\partial_i\phi_2)\Big]\Big\}\mathrm{d}x\,. \tag{23}$$

Following the procedure mentioned in Sec. 3, we can quantize the O(2) nonlinear sigma model using the functional Schrödinger quantization procedure,

$$\hat{H}'_{c_0}\Psi(\phi_1,\phi_2,\lambda,t) = 0\,, \tag{24}$$

$$\hat{H}'_\lambda\Psi(\phi_1,\phi_2,\lambda,t) = 0\,. \tag{25}$$

In Eq. (24) the momenta π_1, π_2 are replaced by the following functional derivatives,

$$\pi_1(x) \to -\mathrm{i}\frac{\delta}{\delta\phi_1(x)}\,, \qquad \pi_2(x) \to -\mathrm{i}\frac{\delta}{\delta\phi_2(x)}\,. \tag{26}$$

One confirms that $\Psi(\phi_1,\phi_2,\lambda,t) = \Psi(\phi_1,\phi_2,t)$ in Eqs (24) and (25). Hence, $\Psi(\phi_1,\phi_2,t)$ satisfies the Schrödinger equation

$$\hat{H}'_{c_0}\Psi(\phi_1,\phi_2,t) = 0\,, \tag{27}$$

which satisfies the following functional differential equation,

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(\phi_1,\phi_2,t) = \hat{H}_{c_0}\Psi(\phi_1,\phi_2,t)\,, \tag{28}$$

where $\hat{H}_{c_0}$ is given by

$$\hat{H}_{c_0} = \frac{1}{2}\int\Big[\Big(-\mathrm{i}\frac{\delta}{\delta\phi_1}\phi_2 + \mathrm{i}\frac{\delta}{\delta\phi_2}\phi_1\Big)^2 - \frac{1}{\phi_1^2+\phi_2^2}(\phi_1\partial_i\partial_i\phi_1 + \phi_2\partial_i\partial_i\phi_2)\Big]\mathrm{d}x\,. \tag{29}$$

One should notice that the O(2) symmetry of our sigma model and the functional Schrödinger equation (28) will be simplified using the polar transformation from the original fields (ϕ_1,ϕ_2) to new fields (R,Θ),

$$\phi_1 = R\sin\Theta\,, \qquad \phi_2 = R\cos\Theta\,. \tag{30}$$

$\hat{H}_{c_0}$ is written as

$$\hat{H}_{c_0} = \frac{1}{2}\int\Big(-\frac{\delta^2}{\delta\Theta^2} - \Theta\partial_i\partial_i\Theta - \frac{1}{R}\partial_i\partial_i R\Big)\mathrm{d}x\,. \tag{31}$$

The factor-ordering ambiguity in Eq. (31) is solved using the so-called "symmetric factor-ordering".[21]

Since $\hat{H}_{c_0}$ does not depend on time, we may separate out the time dependence of the wave functional,

$$\Psi(R,\Theta,t) = \mathrm{e}^{-\mathrm{i}Et}\Psi(R,\Theta)\,. \tag{32}$$

$\Psi(R,\Theta)$ satisfies the time-independent Schrödinger functional equation,

$$\int\Big(-\frac{\delta^2\Psi}{\delta\Theta^2} - \Theta\partial_i\partial_i\Theta\Psi - \frac{1}{R}\partial_i\partial_i R\Psi\Big)\mathrm{d}x = 2E\Psi\,. \tag{33}$$

Following Ref. [22], we obtain the ground state wavefunction

$$\Psi_0[\tilde{\Theta}(k)] = \prod_k\Big(\frac{k}{\pi}\Big)^{1/4}\exp\Big[\frac{-1}{4\pi}k\tilde{\Theta}^2(k)\Big]. \tag{34}$$

This is just the infinite product of harmonic oscillators' ground state wavefunctions, one wavefunction for each k.

The first excited state is calculated as

$$\Psi_1[R,\Theta] = \Big(\frac{k_1}{\pi}\Big)^{1/2}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}k_1 y}\Theta(y)\Psi_0[\Theta]\exp\Big[\int \mathrm{d}z\frac{1}{R(z)}\partial_i\partial_i R(z)\Big]. \tag{35}$$

It agrees with Ref. [23].

Now we would like to quantize our model using the canonical path integral formalism.[3–5]

From Eqs (21) and (22) and taking into account Eq. (6) we get the action integral in terms of the canonical phase-space coordinates as follows:

$$z = \int \mathrm{d}x\mathrm{d}t\Big\{\pi_1\dot{\phi}_1 + \pi_2\dot{\phi}_2 - \frac{1}{2}\Big[(\pi_1\phi_2 - \pi_2\phi_1)^2 - \frac{1}{\phi_1^2+\phi_2^2}(\phi_1\partial_i\partial_i\phi_1 + \phi_2\partial_i\partial_i\phi_2)\Big]\Big\}. \tag{36}$$

Making use of Eqs (14) and (36), the path integral for O(2) nonlinear sigma model is put in the following expression

$$Z = \int\prod\mathcal{D}\phi^1\mathcal{D}\phi^2\mathcal{D}\pi^1\mathcal{D}\phi^2\exp\int\mathrm{d}x\mathrm{d}t\Big\{\pi_1\dot{\phi}_1 + \pi_2\dot{\phi}_2 - \frac{1}{2}\Big[(\pi_1\phi_2 - \pi_2\phi_1)^2 - \frac{1}{\phi_1^2+\phi_2^2}(\phi_1\partial_i\partial_i\phi_1 + \phi_2\partial_i\partial_i\phi_2)\Big]\Big\}. \tag{37}$$

The path integral representation (37) is an integration over the canonical phase-space coordinates (ϕ^1, π^1) and (ϕ^2, π^2).

5 Conclusion

In this work we have studied the $\mathrm{O}(N)$ nonlinear sigma model by using the Hamilton–Jacobi formalism.[6−9]

Following the prescriptions of this method, we obtained the set of Hamilton–Jacobi partial differential equations which have two Hamiltonians H'_0 and H'_λ. The integral conditions lead us to obtain the canonical reduced phase-space coordinates (ϕ^a, π^a) in terms of parameters (t, λ), and the canonical Hamiltonians $(H'_{c_0}, H'{}_\lambda)$ are obtained in terms of the canonical variables.

Then we quantize this model for $(N = 2)$ by making use of the functional Schrödinger representation to compute the ground and excited state wave functional. Our results are in complete agreement with Ref. [23].

Finally, we use the canonical path integral method[3−5] to obtain the path integral quantization for our model as an integration over the canonical phase space coordinates (ϕ^a, π^a).

The advantage of our path integral formalism is that there is not difference between the first- and second-class constraints, and we do not need to enlarge the initial phase space by introducing unphysical auxiliary fields (see Ref. [20] and the references therein). All the needs are the set of Hamilton–Jacobi partial differential equations and the equations of motion. If the system is integrable, one can obtain the reduced canonical phase-space coordinates q_a and p_a $(a = 1, \ldots, n - r)$ in terms of parameters t_α $(\alpha = 0, \ldots, n - r + 1)$. In this case the path integral is obtained directly as an integration over the canonical reduced phase-space coordinates q_a and p_a. Besides, we can quantize this system using the functional Schrödinger representation,

$$\hat{H}'_{c_\alpha} \Psi = 0\,, \tag{38}$$

where $\hat{H}'_{c_\alpha}$ is the canonical reduced Hamiltonian.

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